Rationality of Virasoro Vertex Operator Algebras

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1 Introduction

Vertex operator algebras (VOA) were introduced by Borcherds ([B]) as an axiomatic description of the 'holomorphic part' of a conformal field theory ([BPZ]). An account of the theory of vertex operator algebras may be found in [FLM]. One of the most important examples of VOAs ([FZ]) is the *Virasoro VOAs*, i.e.VOAs corresponding to the representations of the Virasoro algebra L, denoted by V_c in this paper, where V_c is the (unique) irreducible highest weight representation of L with highest weight $(c, 0)$. It is conjectured ([FZ]) that V_c is rational if and only if $c = c_{p,q} = 1 - 6(p-q)^2/pq$, where $p, q \in \{2, 3, 4, ...\}$, and p, q are relatively prime. (The definition of rationality of VOAs is given in Section 2). In this paper we prove this conjecture , and show that when $c = c_{p,q}$, $p, q \in \{2, 3, 4, ...\}$, and $(p, q) = 1$, all the irreducible representations of the *Virasoro VOAs* are precisely those which correspond to irreducible minimal modules of the Virasoro algebra. Then we will prove the fusion rules in the minimal series cases which were stated implicitly in [FF2].

Y.Zhu in [Z] constructed an associative algebra $A(V)$ for a general VOA V and established a 1-1 correspondence between irreducible representations of V and irreducible representations of $A(V)$. This construction enabled I.Frenkel and Y.Zhu to prove the rationality of VOAs associated to the representations of affine Kač-Moody algebras with positive integral level. Here we will prove the rationality of the *Virasoro VOAs* with the help of Zhu's construction.

This paper is organized in the following manner. In Section 2 we recall the definition of VOAs and representations of VOAs. In Section 3 we review Zhu's construction and his theorem on the 1-1 correspondence between irreducible representations of VOA V and irreducible representations of $A(V)$. The proofs of the results in Section 3 can be found in [Z]. In Section 4 we compute $A(V_c)$ and then prove the main results of this paper. In order to organize the material of Section 3 in a neat way, we defer the proof of a statement in Lemma 4.1 of Section 4 to the Appendix.

2 Basic notions

For a rational function $f(z_1, z_2)$, with poles only possible at $z_1 = z_2, z_1 = 0, z_2 = 0$, we denote by $\iota_{z_1,z_2} f(z_1, z_2)$ the power series expansion of $f(z_1, z_2)$ in the domain $\mid z_1 \mid > \mid z_2 \mid$.

Definition 2.1 *A* vertex operator algebra *is a Z_+-graded vector space $V = \bigoplus_{n=0}^{\infty} V_n$ with a sequence of linear operators $\{a(n) \mid n \in Z\} \subset End\ V$ associated to every $a \in V$, such that for fixed $a, b \in V$, $a(n)b = 0$ for n sufficiently large. The generating function $Y(a, z) = \sum_{n \in Z} a(n) z^{-n-1} \in (End\ V)[[z, z^{-1}]]$ is called the* vertex operator of a, *and the following axioms are satisfied:*

(A1) $Y(a, z) = 0$ *iff* $a = 0$.

(A2) *There is a distinguished element called the* vacuum *(which we denote by 1), satisfying*

$$Y(1, z) = I_V. \, (I_V \text{ is the identity in} End\ V)$$

(A3) *There is a distinguished element $\omega \in V$ (called the* Virasoro element*), whose corresponding vertex operator we write in the form*

$$Y(\omega, z) = \sum_{n \in Z} \omega(n) z^{-n-1} = \sum_{n \in Z} L_n z^{-n-2},$$

such that

$$L_0 \mid_{V_n} = n I_{V_n},$$

$$Y(L_{-1}a, z) = \frac{d}{dz} Y(a, z) \quad \textit{for every } a \in V,$$

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m+n,0} \frac{m^3 - m}{12} c,$$

where c is some constant in C, which is called the rank *of V.*

(A4) *The* Jacobi identity *holds, i.e.*

$$\begin{aligned} & \mathrm{Res}_{z-w}(Y(Y(a, z-w)b, w)\iota_{w,z-w}((z-w)^m z^n)) \\ = \ & \mathrm{Res}_z(Y(a, z)Y(b, w)\iota_{z,w}(z-w)^m z^n) - \mathrm{Res}_z(Y(b, w)Y(a, z)\iota_{w,z}(z-w)^m z^n), \end{aligned}$$

for any $m, n \in Z$.

An element $a \in V$ is called *homogeneous* of *degree* n if $a \in V_n$. We write $\deg a = n$.

Definition 2.2 *Given a vertex operator algebra V,* a representation of V (or V-module) *is a Z_+-graded vector space $M = \bigoplus_{n=0}^{\infty} M_n$, such that there is a linear map*

$$V \rightarrow (End\ M)[[z, z^{-1}]],$$

$$a \mapsto Y_M(a, z) = \sum_{n \in Z} a(n) z^{-n-1},$$

satisfying the following axioms:

(R1) $a(n)M_m \subset M_{m+\deg a - n - 1}$ *for every homogeneous a.*

(R2) $Y_M(1, z) = I_M$.

Setting

$$Y_M(\omega, z) = \sum_{n \in Z} L_n z^{-n-2},$$

(R3) $[L_m, L_n] = (m-n)L_{m+n} + \delta_{m+n,0} \frac{m^3-m}{12} c,$

$$Y_M(L_{-1}a, z) = \tfrac{d}{dz} Y_M(a, z) \textit{ for every} a \in V.$$

(R4) *The* Jacobi identity *holds, i.e.*

$$\begin{aligned} & Res_{z-w}(Y_M(Y(a, z-w)b, w)\iota_{w,z-w}((z-w)^m z^n)) = \\ = \ & Res_z(Y_M(a, z)Y_M(b, w)\iota_{z,w}(z-w)^m z^n) - Res_z(Y_M(b, w)Y_M(a, z)\iota_{w,z}(z-w)^m z^n), \end{aligned}$$

for any $m, n \in Z$.

Without loss of generality, we assume that $M_0 \neq 0$ if $M \neq 0$.

The notions of submodules, quotient modules, submodules generated by subsets, direct sums, irreducible modules, completely reducible modules, etc., can be introduced in the usual way. As a module over itself, V is called the adjoint module. A submodule of the adjoint module is called an ideal of V. Given an ideal I in V such that $1 \notin I, \omega \notin I$, the quotient V/I admits a natural VOA structure.

Definition 2.3 *A VOA is called* rational *if it has only finitely many irreducible modules, and moreover every finitely generated module is a direct sum of irreducibles.*

3 Zhu's construction

For every vertex operator algebra V, Zhu defined in [Z] an associative algebra $A(V)$. Moreover for any V-module $M = \bigoplus_{n \in Z_+} M_n$, the top level M_0 admits a natural structure of an $A(V)$-module. The proof of all the statements in this section may be found in [Z].

Definition 3.1 *We define a bilinear operation $*$ on V as follows: for any homogeneous $a \in V$,we let*

$$a * b = Res_z(Y(a,z)\frac{(z+1)^{\deg a}}{z}b), b \in V.$$

Denote by $O(V)$ the linear span of elements

$$Res_z((Y(a,z)\frac{(z+1)^{\deg a}}{z^2}b), b \in V.$$

Then $O(V)$ is a two-sided ideal of the algebra V with the operation $*$ ([Z]). We let $A(V)$ to be the quotient algebra $V/O(V)$.

Convention 3.1 *In this paper,we use $\sim$ in the following way. For $a, b \in V, a \sim b$ means that $a - b \equiv 0 \, (mod \, O(V))$. For $f, g \in End\ V, f \sim g$ means that $f \cdot c \sim g \cdot c$ for any $c \in V$. We use $[a]$ to denote the image of a in V under the projection from V to $A(V)$.*

Proposition 3.1 *Let I be an ideal of V such that $1 \notin I, \omega \notin I$. We have $A(V/I) \cong A(V)/[I]$, where $[I]$ is the image of I under the projection from V to $A(V)$.*

We list some of the properties of $A(V)$ and $O(V)$ below.

Theorem 3.1 *$A(V)$ is an associative algebra with multiplication $*$. Furthermore, we have*

1) [1] is the unit element of the associative algebra $A(V)$.

2) $[\omega]$ is in the center of $A(V)$.

Lemma 3.1 $L_{-1} + L_0 \sim 0$.

Lemma 3.2 *For every homogeneous element a, $b \in V$, and $m \geq n \geq 0$,*

$$Res_z((Y(a,z)\frac{(z+1)^{\deg a+n}}{z^{2+m}}b) \in O(V)$$

Lemma 3.3 *If a and b are homogeneous elements of V, then we have*

$$a * b \sim Res_z(Y(b,z)\frac{(z+1)^{\deg b-1}}{z}a).$$

For a V-module $M = \bigoplus_{n=0}^{\infty} M_n$, among the components $a(n)$ of the vertex operator $Y_M(a,z)$, the component $a(\deg a - 1)$ ($\in End\ M$) is of special interest, since $a(\deg a - 1)$ preserves the gradation of M. We denote $a(\deg a - 1)$ by $o(a)$.

Theorem 3.2 *If $M = \bigoplus_{n=0}^{\infty} M_n$ is a V-module, then the subspace M_0 is a representation of the associative algebra $A(V)$ with the action given as follows: for $[a] \in A(V)$, denote by $a \in V$ a preimage of $[a]$, then $[a]$ acts on M_0 as $o(a)$.*

Theorem 3.3 *Given a representation (W, π) of the associative algebra $A(V)$,there exists a representation $M = \bigoplus_{n \in Z_+} M_n$ of VOA V, such that $M_0 = W$, and $o(a)v = \pi([a])v$ for every $a \in V$, $v \in W$, where $[a]$ is the image of a in $A(V)$. Furthermore, the above construction gives a bijective correspondence between the set of irreducible $A(V)$-modules and the set of irreducible V-modules.*

Remark 3.1 *We define a bilinear operation $\tilde{*}$ on V as follows: for any homogeneous $a \in V$,*

$$a\tilde{*}b = Res_z(Y(a,z)\frac{(1-z)^{\deg a}}{z}b), b \in V.$$

Denote by $\tilde{O}(V)$ the linear span of elements of the form

$$Res_z((Y(a,z)\frac{(1-z)^{\deg a}}{z^2}b), b \in V.$$

Let $\tilde{A}(V)$ to be the quotient space $V/\tilde{O}(V)$. And let $[\tilde{a}] \in \tilde{A}(V)$ act on the top level of $M = \bigoplus_{n \in Z_+} M_n$ as $(-1)^{\deg a} o(a)$. Then everything in this section works in a parallel way.

4 Rationality of $V_{c_{p,q}}$ and the fusion rules

Let us recall first that the Virasoro algebra is the Lie algebra $L = \bigoplus_{n=-\infty}^{\infty} CL_n \bigoplus CC$ with commutation relations

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m+n,0}\frac{m^3-m}{12}C,$$

$$[L_m, C] = 0.$$

Set

$$L_+ = \bigoplus_{n=1}^{\infty} CL_n, \;\; L_- = \bigoplus_{n=1}^{\infty} CL_{-n}.$$

Given complex numbers c and h, the Verma module $M_{c,h}$ over L is a free $U(L_-)$-module generated by 1, such that $L_+1 = 0, L_0 1 = h \cdot 1$ and $C \cdot 1 = c \cdot 1$. There exists a unique maximal proper submodule of $M_{c,h}$, say $J_{c,h}$. Denote the quotient $M_{c,h} \,/\, J_{c,h}$ by $L_{c,h}$.Recall that $v \in M_{c,h}$ is called a singular vector if $L_+ v = 0$ and v is an eigenvector of L_0. For example, $L_{-1}1$ is a singular vector of $M_{c,0}$ for any c. Denote $M_{c,0} \,/ < L_{-1}1 >$ by M_c, where $< L_{-1}1 >$ is the submodule of $M_{c,0}$ generated by the singular vector $L_{-1}1$.

It was proved in ([FZ]) that M_c and $L_{c,0}$ admit natural VOA structures,with the Virasoro elment $\omega = L_{-2}1$. We denote the VOA $L_{c,0}$ by V_c to emphasize the VOA structure on $L_{c,0}$.

There is a natural gradation on $M_{c,0}$, M_c and V_c given by

$$\deg L_{-i_1}L_{-i_2}\cdots L_{-i_n}1 = i_1 + i_2 + \cdots + i_n.$$

Lemma 4.1 *There exists an isomorphism of associative algebras, $F : A(M_c) \cong C[x]$, given by $[\omega]^n \mapsto x^n$,where $C[x]$ is the polynomial algebra in one generator x.*

Proof: By Lemma 3.2 we have

$$(L_{-n-3} + 2L_{-n-2} + L_{-n-1})b = \mathrm{Res}_z(Y(\omega, z)\frac{(z+1)^2}{z^{2+n}}b) \in O(M_c), \tag{4.1}$$

for every $n \geq 0, b \in M_c$. It follows by induction that

$$L_{-n} \sim (-1)^n((n-1)(L_{-2} + L_{-1}) + L_0), \text{for every n} \geq 1. \tag{4.2}$$

Note that the first step of the induction is based on Lemma 3.1.

By Lemma 3.3, we have

$$[b] * [\omega] = [(L_{-2} + L_{-1})b]. \tag{4.3}$$

Note that

$$\begin{aligned} & L_0 L_{-i_1}L_{-i_2}\cdots L_{-i_n}1 = \\ = \ & (i_1 + i_2 + \cdots + i_n)L_{-i_1}L_{-i_2}\cdots L_{-i_n}1. \end{aligned} \tag{4.4}$$

Using (4.2) and (4.4), it is easy to show by induction that

$$[L_{-i_1}L_{-i_2}\cdots L_{-i_n}1] = P([\omega]) \text{ for some } P(x) \in C[x].$$

Since

$$L_{-i_1}L_{-i_2}\cdots L_{-i_n}1,\ i_1 \geq i_2 \geq \cdots \geq i_n \geq 2,$$

generate M_c, the homomorphism of associative algebras

$$F : C[x] \to A(M_c)$$

given by $x^n \mapsto [\omega]^n$ is surjective. (This homomorphism is well-defined since $[\omega]$ is in the center of $A(M_c)$.)

To prove that F is also injective, it suffices to show that the elements in (4.1) generate the whole $O(M_c)$. For a proof of this statement see Appendix. □

Set

$$c_{p,q} = 1 - 6\frac{(p-q)^2}{pq}.$$

Convention 4.1 *For $v \in M_c$, we also write $v \sim P(x)$ if $[v] = P(\,[\omega]\,)$ in $A(M_c)$, by abuse of the symbol $\sim$.*

Whenever we mention $\mathbf{c_{p,q}}$ again, we always assume that $\mathbf{p, q} \in \{\mathbf{2, 3, 4}, ...\}$, and $\mathbf{p, q}$ are relatively prime.

Lemma 4.2 *1) $J_{c,0}$ is generated by the singular vector $L_{-1}1$ if $c \neq c_{p,q}$.*

2) $J_{c,0}$ is generated by two singular vectors if $c = c_{p,q}$. One of them is $L_{-1}1$. The other is denoted by $v_{p,q}$, where $\deg v_{p,q} = (p-1)(q-1)$.

Proof: It follows from Kač's determinant formula ([K]) and the submodule structure theorem of $M_{c,h}$ by B.Feigin and D.Fuchs ([FF1]). □

From this lemma we immediately have the following.

Corollary 4.1 *If $c \neq c_{p,q}$, then V_c is not rational.*

Proof: By Lemma 4.2, $V_c = M_c$ if $c \neq c_{p,q}$. It follows from Lemma 4.1 that $A(V_c) \cong C[x]$. Since there are infinitely many (1-dimensional) irreducible representations of $C[x]$, V_c is not rational by Theorem 3.3. □

From now on, we always assume that $\mathbf{c = c_{p,q}}$.

It follows from Lemma 4.1 that $V_c = M_c / < v_{p,q} >$, where $< v_{p,q} >$ denotes the submodule of M_c generated by $v_{p,q}$.

We rewrite $v_{p,q} = \sigma_{p-1,q-1}1$, where $\sigma_{p-1,q-1} \in U(L_-)$.

Lemma 4.3 *The coefficient of the term $L_{-2}^{\frac{1}{2}(p-1)(q-1)}1$ in $v_{p,q}$ is nonzero.*

Proof: A projective formula of singular vectors was given in [FF1] as follows:

$$(\pi(\sigma_{p-1,q-1}))^2 = \prod_{i=o}^{p-2}\prod_{j=o}^{q-2}[L_{-1}^2 + ((p-2-2i)\theta^{-1} + (q-2-2j)\theta)^2 L_{-2}],$$

where $\theta^2 = -q/p$, π is the projection from $U(L_-)$ to $U(L_-)/U(L_-)L_{-3}$. Then it is easy to see that the coefficient of $L_{-2}^{\frac{1}{2}(p-1)(q-1)}$

$$\prod_{i=o}^{p-2}\prod_{j=o}^{q-2}((p-2-2i)\theta^{-1}+(q-2-2j)\theta)^2 \neq 0. \quad \Box$$

$v_{p,q}$ is unique up to a nonzero scalar. Now we can fix $v_{p,q}$ by letting the coefficient of the term $L_{-2}^{\frac{1}{2}(p-1)(q-1)}1$ be 1.

We call n the length of the monomial

$$L_{-i_1}L_{-i_2}\cdots L_{-i_n}1,\ i_1 \geq i_2 \geq \cdots \geq i_n \geq 2.$$

Thanks to Lemma 4.1 we can assume $v_{p,q} \sim G_{p,q}(x)$ for some $G_{p,q}(x) \in C[x]$. Then we have the following.

Proposition 4.1 *$A(V_c) \cong C[x]/ < G_{p,q}(x) >$, where* $\deg G_{p,q} = \frac{1}{2}(p-1)(q-1)$.

Proof: From the argument of Lemma 4.1, we see that a monomial in M_c of length n corresponds to a polynomial of $\deg n$ in $C[x]$ by means of $F^{-1} : A(M_c) \cong C[x]$. Since $L_{-2}^{\frac{1}{2}(p-1)(q-1)}1$ is the only term in $v_{p,q}$ which has the maximal length $\frac{1}{2}(p-1)(q-1)$ among the monomials in $v_{p,q}$, we have $\deg G_{p,q} = \frac{1}{2}(p-1)(q-1)$.

Using Lemma 3.1 and (4.2), (4.4), it is easy to prove that

$$L_{-i_1}L_{-i_2}...L_{-i_n}v_{p,q} \sim F(x)G_{p,q}(x),\ \ i_1 \geq i_2 \geq \cdots \geq i_n \geq 1$$

for some $F(x) \in C[x]$. Now this proposition follows from Proposition 3.1. $\Box$

Now we need to digress on nilpotent subalgebras of L and the coinvariants.

Recall that the Virasoro algebra is the central extension of $diff\,(S^1)$, the polynomial vector fields on S^1. Let $L_{0,0}$ $(resp.\, L_{1,1})$ be the Lie subalgebra of vector fields of the form $z(z+1)p(z)\frac{d}{dz}, p(z) \in C[z]$ $(\,resp.\, z^2(z+1)^2p(z)\frac{d}{dz})$. $L_{1,1}$ is a two-codimensional ideal of $L_{0,0}$. Set

$$e_0^{'} = z(z+1)^2\frac{d}{dz}, \quad e_0^{''} = z^2(z+1)\frac{d}{dz}\ .$$

Identifying L_n with $z^{-n+1}\frac{d}{dz}$, we have

$$L_{1,1} = \bigoplus_{n\in Z_+} C(L_{-n-3}+2L_{-n-2}+L_{-n-1}),$$

and $e_0^{'} = L_{-2} + 2L_{-1} + L_0$, $e_0^{''} = L_{-2} + L_{-1}$.

An ordered triple of pairs of integers $((m,n),(m^{'},n^{'}),(m^{''},n^{''}))$ is *admissible* [1] if $0 < m, m^{'}, m^{''} < p$, $0 < n, n^{'}, n^{''} < q$, $m + m^{'} + m^{''} < 2p$, $n + n^{'} + n^{''} < 2q$, $m < m^{'} + m''$, $m^{'} < m + m''$, $m'' < m + m'$, $n < n^{'} + n^{''}$, $n^{'} < n + n^{''}$, $n^{''} < n + n^{'}$ and the sums $m + m^{'} + m^{''}$, $n + n^{'} + n^{''}$ are odd. We identify the triples $((m,n),(m^{'},n^{'}),(m^{''},n^{''}))$ and $((m,n),(p-m^{'},q-n^{'}),(p-m^{''},q-n^{''}))$.

Recall that given a Lie algebra g, and a g-module M, the coinvariant (i.e. 0-dimensional homology) of g with coefficients in M, denoted by $H_0(g,M)$, is defined to be the quotient $M\,/\,g\cdot M$.

Let

$$h_{m,n} = \frac{(np-mq)^2-(p-q)^2}{4pq}.$$

$L_{c,h_{m,n}}, 0 < m < p,\ 0 < n < q$ is called the *minimal* module of L.

The following theorem is from [FF2].

Theorem 4.1 *Let* $\{((m,n),\ (m_i^{'},n_i^{'}),\ (m_i^{''},n_i^{''})), i = 1,...,N\}$ *be the set of all admissible triples with the fixed first pair* (m,n)*. Then* $H_0(L_{1,1};\ L_{c,h_{m,n}})$ *can be decomposed into a sum of* N *1-dimensional spaces, such that on the i-th one of these spaces* $e_0^{'}$ *and* $e_0^{''}$ *act as the multiplications by* $h_{m_i^{'},n_i^{'}}$ *and* $h_{m_i^{''},n_i^{''}}$ *respectively.* $N = \frac{1}{2}mn(p-m)(q-n)$.

Remark 4.1 *There is a slight difference between here and [FF2]. In [FF2] ,* $L_{0,0}$, $(resp.\,L_{1,1})$ *appeared to be the Lie subalgebra of vector fields of the form* $z(z-1)p(z)\frac{d}{dz}$, $(resp.\,z^2(z-1)^2p(z)\frac{d}{dz}), p(z) \in C[z]$, *and* $e_0^{'}$ *was* $z(z-1)^2\frac{d}{dz}$, $e_0^{''}$ *was* $z^2(z-1)\frac{d}{dz}$. *But the conclusion and the argument of the theorem remain the same after the changes of signs.*

We consider the case $(m,n) = (1,1)$. $h_{1,1} = 0$, $L_{c_{p,q},\,h_{1,1}} = V_c$. Note that the above admissible triples are exactly $((1,1),\ (m,n),\ (m,n))$, $0 < m < p$, $0 < n < q$. And the action of $e_0^{'}$ and $e_0^{''}$ are the same on $H_0(L_{1,1};\ L_{c_{p,q},\,h_{1,1}})$.

Proposition 4.2 $A(V_c) = H_0(L_{1,1};\ L_{c_{p,q},\,h_{1,1}}) \cong C[x]/ < G_{p,q}(x) >$, *where* $G_{p,q}^2 = \Pi_{m=1}^{p-1}\Pi_{n=1}^{q-1}(x-h_{m,n})$.

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[1] The admissibility condition of [FF2] is incorrect although the argument there is right. And from their argument we can get the admissibility condition given here.

Proof: This follows from the argument above and Lemma 4.1. □

For the following lemma see [DGK] or [FF2] .

Lemma 4.4 *Let S, $S^{'}$ be two minimal L-modules. Then*

$$Ext^1(S,\, S^{'}) = 0.$$

Now we come to our main theorems.

Theorem 4.2 *The vertex operator algebra V_c with $c = c_{p,q}$ is rational. And the minimal modules $L_{c,h_{m,n}}, 0 < m < p, 0 < n < q$ are all the irreducible representations of V_c.*

Proof: It is obvious that all the irreducible representations of the associative algebra $A(V_c) \cong C[x]/ < G_{p,q}(x) >$ are one dimensional with $[\omega]$ acting as multiplications by $h_{m,n}$. $[\omega] \in A(V_c)$ acts on the top level of a irreducible V_c-module via $\omega(\deg \omega - 1) = L_0$ by Theorem 3.2.

If M is a V_c-module, then M is a L-module. And all L-submodules (resp. quotient modules) of M are V_c-modules. Then any irreducible module of V_c should be of the form $L_{c,h}$ for some $h \in C$.

Thus by Proposition 4.2 and Theorem 3.3, we see that the minimal modules $L_{c,h_{m,n}}, 0 < m < p, 0 < n < q$, are exactly all the irreducible representations of V_c.

A highest weight representation $V_{c,h}$ of L is not a representation of V_c unless $V_{c,h}$ is one of the minimal modules. Indeed, if $h \neq h_{m,n}$, then obviously $V_{c,h}$ is not a V_c-module. If $h = h_{m,n}$ but $V_{c,h}$ is reducible, then it contains some highest weight representation $V_{c,h'}$, where $h^{'} \neq h_{m,n}$ for any $0 < m < p, 0 < n < q$, which is not a V_c-module.

Now assume $M = \bigoplus_{n=0}^{\infty} M_n$ is finitely generated by vectors $w_1, \cdots, w_s \in M$. Let k be the maximal degree of $w_1, \cdots, w_s$. There exist finitely many singular vectors in $\bigoplus_{n=0}^{k} M_n$ (because $dim\,(\bigoplus_{n=0}^{k} M_n) < \infty.$). Let v be one of them. Then the highest weight submodule $S = U(L_-)v$ of M must be isomorphic to some $L_{c,h_{m,n}}$, $0 < m < p$, $0 < n < q$. Denote by P the projection from M to $M^1 = M/S$. $M^1 = \bigoplus_{n=0}^{\infty} M_n^1$ is also a V_c-module with finitely many generators $P(w_1), \cdots, P(w_s)$.

(Note that some of $P(w_i)$ may be 0.) Hence the maximal degree of $P(w_1), \cdots, P(w_s)$ is at most k. M^1 has also finitely many singular vectors in $\bigoplus_{n=0}^{k} M_n^1$. Furthermore $dim\,(\bigoplus_{n=0}^{k} M_n^1) < dim(\bigoplus_{n=0}^{k} M_n) < \infty$. By induction on $dim\,(\bigoplus_{n=0}^{k} M_n)$ we can assume that M^1 is a completely reducible module whose irreducible submodules are the irreducible minimal ones. By Lemma 4.4, so is M. □

Now let us sketch the proof of the fusion rules between the minimal modules over V_c. For the definition of fusion rules see [FHL]. For any vertex operator algebra V and a V-module M, a bimodule $A(M)$ over the associative algebra $A(V)$ can be constructed ([FZ]). Relations between the fusion rules and the bimodule $A(M)$ were given in Theorem 1.5.2 and 1.5.3 of [FZ]. Using a similar argument to the one which leads to Proposition 4.2, we see that

$$A(L_{c,h_{m,n}}) = H_0(L_{1,1};\, L_{c,h_{m,n}})$$

with $[\omega] \in A(V_c)$ acting on the left as $e_0^{'}$ and on the right as $e_0^{''}$. Then we get the fusion rules by combining Theorem 4.1 in this paper and Theorem 1.5.2, 1.5.3 in [FZ].

Theorem 4.3 *The fusion rules between modules $L_{c,h_{m^{'},n^{'}}}$ and $L_{c,h_{m^{''},n^{''}}}$ are*

$$L_{c,h_{m^{'},n^{'}}} \times L_{c,h_{m^{''},n^{''}}} = \sum_{(m,n)} N_{(m^{'},n^{'}),(m^{''},n^{''})}^{(m,n)} L_{c,h_{m,n}},$$

where $N_{(m^{'},n^{'}),(m^{''},n^{''})}^{(m,n)}$ is 1 iff $((m,n),(m^{'},n^{'}),(m^{''},n^{''}))$ is an admissible triple of pairs, and 0 otherwise.

It is desirable to get the identity $G_{p,q}^2 = \Pi_{m=1}^{p-1}\Pi_{n=1}^{q-1}(x - h_{m,n})$ directly from the singular vector $v_{p,q}$. But unfortunately explicit formulas for singular vectors in general are still unknown.

In the case $(p,q) = (2, k+2)$, $k = 2K+1, K \geq 1$, a formula was given by L.Benoit and Y.Saint-Aubin ([BS]) and recast into a recursion form by M.Bauer *et al* ([BFIZ]) as follows.

Let $f = (u_k, \cdots, u_1, 1)^T$, where $u_j \in V_c$ is an element of degree j, ($1 \in V_c$ is the vacuum). And let $F = (v_{p,q}, 0, \cdots, 0)^T$. We introduce the following $(k+1) \times (k+1)$ matrices

$$J_- = \begin{pmatrix} 0 & 0 & \cdots & & 0 \\ 1 & 0 & & & 0 \\ 0 & 1 & & & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & & \cdots & 1 & 0 \end{pmatrix},$$

$$J_+ = \begin{pmatrix} 0 & 1\cdot k & 0 & \cdots & & 0 \\ 0 & 0 & 2(k-1) & & & 0 \\ 0 & 0 & 0 & 3(k-2) & & 0 \\ \cdots & & \ddots & \ddots & & \cdots \\ 0 & \cdots & & 0 & 0 & k\cdot 1 \\ 0 & & \cdots & & 0 & 0 \end{pmatrix}$$

Then the singular vector $v_{p,q}$ is determined by the following recursive equation:

$$F = (-J_- + \sum_{i=0}^{k} L_{-i-1}(tJ_+)^j)f := M \cdot f\,,$$

where $t = -2/(k+2)$.

Using (4.2) we can define a matrix $M'(x)$ from M by replacing $L_{-i}(1 \le i \le k+1)$ in the j-th column of M by the number $(-1)^i((i-1)x+k+1-j)$. By Proposition 4.1 $G_{p,q}(x)$ is supposed to be the same as the determinant of $M'(x)$ up to a nonzero constant multiplication. Although we know $G_{p,q}(x)$ from Proposition 4.2, it will be still nice to calculate $M'(x)$ directly to get the explicit expression of $G_{p,q}(x)$.

We also define another matrix $\tilde{M}(x)$ from M by replacing $L_i(1 \le i \le k+1)$ in the j-th column of M by the number $(i-1)x+k+1-j$. Note that if we use the construction in Remark 3.1 instead of the original one given by Zhu, then we get the matrix $\tilde{M}(x)$ instead of $M'(x)$. It is easy to see that $det\, M'(x)$ is the same as $det\, \tilde{M}(x)$. The matrix $\tilde{M}(x)$ has already appeared in [FNO] through some other way of understanding. [2]

---

[2] But the computation in [FNO] of $det\, \tilde{M}(x)$ appeared incorrect.

5 Appendix

Let

$$O^{'}(M_c) = \{(L_{-n-3} + 2L_{-n-2} + L_{-n-1})b,\ \ b \in M_c, n \geq 0\}.$$

We have shown that $O^{'}(M_c) \subset O(M_c)$. Now we will prove that $O(M_c) \subset O^{'}(M_c)$.

We say $a \approx b$ iff $a \equiv b\ (mod(O^{'}(M_c))$.

We prove by induction on $\deg(L_{-n}a)$ that

$$T = Res_w(Y(L_{-n}a, w)\frac{(w+1)^{n+\deg a}}{w^k}b) \in O^{'}(M_c),\ n,\ k \geq 2.$$

Indeed,

$$\begin{aligned} T &= Res_{z-w}Res_w(Y(Y(\omega, z-w)a, w)\iota_{w,z-w}(z-w)^{-n+1}\frac{(w+1)^{n+\deg a}}{w^k}b) \\ &= T_1 - T_2, \quad \text{by the Jacobi identity} \end{aligned}$$

where

$$T_1 = Res_z Res_w(Y(\omega, z)Y(a, w)\iota_{z,w}(z-w)^{-n+1}\frac{(w+1)^{n+\deg a}}{w^k}b),$$

and

$$T_2 = Res_z Res_w(Y(a, w)Y(\omega, z)\iota_{w,z}(z-w)^{-n+1}\frac{(w+1)^{n+\deg a}}{w^k}b).$$

Recall that

$$\begin{aligned} \iota_{w,z}(z-w)^{-n+1} &= \sum_{i\geq 0}(-1)^{-n+1-i}\binom{1-n}{i}z^i w^{-n+1-i} \\ &= \sum_{i\geq 1}(-1)^{-n+1-i}\binom{1-n}{i}z^i w^{-n+1-i} + (-1)^{-n+1}w^{-n+1}. \end{aligned}$$

Applying the induction assumption on a we see that

$$Res_z Res_w(Y(a, w)Y(\omega, z)\sum_{i\geq 1}(-1)^{-n+1-i}\binom{1-n}{i}z^i\frac{w^{-n+1-i}(w+1)^{n+\deg a}}{w^k}b)$$

is in $O^{'}(V)$.

So

$$T_2 \approx Res_z Res_w(Y(a, w)Y(\omega, z)(-1)^{-n+1}\frac{(w+1)^{n+\deg a}}{w^{n-1+k}}b)$$

$$
\begin{aligned}
& = && Res_w(Y(a,w)(-1)^{-n+1}\frac{(w+1)^{n+\deg a}}{w^{n-1+k}}L_{-1}b) \\
(5.1) \qquad & \approx && Res_w(Y(a,w)(-1)^{-n+1}\frac{w(w+1)^{\deg a}}{w^k}L_{-1}b) \\
& \approx && Res_w(Y(a,w)(-1)^{-n+1}\frac{(w+1)^{\deg a+1}}{w^k}L_{-1}b) \quad \text{by induction on } a. \\
& = && T_{21} - T_{22},
\end{aligned}
$$

where

$$
\begin{aligned}
T_{21} &= L_{-1}Res_w(Y(a,w)(-1)^{-n+1}\frac{(w+1)^{\deg a+1}}{w^k}b), \\
T_{22} &= Res_w(Y(L_{-1}a,w)(-1)^{-n+1}\frac{(w+1)^{\deg a+1}}{w^k}b).
\end{aligned}
$$

$T_{22} \sim 0$ by applying induction assumption to $L_{-1}a$.

Equation (5.1) holds by induction since

$$
\frac{(w+1)^{n+\deg a}}{w^{n-1+k}} = \frac{w(w+1)^{\deg a}}{w^k} + \sum_{i=1}^{n}\binom{n}{i}\frac{(w+1)^{\deg a}}{w^{k+i-1}},
$$

And

$$
\begin{aligned}
T_1 &= Res_zRes_w(Y(\omega,z)Y(a,w)\iota_{z,w}(z-w)^{-n+1}\frac{(w+1)^{n+\deg a}}{w^k}b) \\
&= \sum_{i=0}^{\infty}\binom{1-n}{i}Res_zRes_w(Y(\omega,z)Y(a,w)(-1)^iz^{-n-i+1}\frac{w^i(w+1)^{n+\deg a}}{w^k}b) \\
&= \sum_{i=0}^{\infty}(-1)^i\binom{1-n}{i}Res_w(L_{-n-i}Y(a,w)\frac{w^i(w+1)^{n+\deg a}}{w^k}b) \\
&\approx \sum_{i=0}^{\infty}(-1)^i\binom{1-n}{i}(-1)^{n+i}Res_w(((n+i-1)(L_{-2}+L_{-1})+L_0)Y(a,w)\frac{w^i(w+1)^{n+\deg a}}{w^k}b) \\
&= T_{11} + T_{12},
\end{aligned}
$$

where

$$
\begin{aligned}
T_{11} &= \sum_{i=0}^{\infty}(-1)^n\binom{1-n}{i}Res_w(((n+i-1)(L_{-2}+L_{-1}))Y(a,w)\frac{w^i(w+1)^{n+\deg a}}{w^k}b) \\
&= (-1)^n(n+1)(L_{-2}+L_{-1})Res_w(Y(a,w)\frac{(w+1)^{\deg a}}{w^k}b) \ \approx 0,
\end{aligned}
$$

by (4.3) and the induction assumption on a and the identity

$$
\sum_{i=0}^{\infty}\binom{1-n}{i}(n+i-1)w^i = (n-1)(w+1)^{-n}.
$$

And

$$T_{12} = (-1)^n L_0 Res_w Y(a,w)\frac{(w+1)^{\deg a+1}}{w^k}b),$$

by using the identity

$$\sum_{i=0}^{\infty}\binom{1-n}{i}w^i = (w+1)^{-n+1}.$$

Then

$$T = T_{11} + T_{12} - T_{21} - T_{22} \approx (-1)^n(L_{-1}+L_0)Res_w Y(a,w)\frac{(w+1)^{\deg a+1}}{w^k}b) \approx 0,$$

by Lemma 3.1.

This completes the proof that $O(M_c) \subset O^{'}(M_c)$. $\square$

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2. After this work was completed, I received a joint paper [DMZ] from Y.Zhu. Among some other results in [DMZ], C. Dong, G. Mason and Y. Zhu proved Theorem 4.2 for the case $q = p + 1$ i.e. the unitary discrete series.

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