

ON DISK EMBEDDING UP TO S -COBORDISM

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ABSTRACT. The disk embedding lemma is a technique underlying the topological classification results in 4-manifold topology for good fundamental groups. The purpose of this paper is to develop new tools for disk embedding that work up to s -cobordism, without restrictions on the fundamental group. As an application we show that a surgery problem gives rise to a collection of capped gropes that fit in the framework of control theory.

The disk embedding lemma is a technique underlying the topological classification results in 4-manifold topology. It is conjectured to fail, in general, when the fundamental group of the 4-manifold is free. A weaker lemma, disk embedding up to s -cobordism, would imply 4-dimensional surgery but not the 5-dimensional s -cobordism conjecture. Recall its statement:

Conjecture. *Suppose A is an immersion of a 2-sphere into a 4-manifold M , and there is a framed immersed 2-sphere B with trivial algebraic self-intersection, and algebraic intersection 1 with A . Then there is an s -cobordism of the immersion A to a topological embedding.*

The advantage of this formulation is the extra flexibility in changing the ambient 4-manifold, which could make it applicable in contexts where the disk embedding lemma is unavailable. We refer the reader to [Q1] for a discussion of the consequences of this conjecture. It has been established [FQ] under an additional assumption that the transverse pair of spheres is π_1 -null in M (in other words, the inclusion $A \cup B \rightarrow M$ induces the trivial map on fundamental groups.)

The purpose of this paper is to develop new techniques for disk embedding that work up to s -cobordism, for any fundamental group. After reviewing the foundational material – the construction of s -cobordisms and splitting of gropes – the following constructions are presented here. Sections 3, 4: the grope splitting technique [K], [KQ] is extended, up to s -cobordism, to transverse pairs of spheres, and more generally to pairs of Whitney towers. Section 6: the s -cobordism construction for transverse pairs of spheres [FQ] is adapted to the setting of capped gropes, so it can be applied at every surface stage. Section 5: A technique for eliminating cycles,

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which translates the algebraic-combinatorial data into a geometric configuration of gropes. In particular, the following statement is proved.

Theorem 1. *Given a surgery problem, there is a procedure for converting it, up to an s -cobordism, into a large collection of capped gropes whose intersections (measured in $\pi_1 M^4$) are encoded by a tree, up to any given scale.*

This is a refinement of the outcome of grope splitting in [KQ]. In addition to the algebraic manipulations of the group elements represented by the double point loops of the gropes (which were sufficient for the proof of the disk embedding lemma in the subexponential growth case in [KQ]), this enables one to use the geometric techniques of control theory (cf [FQ], chapter 5.4.)

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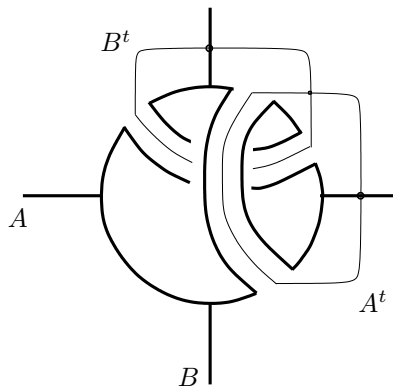
1. CONSTRUCTION OF s -COBORDISMS

The purpose of this section is to review the construction of s -cobordisms in [FQ], Chapter 6. We include it here for convenience of the reader, and also to fix the terminology, since this will be used throughout the paper. Recall that this construction is used in [FQ] to prove that a π_1 -null immersion of a union of transverse pairs, with algebraically trivial intersections, is s -cobordant to an embedding.

The starting point for the construction is a transverse pair of framed spheres (A, B) in a 4-manifold M . Typically in the applications the transverse pairs have algebraically trivial intersections. This means that A, B have a “preferred” intersection point, and all other intersection and self-intersection points are paired up with Whitney disks.

Let D denote a ball in M around the preferred intersection point between A and B , and consider the 4-manifold $M' = M \setminus D$. The intersection of A and B with ∂D is the Hopf link H . Consider parallel copies A', B' of the spheres, and let S_1, S_2 denote the circles of their intersection with ∂D . S_1, S_2 are parallel copies of the components of H . Attach 5-dimensional 2-handles to $M \times I$ along $S_1 \times D^3, S_2 \times D^3$ – thus surgering M along these circles. Figure 1 shows the effect of surgery on M' . The cores of the handles (viewed in surgered M') are capped off with the disks bounded by S_1, S_2 in ∂D , slightly pushed out of the ball D .

This gives two spheres A^t, B^t , transverse to A, B respectively. They are embedded, but intersect in a single point, figure 1. The strategy for using them in [FQ] is the following. Consider the Clifford linking tori, one for each pair of “extra” double points of A and B . These tori are geometrically dual to the Whitney disks for A, B , and they have caps provided by A^t, B^t . When $A \cup B$ is assumed to be π_1 -null in M , the resulting capped surfaces are used embedded Whitney disks, thus solving

FIGURE 1. s -cobordism construction: transverse spheres

the embedding problem for A, B . The crucial observation is that the cores of the 2-handles attached to M may be capped off by disks parallel to A, B , which are now embedded. The resulting embedded 2-spheres intersect the respective co-cores in a single point, and are used to attach 3-handles. The union of $M \times I$ and 2- and 3-handles is an s -cobordism since the boundary map computing the relative homology with $\mathbb{Z}\pi_1$ -coefficients is given by the identity matrix.

2. GROPE SPLITTING

We refer the reader to [FQ] for the definition and basic properties of capped gropes. The (iterated) splitting operation from [KQ] will be used later in the paper, and is briefly reviewed in this section.

A disk-like grope is *dyadic* if all component surfaces are either disks or punctured tori. This means each non-cap surface has exactly one pair of dual subgrope attached to it. A grope has *dyadic branches* if all of the subgrope above the base level are dyadic.

In a grope with dyadic branches, the caps are in one-to-one correspondence with the *dyadic labels*: For each dual pair of surfaces in the grope, label one by 0 and the other by 1. A cap gets a label by reading off the sequence of 0s or 1s encountered in a path going from the base surface to the cap.

The splitting operation splits a surface into two pieces, at the cost of doubling the dual subgrope. It can be used to decompose branches into dyadic branches, and can separate intersection points distinguished by properties unaffected by the dual doubling. In particular, it is used to separate double points that have different dyadic labels, or that represent different group elements in $\pi_1(M^4)$.

Suppose A is a component surface of a grope, not part of the base. Let B be the surface it is attached to, and H the dual subgrope. Now suppose α is an embedded

arc on A , with endpoints on the boundary, and disjoint from attaching circles of higher stages. In the 3-dimensional model, sum B with itself by a tube about α (the normal S^1 bundle), and discard the part of A that lies inside the tube. This splits A into two components. H is a dual for one component; obtain a dual for the other by taking a parallel copy of H . The following lemma is proved by an inductive application of this idea, descending from the caps downward to the first stage surface.

Lemma 2. (*Splitting*) *Any grope can be transformed by iterated splitting to one with dyadic branches, and such that each cap satisfies:*

- (1) *there are no self-intersections;*
- (2) *all caps intersecting the given one have the same label; and*
- (3) *the fundamental group classes of the loops through the intersection points are the same.*

Further, the subset of $\pi_1 M$ occurring as double point loops is the same as that of the original grope.

We will need an iterated version of this lemma, where the intersections are uniformized to distance n . This is made precise by the notion of an n -type. We refer to [KQ] for the formal definition of an n -type, and of a *collision* at distance n .

Lemma 3. (*Splitting to distance n*) *Given a positive integer n and a capped grope, there is a splitting so that*

- (1) *every branch has an n -type,*
- (2) *there are no collisions at distance $\leq n$.*

Given an integer n and a capped grope, its splitting to distance n (outcome of lemma 3) has the following algebraic-combinatorial description. Consider any cap of the split grope. Its intersections going out through chains of n branches are encoded by a tree, whose valence equals the class of the grope, and whose edges are labelled by a finite collection of group elements represented by the intersections of the original grope.

The paths in this tree are mimicked by geometric moves on the caps: the resulting double point loop represents the group element equal to the product of the labels along the path in the tree. However, this combinatorial graph is not, in general, mirrored geometrically as a tree of gropes, due to the existence of cycles. This is the subject of section 5.

3. SPLITTING UP TO S -COBORDISM OF TRANSVERSE PAIRS

This is a preliminary step showing that there is an analogue of splitting of capped surfaces (lemma 2 above) for transverse pairs of spheres, up to s -cobordism. The s -cobordism is completed when eventually the immersed spheres are homotoped to embeddings. An extension to transverse pairs with higher Whitney disk data is discussed in section 4.

Let A, B be a framed transverse pair in M^4 , and suppose A has two intersection points a', a'' that need to be separated. Consider two parallel copies B', B'' of B , and let x', x'' be the distinguished intersections between B' and A , and between B'' and A , respectively. Let α be a simple closed curve in A , dividing it into two disks D', D'' , so that $x', a' \in D'$ and $x'', a'' \in D''$. Attach a 2-handle to $M^4 \times I$ along $\alpha \times D^3$, thus surgering the 4-manifold along the curve α . Now both disks D', D'' are capped with copies of the core of the attached handle, giving immersed spheres A', A'' . This is similar to the argument in section 6.3 of [FQ], which converts a disk-embedding problem into a sphere problem.

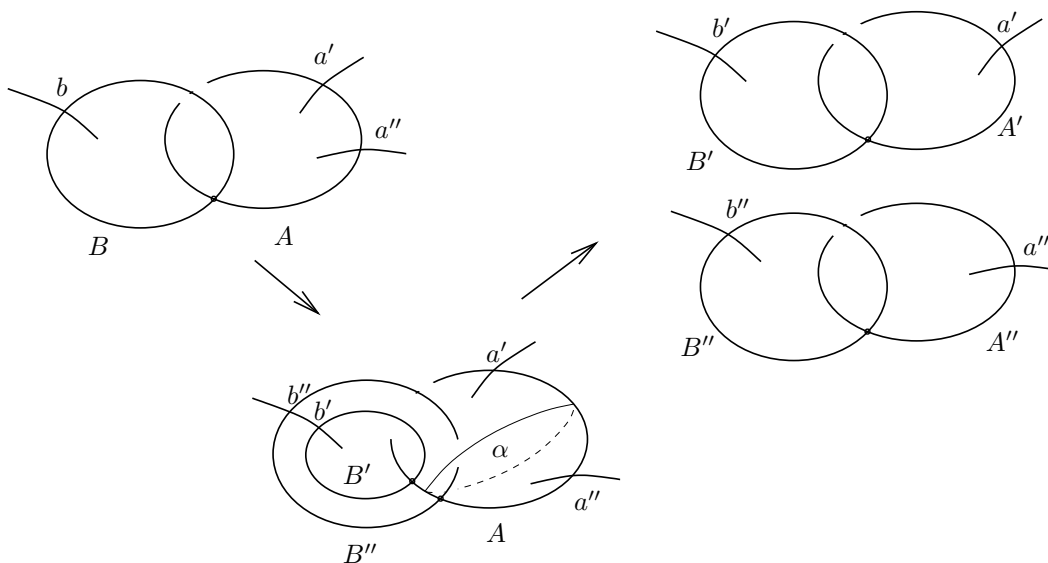


FIGURE 2. Splitting of a transverse pair

This creates two transverse pairs (A', B') and (A'', B'') , where A' and A'' each inherits one of the intersections a', a'' . The original intersection point b is doubled, so both B', B'' have an intersection. If one of the spheres A', A'' is homotopic to an embedding, then we will attach to it a 3-handle, which together with the 2-handle attached earlier yields an s -cobordism. To complete this s -cobordism, and also to solve the original surgery problem, one needs to work with both pairs of transverse spheres $(A', B'), (A'', B'')$.

As in lemma 3, given any $n \geq 1$, there is an iterated splitting of (A, B) so that each sphere has an n -type, and there are no collisions at distance $\leq n$. This is the analogue of splitting of capped surfaces.

4. SPLITTING OF WHITNEY TOWERS.

Here we show that there is an extension of the arguments above from transverse pairs of spheres (dually: capped surfaces) to transverse pairs together with layers of Whitney disks (dually: capped gropes). Let (A, B) be a transverse pair with algebraically trivial intersections, so all extra intersection points are paired up by Whitney disks. Given $k \geq 1$, it may be arranged that: there are k layers of Whitney disks for the intersections of the spheres, and for each $1 \leq i \leq k$ the interiors of Whitney disks at height i are disjoint from all surfaces below that height (i.e. we have a transverse pair of Whitney towers of height k .) This may be done, for example, by converting (A, B) into a transverse pair of capped gropes of height k , and then contracting some of the surface stages.

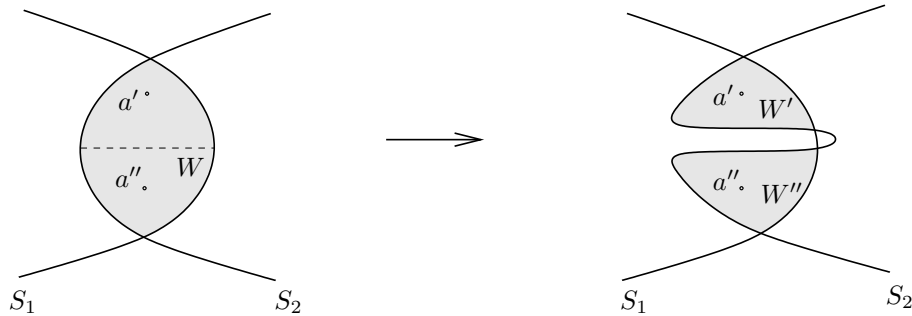


FIGURE 3. Splitting of a Whitney disk

The splitting of transverse pairs described in section 3 will be applied to *pairs* of intersections, rather than individual intersections. Thus at all times the intersections will stay paired up, and whenever a parallel copy of a surface (sphere or Whitney disk) is taken, it acquires also a parallel copy of the whole corresponding higher Whitney data.

The remaining ingredient is the splitting of Whitney disks. Suppose two intersections of some surfaces S_1, S_2 are paired up with a Whitney disk W , and the interior of W contains two collections a', a'' of points that need to be separated. Consider a

curve α in W with one endpoint in S_1 and the other one in S_2 , separating a' and a'' . Perform a finger move on one of the surfaces, say S_1 , along α , so that two new $S_1 - S_2$ intersections are created. The disk W is replaced by two Whitney disks, W' and W'' , containing a' , a'' respectively. This pushes the splitting problem to a lower stage. Whitney towers are split, using an inductive application of this step together with the splitting of spheres described in section 3.

5. UNRAVELING OF SHORT CYCLES.

Before introducing the general construction for capped gropes in section 6, we illustrate the problem, and our approach to it, in the setting of transverse pairs of spheres. Start with a transverse pair of spheres with algebraically trivial intersections, and split it to distance n . In this setup, the splitting of transverse pairs described above will be applied to *pairs* of intersections, rather than individual intersections. Thus at all times the intersections will stay paired up, and whenever a parallel copy of a sphere is taken, its intersections acquire also a parallel copy of the corresponding Whitney disks.

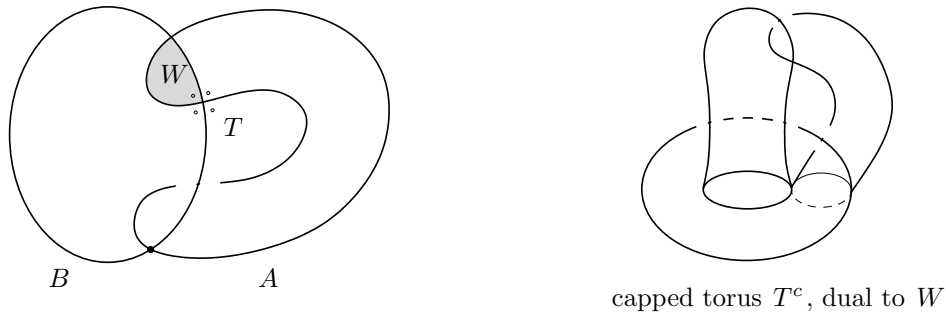


FIGURE 4. A cycle

Perform the s -cobordism construction (attach a pair of 2-handles) in a neighborhood of the distinguished intersection point of each transverse pair of spheres, as described in section 1. Now each Whitney disk has a dual capped torus. We are going to focus on this collection $\{T^c\}$ of capped tori, in the complement of the original spheres. Algebraically we have a line, subdivided into intervals labelled by group elements, encoding the intersections of the spheres, as in [KQ]. (Note that this line also encodes, algebraically, the intersections among the $\{T^c\}$.)

One cannot assume however that this picture holds geometrically as well, i.e. that there is a different capped torus for each vertex of the line. This is because of

the presence of cycles. By definition, a *cycle* is a path, embedded in the graph (in this context, in the line) whose endpoints correspond to the same capped torus. For example, suppose two spheres A , B forming a transverse pair intersect each other, as in figure 4, and let $g \in \pi_1 M$ be the group element corresponding to the intersections.

No splitting is necessary: both spheres have an n -type for each n , and the corresponding graph is a line, with each edge labelled by the same group element g . However, there is no line of tori geometrically: the caps of the dual capped torus intersect each other, this is a cycle of length 1.

The following construction is used to unravel cycles, without changing the uniform algebraic pattern of intersections. Before considering the general problem, here is the description for the cycle in figure 4. Consider n copies of one of the spheres, say B , denote them B_1, \dots, B_n . Now there are n Whitney disks, W_i , $1 \leq i \leq n$, and let T_i denote the corresponding Clifford torus, dual to W_i .

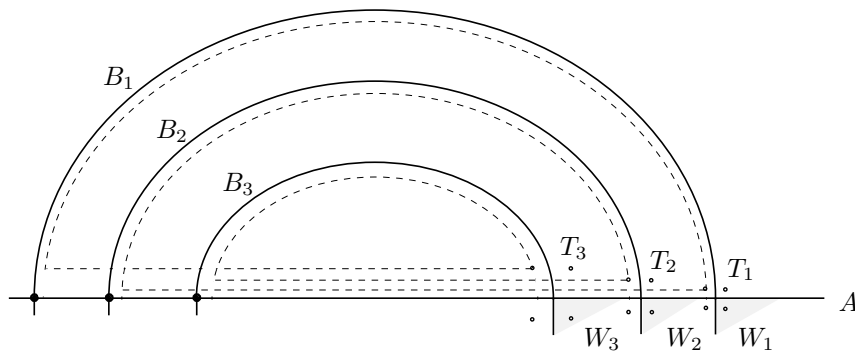


FIGURE 5. Unraveling of a cycle

Perform the construction of section 1 (attach a pair of 2-handles) in a neighborhood of the distinguished intersection point of each pair (A, B_i) , $1 \leq i \leq n$. If all spheres A, B_1, \dots, B_n are eventually shown to be homotopic to embedding, their parallel copies will be used as attaching maps for 3-handles. These, together with the 2-handles attached earlier, form an s -cobordism, since the boundary map $H_3 \rightarrow H_2$ is given by an upper triangular matrix.

Continue the construction of the capped tori: there is a unique choice for the cap of T_i^c , intersecting B_i : it has to go over the dual sphere to B_i , provided by the corresponding 2-handle. There are, however, n choices for the other cap of each T_i^c , intersecting A . We pick a cyclic shift, and it is this choice that is at the core of the construction. For each $i = 1, \dots, n-1$ let the cap of T_i^c go over the 2-sphere, dual to A near the intersection with B_{i+1} , and finally the cap of T_n^c goes over the sphere dual to A near the intersection with B_1 . All caps are disjoint from each other,

except for the unavoidable intersections that are inherited from the dual spheres in the s -cobordism construction. Figure 5 illustrates the case $n = 3$.

The outcome is a cyclic chain of capped surfaces T_1^c, \dots, T_n^c , with the caps of T_i^c intersecting T_{i-1}^c and T_{i+1}^c – the n -cyclic cover of the capped torus in figure 4. This is a geometric realization, with injectivity radius $n/2$, of the algebraic graph encoding the intersections of the spheres.

We now continue the construction in the general case, for an arbitrary transverse pair of spheres, split to distance n . The solution here is going to be an implant of a segment of length n of the universal cover of figure 5. More precisely, pick one of the transverse pairs, (A^1, B^1) , and look at the algebraic graph of intersections up to radius n . It corresponds to a sequence of pairs of spheres, $(A^1, B^1), \dots, (A^n, B^n)$ (actually, multiple parallel copies due to splitting). We assign the labels $A_1, B_1, A_2, B_2, \dots$ to the spheres respecting their linear order. The edges of the graph correspond to the intersections between B^i and A^{i+1} . Moreover, some of the pairs coincide (precisely in the presence of cycles.)

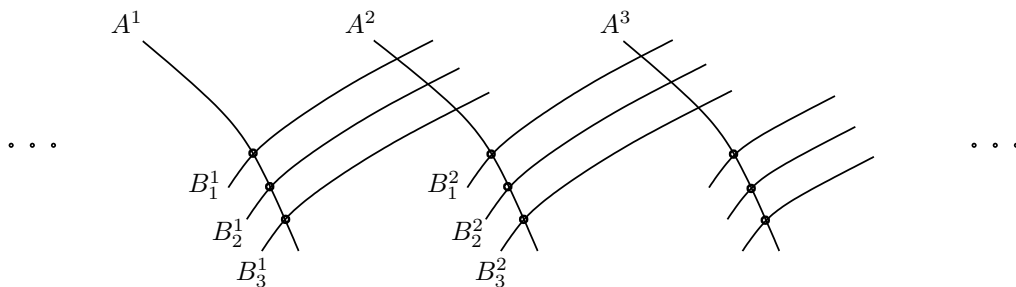


FIGURE 6

Recall that just like in the capped surface case, the spheres in a pair are assigned dyadic labels. The notation may be misleading, as A, B do not correspond to such labels: B^1, A^2 may have the same dyadic label.

Replace B^1 with n copies: B_1^1, \dots, B_n^1 . Since B^1 may occur elsewhere in the chain, say $B^1 = B^i$ for some i , all such B^i are replaced with n copies as well. Note that B^i can never coincide with A^j for any i, j , as this would imply a collision at distance $< n$, and would contradict the splitting assumption. If B^2 has not been affected, replace it with n copies. Continue this procedure until each B^1, \dots, B^n is replaced with n copies.

Now we have a segment of the infinite cyclic cover of figure 5, and we follow the same construction as in that example: assign caps to the capped tori with a cyclic shift. The claim is that the result is a sequence of length n of capped tori, *without cycles*,

exactly mimicing the algebraic line of intersections of the spheres. If there were no cycles to begin with, our construction just replicates the chain of capped surfaces, without altering the algebraic intersections. If there were cycles, they are eliminated.

6. S-COBORDISM CONSTRUCTION FOR CAPPED GROPEs

This is the analogue of the s -cobordism construction for transverse pairs of spheres (described in section 1.) The new feature of the construction for capped gropes is that it can be applied at every surface stage. At the end of this section we give the proof of theorem 1.

Recall that every surface (including the caps), above the first stage, in a capped grope has a transverse grope. It is constructed using two copies of the dual surface, cf section 1.4 in [KQ]. To fix notations, suppose S is a surface in a capped grope, and let A, B be surfaces attached to a symplectic pair of circles α, β in S . Then the transverse capped grope g_a for A is built of two parallel copies of B (and everything attached to B), and analogously the transverse grope g_b for B is made of two copies for A . The transverse gropes g_a, g_b are used to resolve intersections of surfaces with A and B (and with other surface stages and caps attached to A, B) respectively.

The base surfaces S_a, S_b of g_a, g_b intersect in two points near the intersection point p of the circles α, β in S . Consider a 4-ball D around the intersection point p in S . The ball D is chosen so that $D_a = D \cap S_a$ and $D_b = D \cap S_b$ are disks, and D contains the two intersection points $S_a \cap S_b$. Consider the circles C_a, C_b of intersection of S_a, S_b with ∂D . As in the construction of section 1, attach two 2-handles to $M \times I$ along $C_a \times D^3, C_b \times D^3$. The cores of the 2-handles (viewed in the surgered 4-manifold) are capped with the disks D_a, D_b . The result is a pair of spheres, geometrically dual to the surfaces S_a, S_b . Each of these two spheres is embedded, but they intersect in two points.

Consequently, the spheres may be used to resolve intersections with A, B and with other surfaces attached to A, B , but the price is the presence of new intersections between the surfaces pushed off A and B . These spheres are useful for finding embedded caps for the grope. The crucial observation is that once the caps of the grope are improved to embeddings, then the 3-handles can be attached to complete the s -cobordism. The attaching 2-spheres are constructed as follows: a parallel copy of the core of each 2-handle is capped off by an embedded disk built of two parallel copies of the dual capped grope (an embedded disk may be found now, since the caps are assumed to be embedded.) Attaching the 3-handles completes the construction of an s -cobordism, since the boundary map $C_3 \rightarrow C_2$ computing the relative homology with $\mathbb{Z}\pi_1$ -coefficients is the identity matrix.

We conclude with the proof of theorem 1. Starting with a transverse pair of spheres, convert one of them into a capped grope – a capped grope of height 2 suffices, and we assume for simplicity that this is the case. Let n be a positive integer. Split the grope to the distance n (lemma 3.) Fix a cap C of the split grope; as explained in section 2, the intersections among the caps of the split grope are encoded algebraically by a tree T , up to the distance n from C . The goal is to convert this into a geometric picture where the capped gropes correspond to the vertices of the tree, and the intersections between their caps are encoded by the edges. Consider all genus 1 pieces of the base surface (branches) that appear in the tree T .

The gropes have two surface stages and caps. The split grope has dyadic branches; fix a genus one piece of the base surface. It branches into two second stage surfaces, and then into four caps. Adapting to the setting of gropes the cyclic construction of section 5, consider n parallel copies of one of the second stage surfaces, and then n parallel copies of one in each dual pair of caps. The s -cobordism construction described above will be applied at each intersection point of the attaching circles of all surfaces (second stage surfaces and caps) present in the picture. This is the model that will be applied at each branch of the split capped grope. Finally, the construction for resolving cycles in section 5 is generalized from the the context of transverse pairs of spheres (intersections encoded by a line) to capped gropes (intersections encoded by a tree.) The intersections between the caps are pushed down and off the grope using the transverse spheres provided by the s -cobordism construction. This is done with cyclic shift with respect to the numbering of the parallel copies of the caps, and of the dual spheres. The stage, to which the intersection is pushed down to, is determined based on the dyadic labels of the caps. The proof that the cycles are resolved is analogous to the sphere case, in section 5.

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