

# LINK GROUPS OF 4-MANIFOLDS

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ABSTRACT. We introduce the notion of a flexible 2-cell and use it to define invariants – link groups – of 4-manifolds. Flexible cells combine some features of both surfaces and handlebodies, therefore the link group  $\lambda(M)$  measures certain aspects of the handle structure of a 4-manifold  $M$ . This group is a quotient of the fundamental group, and we construct examples of manifolds  $M$  with  $\pi_1(M) \neq \lambda(M)$ . More detailed information is contained in a two-parameter family of link groups  $\lambda_{i,j}(M)$ . A generalization of the Milnor group is developed to formulate an obstruction to embeddability of flexible cells into 4-space.

## 1. INTRODUCTION

Maps of surfaces and more general 2-complexes have classically been used to define invariants of topological spaces, for example the fundamental group and the first homology group of a space. More generally, one gets the quotients of  $\pi_1 X$  by the terms of its lower central, and also derived, series if one considers based loops in a space  $X$  modulo loops bounding maps of certain special 2-complexes, gropes [3].

On the geometric side, surfaces play a central role in 4-dimensional topology, in both smooth and topological categories. However it has been a common theme that the geometry of surfaces and of more general 2-complexes is not sufficiently flexible to reflect the rich structure of 4-manifolds and their invariants. For example, consider a 4-manifold  $(M, \partial M)$  with a handle decomposition with 0-, 1- and 2-handles. Rather different 4-manifolds of this type may be homotopy equivalent to a given fixed 2-complex. A basic feature that enables this (and makes the structure of 4-manifolds so rich) is that the attaching maps of 2-handles may be homotopic but not isotopic.

This paper introduces the notion of a *flexible cell*, designed to enrich the standard 2-cell with some of the features of 4-dimensional handlebodies. In an analogy with the fundamental group, we define the *link group*  $\lambda(M)$  of a 4-manifold  $M$  as based loops in  $M$  modulo loops bounding flexible cells. Therefore the link group

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$\lambda(M)$  reflects certain aspects of the handle structure of a 4–manifold  $M$ . Although their definition makes sense in any dimension, the link groups are a non-trivial theory only in dimension 4, and they are *not* in general a homotopy invariant of a 4–manifold.

Since  $D^2$  is a trivial example of a flexible cell, the link group  $\lambda(M)$  is a quotient of the fundamental group of  $M$ . One can easily find examples of 4–manifolds with  $\pi_1(M) \neq \lambda(M)$ . Flexible cells may be given geometric and algebraic gradings: *height* and *nilpotent class*, leading to a two-parameter collection of link groups  $\lambda_{i,j}(M)$ , where  $j > i$ . In this notation, the group  $\lambda(M)$  above corresponds to  $\lambda_{\infty,\infty}$ . We show that given a surjection of finitely presented groups  $\pi \rightarrow \lambda$ , where  $\pi$  is aspherical, there are 4-manifolds  $M$  with  $\pi_1(M) \cong \pi$ ,  $\lambda_{1,2}(M) \cong \lambda$ .

The main motivation for this work is the question of whether there is a “non-abelian” Alexander duality in dimension 4. This question arises in the analysis of decompositions of the 4–ball in the  $A, B$ –slice problem, a reformulation of the 4–dimensional topological surgery conjecture. An application of link groups in this context is given in [5].

The connection to the  $A, B$ –slice problem is provided by the following result, showing how flexible cells fit in the framework of Milnor’s theory of link homotopy:

**Theorem 9.1.** *If the components of a link  $L \subset S^3 = \partial D^4$  bound disjoint flexible cells in  $D^4$  then  $L$  is homotopically trivial.*

To prove this result, we develop a generalization of the Milnor group and formulate an obstruction to embeddability of a disjoint collection of flexible cells in 4–space.

Sections 2, 3 summarize the relevant background material on presentations of nilpotent quotients and Milnor’s theory of link homotopy and related results on surfaces in 4–space. Section 4 introduces flexible cells and link groups  $\lambda(M)$ , and gives examples of 4–manifolds with  $\pi_1 \neq \lambda$ . Sections 5 – 9 concern embeddings of flexible cells in 4–space,  $(C, \gamma) \hookrightarrow (D^4, S^3)$ . More specifically, the fundamental group of the complement,  $\pi_1(D^4 \setminus C)$ , is analyzed in section 5. Section 6 develops a generalization of the Milnor group in the context of flexible cells. It is used, in particular, to define an algebraic grading of flexible cells and link groups  $\lambda_{i,j}$ . Sections 7, 8, 9 define an obstruction to embeddability of a collection of flexible cells in  $D^4$  with a prescribed boundary, given by a link in  $S^3$ .

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## 2. A PRESENTATION OF NILPOTENT QUOTIENTS

The purpose of this section is to describe a presentation of the quotients  $\pi/\pi^q$  of a group  $\pi$  by the terms of its lower central series, given generators of the first and second homology of  $\pi$ . This technique is well-known (see also [4]), and it will be used often in the paper. The lower central series of a group  $\pi$  is defined inductively by  $\pi^1 = \pi$ ,  $\pi^2 = [\pi, \pi]$ ,  $\dots$ ,  $\pi^q = [\pi, \pi^{q-1}]$ .

To state the lemma, fix a group  $\pi$  and suppose that  $H_1(\pi; \mathbb{Z})$  is generated by  $g_1, \dots, g_n$ ,  $H_2(\pi; \mathbb{Z})$  is generated by  $r_1, \dots, r_m$ , and let  $q \geq 2$  be an integer. Then the result of lemma 2.1 is that, roughly,  $g_1, \dots, g_n$  and  $r_1, \dots, r_m$  provide a set of generators and relations respectively in a presentation of  $\pi/\pi^q$ . To make this precise, consider the quotient homomorphism  $\alpha: \pi/\pi^q \rightarrow \pi/[\pi, \pi]$  and let  $\hat{g}_i \in \pi/\pi^q$  denote some preimage of  $g_i$  under  $\alpha$ ,  $i = 1, \dots, n$ . It is a standard fact in nilpotent group theory [10] that  $\hat{g}_1, \dots, \hat{g}_n$  generate  $\pi/\pi^q$ .

Let  $W \rightarrow K(\pi, 1)$  be a map from the wedge of  $n$  circles  $W$ , inducing an epimorphism  $\beta: \pi_1(W) \rightarrow \pi/\pi^q$  and mapping the  $i$ -th free generator of  $\pi_1(W)$  to  $\hat{g}_i$ . Let  $f_j: \Sigma_j \rightarrow K(\pi, 1)$  be a map of a surface  $\Sigma_j$ , representing the generator  $r_j$  of  $H_2(K(\pi, 1)) \cong H_2(\pi)$ ,  $j = 1, \dots, m$ . We assume here that each space has a fixed basepoint, and all maps preserve them. The standard basis of  $H_1(\Sigma_j)$  pulls back via  $\beta$  to some elements in  $\pi_1(W)$ ; let  $\hat{r}_j \in \pi_1(W)$  be a lift via  $\beta$  of the attaching map of the 2-cell of  $\Sigma_j$ . (In particular, if  $\Sigma_j$  is a 2-sphere then the corresponding word  $\hat{r}_j$  is trivial.)

**Lemma 2.1.** *Suppose  $H_1(\pi; \mathbb{Z})$  is generated by  $g_1, \dots, g_n$ , and  $H_2(\pi; \mathbb{Z})$  is generated by  $r_1, \dots, r_m$ . Then in the notations as above,*

$$\pi/\pi^q \cong \langle \hat{g}_1, \dots, \hat{g}_n \mid \hat{r}_1, \dots, \hat{r}_m, (F_{\hat{g}_1, \dots, \hat{g}_n})^q \rangle$$

where  $F_{\hat{g}_1, \dots, \hat{g}_n}$  denotes the free group on generators  $\hat{g}_1, \dots, \hat{g}_n$ .

To prove the lemma we need a refinement of Stallings theorem [9] due to Dwyer. Given a space  $X$ , the Dwyer's subspace  $\phi_k(X) \subset H_2(X; \mathbb{Z})$  is defined as the kernel of the composition

$$H_2(X) \rightarrow H_2(K(\pi_1 X, 1)) = H_2(\pi_1 X) \rightarrow H_2(\pi_1(X)/\pi_1(X)^{k-1}).$$

**Theorem 2.2.** [1] *Let  $k$  be a positive integer and let  $f: X \rightarrow Y$  be a map inducing an isomorphism on  $H_1(\cdot; \mathbb{Z})$  and mapping  $H_2(X)/\phi_k(X)$  onto  $H_2(Y)/\phi_k(Y)$ . Then  $f$  induces an isomorphism  $\pi_1(X)/(\pi_1(X))^k \cong \pi_1(Y)/(\pi_1(Y))^k$ .*

*Proof of lemma 2.1.* Let  $X$  be the 2-complex obtained from  $W$  by attaching  $m$  two-cells along the words  $\hat{r}_1, \dots, \hat{r}_m$ . The composition  $W \longrightarrow K(\pi, 1) \longrightarrow K(\pi/\pi^q, 1)$  extends to  $X$ , inducing an isomorphism  $H_1(X) \cong H_1(\pi) \cong H_1(\pi/\pi^q)$  and an epimorphism on  $H_2/\phi_q$ . Now an application of Dwyer's theorem 2.2 concludes the proof of Lemma 2.1.  $\square$

### 3. THE MILNOR GROUP: LINKS IN $S^3$ AND SURFACES IN $D^4$

In this section we recall the relevant results on Milnor groups and  $\bar{\mu}$ -invariants [7], [8]. This material is used to set up the definition of flexible cells in section 4. Sections 5 – 9 develop a generalization of the Milnor group and of other aspects of the theory in the context of flexible cells in  $D^4$ .

**3.1. Links in  $S^3$ .** Let  $L = (l_1, \dots, l_n)$  be an oriented link in  $S^3$ , and consider meridians  $m_1, \dots, m_n$  to the components of  $L$ . Observe that  $H_1(S^3 \setminus L)$  is generated by  $m_1, \dots, m_n$ , and a set of generators for  $H_2(S^3 \setminus L)$  is provided by  $n - 1$  tori: the boundary of a regular neighborhood of  $n - 1$  components of  $L$ . By lemma 2.1,  $\pi_1(S^3 \setminus L)/(\pi_1(S^3 \setminus L))^q$  has a presentation

$$(3.1) \quad \langle m_1, \dots, m_n | [m_1, w_1], \dots, [m_{n-1}, w_{n-1}], (F_{m_1, \dots, m_n})^q \rangle .$$

Here  $F_{m_1, \dots, m_n}$  denotes the free group generated by  $m_1, \dots, m_n$ . The Magnus expansion homomorphism  $M: F_{m_1, \dots, m_n} \longrightarrow \mathbb{Z}\{x_1, \dots, x_n\}$  into the ring of formal non-commutative power series in the indeterminates  $x_1, \dots, x_n$  is defined by  $M(m_i) = 1 + x_i$ ,  $M(m_i^{-1}) = 1 - x_i + x_i^2 \pm \dots$  for  $i = 1, \dots, n$ . Denoting by  $w_j$  a word in the free group representing the untwisted  $j$ -th longitude of the link, let

$$M(w_j) = 1 + \sum \mu_L(I, j) x_I$$

be the expansion of  $w_j$ , where the summation is over all multiindices  $I = (i_1, \dots, i_k)$  with entries between 1 and  $n$ , and  $x_I = x_{i_1} \cdot \dots \cdot x_{i_k}$ ,  $k > 0$ . This expansion defines for each such multiindex  $I$  the integer  $\mu_L(I, j)$ . Let  $\Delta_L(i_1, \dots, i_k)$  denote the greatest common divisor of  $\mu_L(j_1, \dots, j_s)$  where  $j_1, \dots, j_s$ ,  $2 \leq s \leq k - 1$  is to range over all sequences obtained by cancelling at least one of the indices  $i_1, \dots, i_k$  and permuting the remaining indices cyclicly.

Let  $\bar{\mu}_L(I)$  denote the residue class of  $\mu_L(I)$  modulo  $\Delta_L(I)$ . Analyzing the indeterminacy caused by the relations in the presentation (3.1), one sees that for each multiindex  $I$  of length  $|I| \leq q$  the residue class  $\bar{\mu}_L(I)$  is an *isotopy invariant* of the link  $L$ , where  $\bar{\mu}_L(I)$  is defined using the quotient  $\pi_1(S^3 \setminus L)/(\pi_1(S^3 \setminus L))^q$ . In

particular, *the first non-vanishing coefficients*  $\mu_L(I)$  *are well-defined.* (By first non-vanishing coefficients we mean  $\mu_L(I)$  such that  $\mu_L(J) = 0$  for all proper subsets  $J \subset I$ .)

**3.2. Link homotopy and Milnor groups.** Two  $n$ -component links  $L$  and  $L'$  in  $S^3$  are said to be *link-homotopic* if they are connected by a 1-parameter family of immersions such that different components stay disjoint at all times.  $L$  is said to be *homotopically trivial* if it is link-homotopic to the unlink.  $L$  is *almost homotopically trivial* if each proper sublink of  $L$  is homotopically trivial.

For a group  $\pi$  normally generated by  $g_1, \dots, g_k$  its *Milnor group* (with respect to  $g_1, \dots, g_k$ )  $M\pi$  is defined to be the quotient of  $\pi$  by its subgroup

$$(3.2) \quad \ll [g_i, g_i^h] : 1 \leq i \leq k, \quad h \in \pi \gg .$$

$M\pi$  is nilpotent of class  $\leq k + 1$ , in particular it is a quotient of  $\pi/(\pi)^{k+1}$ , and is generated by the quotient images of  $g_1, \dots, g_k$ . The Milnor group  $M(L)$  of a link  $L$  is defined to be  $M\pi_1(S^3 \setminus L)$  with respect to its meridians  $m_i$ .

Milnor showed in [7] that the Magnus expansion induces a well defined injective homomorphism  $MM: M(F_{m_1, \dots, m_k}) \longrightarrow R(x_1, \dots, x_k)$  into the ring  $R(x_1, \dots, x_k)$  which is the quotient of  $\mathbb{Z}\{x_1, \dots, x_k\}$  by the ideal generated by monomials  $x_{i_1} \cdots x_{i_r}$  with some index occurring at least twice. Indeed, every term in the Magnus expansion of each defining Milnor relation (3.2) has repeating variables. Let  $\bar{w}_n \in MF_{m_1, \dots, m_{n-1}}$  be a word representing  $l_n$  in  $M\pi_1(S^3 \setminus (l_1 \cup \dots \cup l_{n-1}))$ . Then  $\bar{\mu}$ -invariants of  $L$  with non-repeating coefficients may also be defined by the equation

$$MM(\bar{w}_n) = 1 + \sum \mu_L(I, n) x_I$$

where summation is over all multiindices  $I$  with non-repeating entries between 1 and  $n-1$ , and  $\bar{\mu}_L(I, n)$  is the residue class of  $\mu_L(I, n)$  modulo the indeterminacy  $\Delta_L(I, n)$ , defined above.

The Milnor group of  $L$  is the largest common quotient of the fundamental groups of all links link-homotopic to  $L$ , hence *if  $L$  and  $L'$  are link homotopic then their Milnor groups are isomorphic.* The next result gives an algebraic criterion for a link to be null-homotopic.

**Lemma 3.3.** ([7]) *For an  $n$ -component link  $L$ , the following conditions are equivalent:*

- (i)  $L$  is homotopically trivial,
- (ii) the components of  $L$  bound disjoint immersed disks in  $D^4$ ,

- (iii)  $M(L) \cong M(F_{m_1, \dots, m_n})$  with the isomorphism carrying a meridian to  $l_i$  to the generator  $m_i$  of the free group,  
 (iv) all  $\bar{\mu}$ -invariants of  $L$  with non-repeating coefficients vanish.

It follows from Lemma 3.3 that  $L$  is almost homotopically trivial if and only if all its  $\bar{\mu}$ -invariants with non-repeating coefficients of length less than  $n$  vanish. In particular, if  $L$  is almost homotopically trivial then its  $\bar{\mu}$ -invariants with non-repeating coefficients of length  $n$  are well-defined integers.

**3.4. The link composition lemma.** We will now recall the link composition lemma [2] (see also [6]). The result on flexible cells proved in section 9 contains this theorem as a special case. Given a link  $\widehat{L} = (l_1, \dots, l_{k+1})$  in  $S^3$  and a link  $Q = (q_1, \dots, q_m)$  in the solid torus  $S^1 \times D^2$ , their “composition” is obtained by replacing the last component of  $\widehat{L}$  with  $Q$ . More precisely, it is defined as  $C = (c_1, \dots, c_{k+m}) := (l_1, \dots, l_k, \phi(q_1), \dots, \phi(q_m))$ , where  $\phi: S^1 \times D^2 \hookrightarrow S^3$  is a 0-framed embedding whose image is a tubular neighborhood of  $l_{k+1}$ . The meridian  $\{1\} \times \partial D^2$  of the solid torus will be denoted by  $\wedge$  and we set  $\widehat{Q} := Q \cup \wedge$ .

**Theorem 3.5.** *If both  $\widehat{L}$  and  $\widehat{Q}$  are homotopically essential in  $S^3$  then so is their composition  $L \cup \phi(Q)$ .*

**3.6. Links in  $S^1 \times D^2$ .** Let  $L$  be a link in  $S^1 \times D^2$ . We denote by  $\wedge$  the meridian  $\{p\} \times \partial D^2$  and set  $\widehat{L} = L \cup \wedge$ . Consider  $\widehat{L}$  as a link in  $S^3$ , using a standard embedding  $S^1 \times D^2 \subset S^3$ . Links in the solid torus will be used as attaching regions for 2–handles in the definition of flexible cells (see next section), and we need to specify the class of links necessary for the definition. Let  $\wedge'$  denote another meridian  $\{q\} \times \partial D^2$ ,  $p \neq q$ .

**Definition 3.7.** We will say that a link  $L = (l_1, \dots, l_n) \subset S^1 \times D^2$  is essential and (almost trivial)<sup>+</sup> if  $\widehat{L}$  is homotopically essential, and for each  $i$ ,  $(L \setminus l_i) \cup \wedge \cup \wedge'$  is homotopically trivial.

An important example is given by  $L = \text{Bing double of the core } S^1 \times \{0\}$ , and more generally by  $L = \text{iterated Bing double of the core}$ . The definition also allows the trivial example:  $L = \text{core of the solid torus}$ .

The second condition is slightly stronger than just the requirement that  $\widehat{L}$  is almost homotopically trivial. We include it since it is technically convenient for the proofs of the properties of flexible cells. We need to reformulate the conditions on  $L$  in

terms of  $\bar{\mu}$  invariants. Consider the solid torus  $S^1 \times D^2$  as the complement in  $S^3$  of an unknotted circle and note that

$$\pi_1(S^1 \times D^2 \setminus L)/(\pi_1(S^1 \times D^2 \setminus L))^q \cong \pi_1(S^3 \setminus (L \cup \Lambda'))/\pi_1(S^3 \setminus (L \cup \Lambda'))^q.$$

These groups are generated by the meridians  $m_1, \dots, m_n$  to the components of  $L$  and by the longitude  $l = S^1 \times \{x\}$  of the solid torus (respectively the meridian  $m$  to  $\Lambda'$  for the second group.) Consider the free group  $F_{m_1, \dots, m_n, m}$  mapping onto these groups, and the Magnus expansion

$$M: F_{m_1, \dots, m_n, m} \longrightarrow \mathbb{Z}\{x_1, \dots, x_n, y\}, \quad M(m_i) = 1 + x_i, \quad M(m) = 1 + y.$$

Let  $W$  be a word representing  $\wedge$  in the free group. Assuming that  $L$  satisfies the conditions in the definition above, observe that all terms with *non-repeating variables* in the expansion  $M(W)$  are either of the form  $x_{i_1} \cdots x_{i_n}$  or they contain all variables  $x_1, \dots, x_n$  and  $y$ . Since the link  $\widehat{L}$  is homotopically essential, renumbering the components of  $L$  if necessary, one can assume that the term  $\mu x_1 \cdots x_n$  in the Magnus expansion  $M(W)$  has the coefficient  $\mu \neq 0$ . It is important to note that there are no terms that contain  $y$  and just a proper subset of the variables  $x_1, \dots, x_n$ .

**3.8. Surfaces in  $D^4$ .** Let  $\Delta = \cup \Delta_i$  be a collection of immersed disks in  $(D^4, \partial D^4)$ . By Alexander duality,  $H_1(D^4 \setminus \Delta)$  is generated by the meridians to the components of  $\Delta$ , and  $H_2(D^4 \setminus \Delta)$  is generated by the *Clifford tori* linking the double points of  $\Delta$ .

More precisely, a local model for the surfaces near a double point is given by  $\mathbb{R}^2 \times \{0\} \cap \{0\} \times \mathbb{R}^2 \subset \mathbb{R}^2 \times \mathbb{R}^2$ . The Clifford torus is the product of the unit circles  $S^1 \times S^1$ . The linking number of classes  $a \in H_1(\Delta)$  and  $b \in H_2(D^4 \setminus \Delta)$  may be computed as the intersection number of  $\Sigma \cdot b$  where  $a = \partial \Sigma \subset D^4$ .  $H_1(\Delta)$  is generated by the double point loops – loops in  $\Delta$  passing exactly once through a double point. It is clear from the local model that the double point loops are paired up  $\delta_{i,j}$  with the Clifford tori.

Suppose the disks  $\Delta_i$  are disjoint, so all double points are self-intersections. According to lemma 2.1,  $\pi_1(D^4 \setminus \Delta)/(\pi_1(D^4 \setminus \Delta))^q$  is generated by the meridians  $m_1, \dots, m_n$  to the components of  $\Delta$ , and the relations (corresponding to the Clifford tori) are all of the form  $[(m_i)^f, (m_j)^g] = 1$  for some  $f, g$ . In particular, the Milnor group  $M\pi_1(D^4 \setminus \Delta)$  (with respect to the meridian generators) is isomorphic to the free Milnor group  $MF_{m_1, \dots, m_n}$ .

This gives a useful perspective on the relation between (i) and (ii) in lemma 3.3: if a link  $L$  is essential then  $M(L)$  is not isomorphic to the free Milnor group. This implies that the components of  $L$  do not bound disjoint maps  $\Delta$  of disks in  $D^4$ : otherwise the inclusion map  $S^3 \setminus L \rightarrow D^4 \setminus \Delta$  would induce a homomorphism  $M(L) \rightarrow MF_{m_1, \dots, m_n}$  – a contradiction.

#### 4. FLEXIBLE CELLS AND LINK GROUPS

**Definition 4.1.** A *model flexible cell* ( $f$ -cell) of height 1 is a smooth 4-manifold  $C$  with boundary and with a specified attaching curve  $\gamma \subset \partial C$ , defined as follows. Consider a planar surface  $P$  with  $k + 1$  boundary components  $\gamma, \alpha_1, \dots, \alpha_k$  ( $k \geq 0$ ), and set  $\bar{P} = P \times D^2$ . Let  $L_1, \dots, L_k$  be a collection of links,  $L_i \subset \alpha_i \times D^2$ ,  $i = 1, \dots, k$ . We assume that for each  $i$ ,  $\widehat{L}_i$  is essential and (almost trivial)<sup>+</sup> (see definition 3.7). Then  $C$  is obtained from  $\bar{P}$  by attaching zero-framed 2-handles along the components of  $L_1 \cup \dots \cup L_k$ .

The surface  $S$  (and its thickening  $\bar{S}$ ) will be referred to at the *body* of  $C$ , and the 2-handles are the *handles* of  $C$ .

A *model  $f$ -cell*  $C$  of height  $h$  is obtained from an  $f$ -cell of height  $h-1$  by replacing its handles with  $f$ -cells of height one. The *body* of  $C$  consists of all (thickenings of) its surface stages, except for the handles.

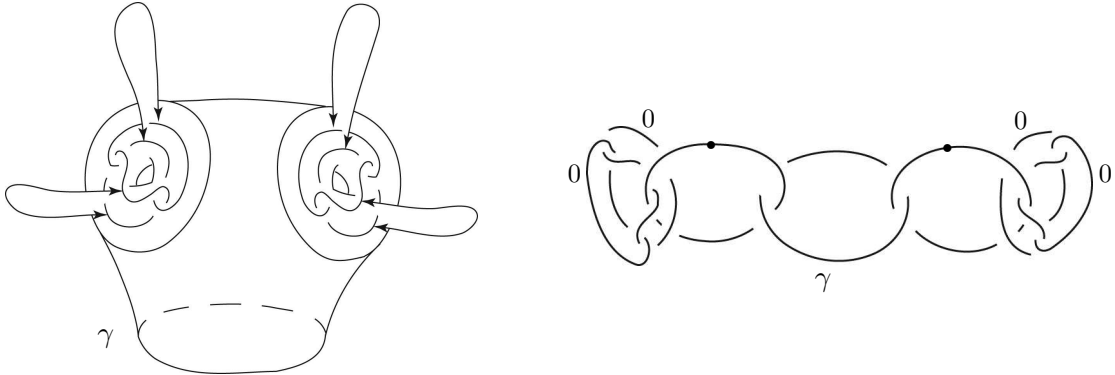


FIGURE 1. Example of a model  $f$ -cell of height 1: a schematic picture and a Kirby diagram

Figure 1 gives an example of an  $f$ -cell of height 1: a schematic picture and a precise description in terms of a Kirby diagram. Here  $P$  is a pair of pants, and each link  $L_i$  is the Bing double of the core of the solid torus  $\alpha_i \times D^2$ ,  $i = 1, 2$ . The reader is urged to keep this example in mind while reading the paper. This is an

important example, and the theory already exhibits many of its interesting features in this case.

*Remarks.* 1. The standard 2-handle  $H = D^2 \times D^2$  with  $\gamma = \partial D^2 \times \{0\}$  provides a trivial example of an  $f$ -cell (of any height) – corresponding to the case  $k = 0$  in the definition above. Alternatively, one gets the 2-handle  $H$  by considering the links  $L =$  cores of the corresponding solid tori. Similarly, an  $f$ -cell of height  $h$  also satisfies the definition of an  $f$ -cell of any height  $> h$ .

2. One may assume that no body surface of  $C$  above the first stage is an annulus: suppose an annulus  $A$  is present,  $\partial A = \gamma_A \cup \alpha_A$ . Then  $A$  may be used to deform the attaching maps of handles or higher stages from  $\alpha_A \times D^2$  to  $\gamma_A \times D^2$ . This eliminates  $A$  (and increases the number of components of the link one stage below – note that it is still essential and (almost trivial)<sup>+</sup>, see link composition lemma 3.5 and also section 9). So while technically annuli are allowed by the definition, only planar surfaces with  $\geq 3$  boundary components above the first stage contribute to the “non-trivial” increase of the height of  $C$ .

3. If the links  $L$  defining  $C$  have at least two components, then  $C$  is homotopy equivalent to the wedge of a collection of circles and of a collection of 2-spheres (one for each handle of  $C$ ). Of course all information about the attaching maps of the 2-handles is then lost. Note that  $\gamma$  is non-trivial in  $H_1(C)$ .

4. In the definition above we used zero framed 2-handles. In fact, the framing is not going to be important for our applications, in light of definition 4.4.

5. Recall the assumptions on each link  $L$  in definition 4.1: (i)  $\widehat{L}$  is homotopically essential, and (ii)  $L \cup \wedge \cup \wedge'$  is almost trivial. It is crucial for the applications of  $f$ -cells that the link  $L \cup \wedge$  is essential – this is made precise using the Magnus expansion  $M(\wedge)$ , see section 3.6. Therefore the basic requirements on  $L$  should be:  $\widehat{L}$  is homotopically essential and almost trivial. We make a slightly stronger assumption:  $L$  is (almost trivial)<sup>+</sup> since this makes the proofs of the properties of  $f$ -cells technically easier. It is an interesting question whether the extra condition may be removed.

**4.2. The associated tree.** It is useful to encode the branching of an  $f$ -cell  $C$  using the associated tree  $T_C$ . Define  $T_C$  inductively: suppose  $C$  has height 1. Then we assign to the body surface  $P$  (say with  $k + 1$  boundary components) of  $C$  the cone  $T_P$  on  $k + 1$  points. We consider the vertex corresponding to the attaching circle  $\gamma$  of  $C$  as the root of  $T_P$ , and the other  $k$  vertices as the leaves of  $T_P$ . For each handle of  $C$  we attach an edge to the corresponding leaf of  $T_P$ . The leaves of the resulting tree  $T_C$  are in 1 – 1 correspondence with the handles of  $C$ .

Suppose  $C$  has height  $h > 1$ , then it is obtained from an  $f$ -cell  $C'$  of height  $h - 1$  by replacing the handles of  $C'$  with  $f$ -cells  $\{C_i\}$  of height 1. Assuming inductively that  $T_{C'}$  is defined, one gets  $T_C$  by replacing the edges of  $T_{C'}$  associated to the handles of  $C'$  with the trees corresponding to  $\{C_i\}$ . Figure 4 gives an example of an  $f$ -cell of height 2 and its associated tree.

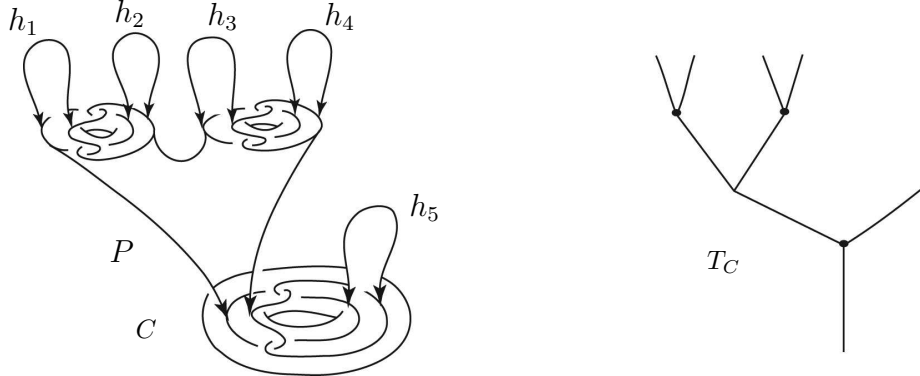


FIGURE 2. A schematic picture of an  $f$ -cell of height 2 and the associated tree

We divide the vertices of  $T_C$  into two types: the cone points corresponding to body (planar) surfaces are *unmarked*; the rest of the vertices are *marked*. Therefore the valence of an unmarked vertex equals the number of boundary components of the corresponding planar surface. The marked vertices are in 1-1 correspondence with the links  $L$  defining  $C$ , and the valence of a marked vertex is the number of components of  $L$  plus 1. It is convenient to consider the 1-valent vertices of  $T_C$ : its root and leaves (corresponding to the handles of  $C$ ) as unmarked. This terminology is useful in defining the maps of  $f$ -cells below. The height of an  $f$ -cell  $C$  may be read off from  $T_C$  as the maximal number of marked vertices along a geodesic joining a leaf of  $T_C$  to its root.

4.3. *Convention:* Recall from section 3.7 that each link  $L$  has a “preferred” ordering of its components, reflecting a non-trivial coefficient in the expansion  $M(\wedge)$ . We fix a specific planar embedding of  $T_C$ , reflecting this order. This applies to marked vertices – there is a flexibility in embedding the tree at the unmarked vertices.

**Definition 4.4.** An  $f$ -cell is a model  $f$ -cell with a finite number of self-plumbings and plumbings among the handles and body surfaces of  $C$ , subject to the following disjointness requirement:

- Consider two surfaces  $A, B$  (they could be handles or body stages) of  $C$ . Let  $a, b$  be the corresponding vertices in  $T_C$ . (For body surfaces this is the corresponding unmarked cone point, for handles this is the associated leaf.) Consider the geodesic joining  $a, b$  in  $T_C$ , and look at its vertex  $c$  closest to the root of  $T_C$  – in other words,  $c$  is the first common ancestor of  $a, b$ . If  $c$  is a marked vertex then  $A, B$  are disjoint.

In particular, self-plumbings of any handle and body surface are allowed. In the example shown in figure 2 above, the handle  $h_1$  is required to be disjoint from  $h_2$ ,  $h_3$  is disjoint from  $h_4$ ;  $h_1 - h_4$  and  $P$  are disjoint from  $h_5$ .

An  $f$ -cell in a 4-manifold  $M$  is an embedding of an  $f$ -cell into  $M$ . We say that its image is a realization of  $C$  in  $M$ , and abusing the notation we denote its image in  $M$  also by  $C$ .

*Remarks.* 1. A model  $f$ -cell  $C$  is a thickening of 2-manifolds except for the neighborhoods  $N$  of the solid tori  $\alpha \times D^2$ , containing the links  $L$  which serve as the attaching regions for higher stages. In the definition of an  $f$ -cell we require that all singularities of a map  $C \rightarrow M^4$  are double points between the surfaces, and the neighborhoods  $N$  are embedded and disjoint from everything else. This requirement is important and is included to prevent trivial unlinking. Note that conversely given a map  $C \rightarrow M$  which is an embedding near the aforementioned solid tori, it may be perturbed so that all its singularities are double points of surface stages of  $C$ .

The difference between continuous and smooth maps of  $f$ -cells in a smooth 4-manifold is not significant: if  $\phi: C \rightarrow M$  is continuous, it may be perturbed to a smooth map  $\phi'$ . The solid tori  $\alpha \times D^2$  discussed above are thickenings of circles, so by general position their image under  $\phi'$  is embedded and disjoint from thickenings of the surface stages.

**Definition 4.5.** Let  $M$  be a 4-manifold with a basepoint. Consider the following relation on based loops in  $M$ :  $\gamma \sim \gamma'$  if there is a based homotopy from  $\gamma(\gamma')^{-1}$  to a based loop which bounds an  $f$ -cell of height  $n$  in  $M$ . The  $n$ -th link group  $\lambda_n(M)$  is defined as  $\{\text{based loops in } M\}/\sim$ .

An immersion  $D^2 \rightarrow M$  is an example of an  $f$ -cell of any height, so if two embedded loops  $\gamma, \gamma'$  cobounded a map of the disk, then perturbing the map, one sees that they also cobound an  $f$ -cell. However we include the homotopy into the relation to deal with the loops which are not embedded. Alternatively, one could address this issue by allowing arbitrary singularities near the attaching curve in the definition of an  $f$ -cell.

**Proposition 4.6.** *The relation  $\gamma \sim \gamma'$  in definition 4.5 is an equivalence relation.*

*Proof.* One needs to verify that if  $\gamma_1 \sim \gamma'_1$  and  $\gamma_2 \sim \gamma'_2$  then  $\gamma_1\gamma_2 \sim \gamma'_1\gamma'_2$ . It follows from the definition that for each  $i = 1, 2$ ,  $\gamma_i(\gamma'_i)^{-1}$  is homotopic to a loop  $\alpha_i$  which bounds an  $f$ -cell  $C_i$  of height  $n$ . It suffices to prove that  $\alpha_1(\alpha_2)^{-1}$  is homotopic to a loop which bounds an  $f$ -cell of height  $n$ . We have the wedge  $(C_1, \alpha_1) \vee_p (C_2, \alpha_2)$  of two  $f$ -cells of height  $n$ , where the identification point  $p$  is the base point.  $C_1 \vee C_2$  in fact may be viewed as the image of a map  $\phi: C \rightarrow M$ , where  $C$  is an  $f$ -cell of height  $n$ . Here the first stage surface  $P$  of  $C$  is the boundary connected sum, along an arc  $\beta$ , of the first stage surfaces of  $C_1, C_2$ , and the map  $\phi$  send the arc  $\beta$  to  $p$ .

Perturb the map  $\phi$ . This changes the boundary  $\alpha_1(\alpha_2)^{-1}$  by a homotopy. Note that the solid tori  $\alpha \times D^2$  in the definition of  $C$  are neighborhoods of circles, and it may be assumed that all intersections  $C_1 \cap C_2$  are double points among surface stages (or handles). Furthermore, all such intersections are allowed by the definition of an  $f$ -cell: the common ancestor of all double points in  $C_1 \cap C_2$  in the tree  $T_C$  is the cone point corresponding to  $P$ . According to the definition,  $C$  is an  $f$ -cell.  $\square$

*Remarks.* 1. It follows from the definition that  $\pi_1(M)$  surjects onto  $\lambda_1(M)$ . Moreover, since an  $f$ -cell of height  $n$  satisfies the definition of an  $f$ -cell of height  $n+1$  (see remark 1 after definition 4.1),  $\lambda_n(M)$  maps onto  $\lambda_{n+1}(M)$ . In section 6 we introduce an additional grading on  $f$ -cells, leading to a two-parameter family of groups  $\lambda_{i,j}(M)$ .

2. Note that the definition of  $\lambda_n(M)$  makes sense for a manifold of any dimension (and in fact for any topological space), but the theory is non-trivial only in dimension 4. If  $\dim M \geq 5$  then the disjointness requirement is satisfied by general position. If  $\dim M < 4$  then one doesn't expect it hold due to the dimension count.

3. A few words on the terminology: the name *flexible cell* refers to the fact that certain loops in 4-manifolds bound them when they don't bound the standard 2-cell. The groups  $\lambda$  are called *link groups* since flexible cells  $C$ , used to define  $\lambda$ , are determined by a collection of links (the attaching regions for the handles of  $C$ .)

It is easy to find examples of 4-manifolds with  $\pi_1 \neq \lambda_1$ . We start with an example of  $M^4$  with  $\pi_1 M \cong \mathbb{Z}$  and  $\lambda_1(M) = 0$ :

**Example 4.7.** Consider  $M = S^1 \times D^2 \times I \cup_L 2$ -handles where  $L$  is the Bing double of the solid torus  $S^1 \times D^2 \times \{1\}$ , or any other link satisfying the conditions 3.7. Clearly  $\pi_1 M \cong \mathbb{Z}$  and  $\lambda_1(M) = 0$ . On the other hand, it follows from lemma 8.1

that the manifold  $N = S^1 \times D^3$  provides an example with  $\lambda_1(N) \cong \pi_1(N) \cong \mathbb{Z}$ . See lemma 6.5 for a more detailed discussion.

## 5. FLEXIBLE CELLS IN 4-SPACE

We begin the section by showing that any  $f$ -cell  $(C, \gamma)$  has a realization in  $(D^4, \partial D^4)$ . The main purpose of the section is to analyze the fundamental group of the complement,  $\pi_1(D^4 \setminus C)$ . In particular, we will use the technique presented in section 2 to find a presentation of the nilpotent quotients  $\pi_1(D^4 \setminus C)/(\pi_1(D^4 \setminus C))^q$ . These results will be used in sections 6 – 8 to formulate invariants which depend only on the underlying model  $f$ -cell  $C$  and not on its particular realization in the 4-ball. To fix the notation, recall that a model  $f$ -cell  $(C, \gamma)$  of height 1 is determined by the following data:

- the number of boundary components of the body surface  $P$ :  $\partial P = \gamma \cup \alpha_1 \cup \dots \cup \alpha_k$ ,
- A collection of links  $L_1, \dots, L_k$  where  $L_i \subset \alpha_i \times D^2$ .

**Lemma 5.1.** *Let  $(C, \gamma)$  be a model  $f$ -cell. Then there is a realization of  $(C, \gamma)$  in  $(D^4, \partial D^4)$ .*

*Proof.* The proof is inductive, starting with the base surface of  $C$  and moving up. Start with an unknotted circle  $\gamma$  in  $S^3$  and let  $\gamma \times D^2$  bound a 2-handle  $D^2 \times D^2$  in  $D^4$ . Puncture the core of the handle to get an embedding of the first stage planar surface  $P$ . Note that for each  $i$ ,  $\alpha_i \times D^2$  bounds a (just removed) 2-handle  $H_i$  in the complement of  $P$  in  $D^4$ . The link  $L_i \subset \alpha_i \times D^2$  is homotopically trivial, so it bounds disjoint immersed disks  $\{\Delta\}$  in  $H_i$ . The self-intersections of handles and of body surfaces are allowed in the definition of  $f$ -cells. If the height of  $C$  is greater than 1, repeat the construction. (The disks  $\Delta$  are converted into second stage surfaces by puncturing them in the complement of double points, etc.)  $\square$

5.2. This construction has a useful generalization: Let  $(C, \gamma)$  be a model  $f$ -cell of height  $n$ . Denote  $C_k$  = bottom  $k$  stages of  $C$ , and  $C_k^i$  = components of its upper  $n - k$  stages. Consider  $\overline{C}_k$  obtained from  $C$  by replacing each  $C_k^i$  with a 2-handle  $H_k^i$ , so  $\overline{C}_k$  is a model  $f$ -cell of height  $k$ . In terms of the associated trees,  $T_{\overline{C}_k}$  is obtained from  $T_C$  by replacing each of the subtrees  $T_{C_k^i}$  with an edge (a leaf). Lemma 5.1, applied to  $C_k^i$  in  $H_k^i$ , implies that there is a realization  $(C, \gamma) \longrightarrow (\overline{C}_k, \gamma)$ .

### 5.3. Filtrations of $\pi_1(D^4 \setminus C)/(\pi_1(D^4 \setminus C))^q$ .

Suppose  $C \subset D^4$  has height  $n$ . For each body surface  $P_i^k$  at height  $k$ , consider a meridian  $m_i^k$ : starting at the basepoint in  $D^4 \setminus C$ , choose an arc  $\beta$  joining it to  $P_i^k$ . Then  $m_i^k$  is an element of  $\pi_1(D^4 \setminus C)$  obtained by traversing  $\beta$  almost to the surface, followed by a small circle linking  $P_i^k$  and then by  $\beta^{-1}$ . The meridian  $m^1$  may be identified with the meridian to  $\gamma$  in  $S^3$ , and the  $\{m^{n+1}\}$  are the meridians to the handles of  $C$ . Denote  $\mathcal{M}^k = \{m_i^k\}$ ,  $\mathcal{M} = \mathcal{M}^{n+1}$ .

*Remark.* The height of  $C$  is defined as the maximum distance from a leaf of  $T_C$  to its root. In case the height of  $C$  is not uniform – that is, the distance from some of the leaves of  $T_C$  to the root is less than  $n$  – we uniformize it by inserting trivial surface stages (annuli). The effect on  $T_C$  is the introduction of new vertices (of valence 2) in certain edges, without changing its homeomorphism type. Then the handles of  $C$  are all at height  $n + 1$ .

By Alexander duality, the meridians  $\mathcal{M}$  generate  $H_1(D^4 \setminus C)$ . According to section 2, they also generate  $\pi_1(D^4 \setminus C)/(\pi_1(D^4 \setminus C))^q$ .

**Definition 5.4.** Let  $\tilde{G}_k$  (respectively  $G_k(C)$ ) denote the subgroup of  $\pi_1(D^4 \setminus C)/(\pi_1(D^4 \setminus C))^q$  generated (respectively normally generated) by the elements of  $\mathcal{M}^k$ .

*Remarks.* Note that choosing a different path  $\beta$  in the definition of a meridian  $m_j^i$  results in a conjugate element  $gm_j^i g^{-1}$ ,  $g \in \pi_1(D^4 \setminus C)$ . Conversely, any conjugate element is represented by a meridian, for some choice of  $\beta$ . Given  $C \subset D^4$ , the group  $G_k(C)$  is well-defined while  $\tilde{G}_k(C)$  depends on the choice of the meridians. Note that, even though  $G_k(C)$  is nilpotent, it is not in general generated by the elements of  $\mathcal{M}^k$ . These groups form filtrations

$$\begin{aligned} \pi_1(D^4 \setminus C)/(\pi_1(D^4 \setminus C))^q &= \tilde{G}_{n+1}(C) > \tilde{G}_n(C) > \dots > \tilde{G}_1(C), \\ \pi_1(D^4 \setminus C)/(\pi_1(D^4 \setminus C))^q &= G_{n+1}(C) \triangleright G_n(C) \triangleright \dots \triangleright G_1(C) \end{aligned}$$

Since  $m_1$  may be identified with a meridian to  $\gamma$  in  $S^3$ ,  $\tilde{G}_1(C)$  is the image of  $\pi_1(S^3 \setminus \gamma)/(\pi_1(S^3 \setminus \gamma))^q \cong \mathbb{Z}$ . We will show (theorem 8.1) that in fact  $\tilde{G}_1(C) \cong \mathbb{Z}$ .

To clarify the meaning of the groups  $\tilde{G}_k(C)$ , we will show that for certain embeddings  $C \subset D^4$ ,  $G_k(C)$  is isomorphic to  $\pi_1(D^4 \setminus C')/(\pi_1(D^4 \setminus C'))^q$  for some  $f$ -cell  $C'$  of height  $k$ . The dependence on a particular embedding into  $D^4$  will be removed in the next section with the introduction of the generalized Milnor group.

Given a model  $f$ -cell  $C$ , consider a model  $f$ -cell  $\overline{C}_k$  of height  $k$  such that there is a realization  $C \subset \overline{C}_k$  (see 5.2.) By lemma 5.1, we may consider a realization of  $\overline{C}_k$  in  $D^4$ . This requires an introduction of self-intersections in various body surfaces

and handles of  $\overline{C}_k$ . These translate into the corresponding self-intersections of body surfaces of  $C$ , and abusing the notation we will denote the realizations by the same symbols as the model  $f$ -cells:  $C \subset \overline{C}_k \subset D^4$ .  $\pi_1(D^4 \setminus \overline{C}_k)/(\pi_1(D^4 \setminus \overline{C}_k))^q$  is generated by the meridians to the handles of  $\overline{C}_k$ . Under the homomorphism  $i_*$  induced by the inclusion  $D^4 \setminus \overline{C}_k \hookrightarrow D^4 \setminus C$ , they map to elements of  $\mathcal{M}^k$ , therefore  $i_*$  is a surjection:

$$i_*: \pi_1(D^4 \setminus \overline{C}_k)/(\pi_1(D^4 \setminus \overline{C}_k))^q \twoheadrightarrow \tilde{G}_k(C).$$

**Lemma 5.5.** *The map  $i_*$  is an isomorphism.*

*Proof.* First suppose there is a curve  $\delta \subset D^4 \setminus \overline{C}_k$  which is null-homotopic in  $D^4 \setminus C$ , so  $\delta = \partial\Delta$  where  $\Delta$  is an immersion  $D^2 \looparrowright D^4 \setminus C$ . Note that  $\Delta$  is disjoint from the body of  $\overline{C}_k$  (since it equals the bottom  $k$  stages of  $C$ , and  $\Delta \cap C = \emptyset$ ), and it may only intersect the handles of  $\overline{C}_k$ . Perturb  $\Delta$  so it is in general position with respect to the handles  $H_k^i$  of  $\overline{C}_k$ , so  $\Delta \cap \overline{C}_k$  is a collection of meridional disks of the 2-handles. Let  $\Sigma \subset \Delta$  be one of these disks, denote  $\sigma = \partial\Sigma$ . Connecting it to the basepoint, consider  $\sigma$  as a meridian in  $\mathcal{M}^k$ . It is proved in theorem 8.1 below (applied to  $C_k^i \subset H_k^i$  using the notation of 5.2) that any such meridian is non-trivial in  $\pi_1(H_k^i \setminus C_k^i)$ . Consequently  $\Sigma$  intersects  $C_k^i$ , a contradiction. Therefore  $\Delta$  is disjoint from the handles  $H_k^i$  and so  $\delta$  is null-homotopic in  $D^4 \setminus \overline{C}_k$ . In other words, the map

$$\pi_1(D^4 \setminus \overline{C}_k) \longrightarrow \pi_1(D^4 \setminus C)$$

is injective. The assertion of the lemma deals with the quotients by the  $m$ -th terms of the lower central series, so assume  $\delta$  is a curve in  $D^4 \setminus \overline{C}_k$  which is trivial in  $\pi_1(D^4 \setminus C)/(\pi_1(D^4 \setminus C))^q$ . Then  $\delta$  bounds a map of an  $q$ -grope  $G$  in  $D^4 \setminus C$ . The subset of  $G$  which is not locally a 2-manifold is 1-dimensional, so we can again arrange that  $G \cap \overline{C}_k$  is a collection of meridional disks of the 2-handles, or in other words  $\delta$  already bounds an  $m$ -grope in  $D^4 \setminus \overline{C}_k$ .  $\square$

Another filtration,  $\pi_1(D^4 \setminus C)/(\pi_1(D^4 \setminus C))^q = H_{n+1}(C) \triangleright \dots \triangleright H_1(C)$ , with  $G_k(C) \triangleleft H_k(C)$ , is defined in section 7 in terms of the Magnus expansion.

### 5.6. A presentation of $\pi_1(D^4 \setminus C)/(\pi_1(D^4 \setminus C))^q$ .

Given  $(C, \gamma) \subset (D^4, \partial D^4)$ , let  $m$  denote a meridian to  $\gamma$  in  $S^3$ . First assume  $C$  has height 1 and  $P$  is a pair of pants. Notations:  $\partial P = \gamma \cup \alpha_1 \cup \alpha_2$ ,  $\overline{P} = P \times D^2$ ,  $C = \overline{P} \cup_{L_1 \cup L_2}$  2-handles where  $L_i \subset \alpha_i \times D^2$  are links satisfying the conditions in definition 4.1. Let  $I_1, I_2$  be the index sets for the components of  $L_1, L_2$  respectively.  $\Lambda_1, \Lambda_2$  will denote the meridional curves  $\{p_i\} \times \partial D^2$  of the solid tori  $\alpha_i \times D^2$ .

$H_1(D^4 \setminus C)$  is generated by  $\mathcal{M} = \{m_i\}$ : the meridians to the handles of  $C$ . The index sets  $I_1, I_2$  also parametrize the handle of  $C$ , and to be specific, divide the set of meridians  $\mathcal{M}$  into two subsets:  $\mathcal{M}_{I_1}, \mathcal{M}_{I_2}$ . Denote by  $F_{\mathcal{M}} = F_{\mathcal{M}_{I_1}, \mathcal{M}_{I_2}}$  the free group generated by the elements of  $\mathcal{M}$ , and consider the Magnus expansion

$$(5.1) \quad \pi_1(D^4 \setminus C)/(\pi_1(D^4 \setminus C))^q \xleftarrow{p} F_{\mathcal{M}} = F_{\mathcal{M}_{I_1}, \mathcal{M}_{I_2}} \xrightarrow{M} \mathbb{Z}\{X\} = \mathbb{Z}\{X_{I_1}, X_{I_2}\}$$

where  $M(m_i) = 1 + x_i$ ,  $i \in I_1 \cup I_2$ . We need to fix a specific word in  $F_{\mathcal{M}}$  representing the meridian  $m$ .  $\Lambda_1$  and  $m$  cobound a cylinder in  $D^4 \setminus C$ : the circle normal bundle of  $P$  in  $D^4$ , restricted to a path in  $P$  joining two points in  $\alpha_1$  and  $\gamma$ . Therefore  $m, \Lambda_1$  are conjugate in  $\pi_1(D^4 \setminus C)$ . Consider  $\Lambda_1$  in  $\pi_1(\alpha_1 \times D^2 \setminus L_1)$  and consider the commutative diagram of Magnus expansions:

$$(5.2) \quad \begin{array}{ccccc} \pi_1(S^1 \times D^2 \setminus L_1)/(\pi_1(S^1 \times D^2 \setminus L_1))^q & \longleftarrow & F_{\mathcal{M}_1} & \xrightarrow{M_1} & \mathbb{Z}\{X_{I_1}, y\} \\ & & \downarrow i_* & & \downarrow \\ \pi_1(D^4 \setminus C)/(\pi_1(D^4 \setminus C))^q & \longleftarrow & F_{\mathcal{M}} & \xrightarrow{M} & \mathbb{Z}\{X\} \end{array}$$

Denote by  $W_1$  some word representing  $\Lambda_1$  in the free group  $F_{\mathcal{M}_1}$ , then  $i_*(W_1)$  represents  $m$  in  $F_{\mathcal{M}}$ . Recall (see section 3.6) that each term in the expansion  $M_1(W_1)$  contains all of the variables  $x_1, \dots, x_n$ , and according to the commutative diagram above, this is also true for  $M(w)$ .

Given an element  $g \in \pi_1(D^4 \setminus C)/(\pi_1(D^4 \setminus C))^q$ , consider a word  $w$  representing it in  $F_{\mathcal{M}}$ . As in the classical case of Milnor's invariants of links, discussed in section 3, the coefficients of the Magnus expansion  $M(w)$  in general are not well-defined invariants of  $g$ . This is due to the choice of the meridians generating the group, and since the kernel of the projection  $p$  is non-trivial. In our case, compared to the classical situation, the kernel involves more relations in  $\pi_1(D^4 \setminus C)/(\pi_1(D^4 \setminus C))^q$ , reflecting the topology of flexible cells.

According to lemma 2.1, to see the relations in  $\pi_1(D^4 \setminus C)/(\pi_1(D^4 \setminus C))^q$  we need to analyze the generators of  $H_2(D^4 \setminus C)$ . By Alexander duality,  $H_2(D^4 \setminus C) \cong H_1(C, \gamma)$ . Note that  $H_1(C, \gamma)$  is generated (1) by double point loops corresponding to the intersections among the handles and body surfaces, subject to the disjointness requirement in definition 4.4, and (2) by  $H_1(P, \gamma)$ . (Here we assume the non-trivial case: each link  $L_i$  consists of at least two components, so  $C$  is homotopy equivalent to the wedge of two circles with a collection of two-spheres, one for

each handle.) We will divide the corresponding dual generators of  $H_2(D^4 \setminus C)$  into four types,  $(R_1) - (R_4)$ , and analyze the resulting indeterminacy in the coefficients of the Magnus expansion (5.1).

**(R<sub>1</sub>)** Clifford tori for the self-intersections of any handle  $H_i$  of  $C$ ,  $i \in I_1 \cup I_2$ .

The corresponding relations are of the form  $[(m_i)^f, (m_i)^g] = 1$ ,  $i \in I_1 \cup I_2$ ,  $f, g \in \pi_1(D^4 \setminus C)$  (see section 3.8), and are familiar from the study of link homotopy and the classical Milnor group (see 3.2). Pulling back the relations to  $F_{\mathcal{M}}$ , consider the ideal  $\mathcal{I}_1$  generated by their images in  $\mathbb{Z}\{\mathcal{X}\}$ . Observe that each term (besides 1) of any element in the ideal  $\mathcal{I}_1$  has repeating variables.

More precisely, note that for any  $a \in F_{\mathcal{M}}$ , the Magnus expansions  $M(a^{-1}m_i a)$  and  $M(a^{-1}(m_i)^{-1}a)$  are of the form  $1 +$  terms containing  $x_i$  (where  $M(m_i) = 1 + x_i$ .) The commutator  $[(m_i)^f, (m_i)^g]$  is a product of  $(m_i)^g$  conjugated by  $(m_i)^f$  and  $(m_i^{-1})^g$ , therefore  $M([(m_i)^f, (m_i)^g]) = 1 +$  terms containing at least two entries of  $x_i$ . Hence the monomials with non-repeating variables are invariant under multiplication by a conjugate of the relation  $(R_1)$ .

According to definition 4.4, any handle attached to  $L_1$  can intersect any handle attached to  $L_2$ . The corresponding generators of  $H_2$  are

**(R<sub>2</sub>)** Clifford tori for the intersections between the 2-handles  $H_{i_1}$  and  $H_{i_2}$ , where  $i_1 \in I_1, i_2 \in I_2$ .

These tori give relations  $[(m_{i_1})^f, (m_{i_2})^g] = 1$ . Each term of any element in the ideal generated by the Magnus expansion of these relations has both variables  $x_{i_1}$  and  $x_{i_2}$ , where  $i_1 \in I_1, i_2 \in I_2$ .

**(R<sub>3</sub>)** Clifford tori for the intersections of any handle  $H_i$  with the body surface  $P$ , and Clifford tori for the self-intersections of  $P$ .

These generators of  $H_2$  impose the relations of the form  $[m_i^f, m^g]$ , and of the form  $[m^f, m^g]$ . Here  $m_i$  is a meridian to a handle  $H_i$ ,  $i \in I_1 \cup I_2$  and  $m$  is a meridian to  $P$ . Recall from the discussion at the beginning of section 5.6 that each term in the expansion  $M(m)$  contains each of the variables  $X_{I_1}$ . If  $i$  above is an element of  $I_1$  then all terms in the expansion of  $[m_i^f, m^g]$  contain a repeating variable (one of the  $X_{I_1}$ ). If  $i$  is an element of  $I_2$  then each term in the expansion of  $[m_i^f, m^g]$  contains both variables  $x_{i_1}$  and  $x_{i_2}$  for some  $i_1 \in I_1, i_2 \in I_2$ . Note that in each case, the indeterminacy has already appeared as a result of relations  $(R_1), (R_2)$ .

There is another generator  $(R_4)$  of  $H_2(D^4 \setminus C)$ , Alexander dual to  $H_1(P, \partial P \cap S^3) \cong \mathbb{Z}$ . Since we assumed each link  $L_i$  has at least two components, the meridian  $\wedge_i = \{p_i\} \times \partial D^2$  of the solid torus  $\alpha_i \times D^2$  bounds a surface  $S_i$  in  $\alpha_i \times D^2 \setminus L_i$ .

(Consider the disk  $\{p_i\} \times D^2$ . Since  $\Lambda_i$  has the trivial linking number with each component of  $L_i$ , the disk may be converted into a surface disjoint from the link.)

A geometric representative for this class of  $H_2(D^4 \setminus C)$  is given by the surface  $S_1 \cup \text{annulus} \cup S_2$ . Here the annulus is cobounded by  $\Lambda_1$  and  $\Lambda_2$ , and is the circle normal bundle of  $P$  in  $D^4$ , restricted to a path in  $P$  joining two points in  $\alpha_1, \alpha_2$ . Denote, as above, by  $W_1, W_2$  some words in the free group representing  $\Lambda_1, \Lambda_2$ . Then the corresponding relation is

$$(\mathbf{R}_4) \quad W_1(W_2)^{-1} = 1.$$

Now consider the *general height = 1 case*:  $\partial P = \gamma \cup \alpha_1 \cup \dots \cup \alpha_n$ . The relations are directly analogous to those described above; in particular there are  $n - 1$  relations of type  $(R_4)$ :  $W_1(W_2)^{-1} = 1, \dots, W_{n-1}(W_n)^{-1} = 1$ .

**The general case** (height  $\geq 1$ ). The double points occur as intersections of handles and body surfaces, subject to the disjointness assumption in definition 4.4. More precisely, the general relations of types  $(\mathbf{R}_1) - (\mathbf{R}_3)$  are represented by the Clifford tori for self-intersections of each handle and body surface of  $C$ , and for intersections of any two handles and/or body surfaces, such that the first common ancestor of the corresponding vertices in  $T_C$  is unmarked. Recall that the generators of  $H_1(D^4 \setminus C)$ , and also the variables  $X$  are in 1-1 correspondence with the handles of  $C$  and also with the leaves of  $T_C$ . The analysis analogous to the above implies that *each term of any element in the ideal generated by the Magnus expansions of the relations  $(R_1) - (R_3)$  either contains repeating variables, or it contains variables  $x_i$  and  $x_j$  whose first common ancestor in  $T_C$  is unmarked.*

There is also a collection of relations  $(\mathbf{R}_4)$  for the body surfaces of  $C$ . Each generator of  $H_1(\text{body of } C, \gamma)$  contributes a relation of type  $W_1(W_2)^{-1}$  as above.

## 6. GENERALIZED MILNOR GROUP

Starting with an  $f$ -cell  $(C, \gamma) \subset (D^4, \partial D^4)$  we will derive invariants of  $(C, \gamma)$  independent of the embedding into  $D^4$ . This feature of the invariants is particularly important for applications to the A-B slice problem [5]. Recall Milnor's work on links in  $S^3$  and a related result for disks in 4-space: If a link  $L \subset S^3$  bounds disjoint immersions of disks  $\Delta$  in  $D^4$  then the Milnor group  $M\pi_1(D^4 \setminus \Delta)$  is the free Milnor group (see section 3.8). In particular, the coefficients in the Magnus expansion  $M\pi_1(D^4 \setminus \Delta) \rightarrow R[X]$  are well-defined. In our setting  $M\pi_1(D^4 \setminus C)$  is *not* the free Milnor group. The goal is to analyze the indeterminacy and to extract useful invariants.

Recall the notation: we fix a collection  $\mathcal{M}$  of meridians  $m$  to the handles of  $C$ , one for each handle. Then the elements of  $\mathcal{M}$  generate any nilpotent quotient of  $\pi_1(D^4 \setminus C)$ .

**Definition 6.1.** The generalized Milnor group  $GM(C)$  denotes  $\pi_1(D^4 \setminus C)$  modulo the normal closure of all elements of the form

$$(6.1) \quad [m^f, m^g], \quad \text{and} \quad [m_1^f, m_2^g], \quad \text{where} \quad f, g \in \pi_1(D^4 \setminus C), \quad m, m_1, m_2 \in \mathcal{M}, \quad \text{and}$$

the first common ancestor of  $m_1, m_2$  is unmarked (see definition 4.4).

In particular,  $GM(C)$  is a quotient of the classical Milnor group  $M\pi_1(D^4 \setminus C)$  defined using the set  $\mathcal{M}$  of normal generators. Consequently,  $GM(C)$  is nilpotent, and so is generated by the elements of  $\mathcal{M}$ .

For example, consider a realization in  $D^4$  of the  $f$ -cell in figure 2. Denoting by  $m_i$  a meridian to the handle  $h_i$ ,  $i = 1, \dots, 5$ , the relations in the definition of  $GM(C)$  are:

$$[m_i^f, m_i^g] = 1, \quad i = 1, \dots, 5, \quad [m_1^f, m_3^g] = 1, \quad [m_1^f, m_4^g] = 1, \quad [m_2^f, m_3^g] = 1, \quad [m_2^f, m_4^g] = 1.$$

The definition of  $M(C)$  incorporates the relations  $(R_1) - (R_3)$  in  $\pi_1(D^4 \setminus C)$ , discussed in the previous section. In the classical Milnor's theory, the free Milnor group has a well-defined representation into (the units of) the ring of polynomials where the terms have non-repeating variables. In the next section we describe the analogous representation for  $GM(C)$ . In the present setup there is also an additional indeterminacy, due to the relations  $(R_4)$ , and this is analyzed in section 8. It is convenient to define, analogously to the classical case, the free Milnor group:

**Definition 6.2.** The *free* generalized Milnor group  $GM(F_{\mathcal{M}})$  is defined to be the free group  $F_{\mathcal{M}}$  modulo the relations (6.1).

It follows that  $GM(C)$  is the quotient of  $GM(F_{\mathcal{M}})$  by the relations  $(R_4)$ .

*Remark.* If one imposes a stronger condition on the links  $L$  in the definition 4.1 (see also 3.7):  $\wedge \in M(L \cup \wedge')$  lies in the subgroup  $M(L)$ , then it is not difficult to show that  $GM(C)$  depends only on  $C$  and not on the embedding  $C \rightarrow D^4$ . In particular, this is true in the important case when each link  $L$  is an (iterated) Bing double of the core of the solid torus. It is an interesting question whether  $GM(C)$  is an invariant of  $C$  in the general case. Analogously to the classical case,  $MC(G)$  has the following property.

**Proposition 6.3.** *Given a model  $f$ -cell  $C$ , there exists a realization  $\overline{C} \subset D^4$  of  $C$  such that  $\pi_1(D^4 \setminus \overline{C}) \cong M(\overline{C})$ .*

*Proof.* Consider any realization  $C' \subset D^4$  of  $C$ .  $GM(C')$  is nilpotent and finitely generated, and is therefore finitely presented. That is,  $MC(G)$  is isomorphic to  $\pi_1(D^4 \setminus C')$  modulo a finite number of relations (6.1). Introduce plumbings and self-plumbings of  $C'$  to get  $\overline{C}$  satisfying the proposition.  $\square$

**6.4. Grading of  $f$ -cells.** Given  $(C, \gamma) \subset (D^4, \partial D^4)$ , let  $m$  denote a meridian to  $\gamma$  in  $S^3$ . There is no relation, in general, between the height of  $C$  and how deep  $m$  is in the lower central series of  $\pi_1(D^4 \setminus C)$ , or of  $GM(C)$ . For example, let  $(C_1, \gamma_1)$  be an  $f$ -cell of height  $k$  where each link is the Bing double of the core of the corresponding solid torus (and the body surfaces are arbitrary – to be specific consider pairs of pants.) Also consider  $(C_2, \gamma_2)$  of height 1 where the body surface is an annulus and the link is the  $k$ -iterated Bing double of the core. Then for each  $i = 1, 2$ ,  $m_i$  is in the  $2^k$ -th term of the lower central series of  $\pi_1(D^4 \setminus C_i)$ .

Define the nilpotency class of  $C$  to be the least  $k$  such that the  $k$ -th term of the lower central series  $GM(C)^k$  is trivial. Assuming that each link in the definition of  $C$  has at least two components, it is clear that the nilpotency class of an  $f$ -cell of height  $k$  is at least  $k + 1$ . Refining definition 4.5, consider  $\lambda_{i,j}(M) = \{\text{based loops in } M\}$  modulo loops bounding  $f$ -cells of height  $i$  and having nilpotency class  $j$ . There is a commutative diagram of surjections

$$\begin{array}{ccccccc}
 \pi_1(M) & \longrightarrow & \lambda_{1,2}(M) & \longrightarrow & \lambda_{1,3}(M) & \longrightarrow & \lambda_{1,4}(M) \longrightarrow \dots \\
 & & & & \downarrow & & \downarrow \\
 & & & & \lambda_{2,3}(M) & \longrightarrow & \lambda_{2,4}(M) \longrightarrow \dots \\
 & & & & & & \downarrow \\
 & & & & & & \vdots
 \end{array}$$

**Lemma 6.5.** *Let  $\pi, \lambda$  be finitely presented groups, where  $\pi$  is aspherical, and suppose  $\pi$  maps onto  $\lambda$ . Then there are 4-manifolds  $M$  with  $\pi_1(M) \cong \pi$  and  $\lambda_{1,2}(M) \cong \lambda$ .*

*Proof.* Consider an aspherical 2-complex  $K$  with  $\pi_1 K \cong \pi$ . Replacing the cells of  $K$  by 0-, 1- and 2-handles, one gets a 4-manifold  $N$  with boundary. Observe that  $\pi_1(N) \cong \lambda_{1,2}(N)$ : suppose there is a loop  $\gamma \subset N$  trivial in  $\lambda_1(N)$  but not in  $\pi_1(N)$ . Then  $\gamma$  is homotopic to a loop  $\gamma'$  which bounds an  $f$ -cell  $C$  of height 1. Denote the body surface of  $C$  by  $P$ ,  $\partial P = \gamma' \cup \alpha_1 \cup \dots \cup \alpha_n$ . It follows that  $\alpha_i \neq 1 \in \pi_1(N)$  for some  $i$ . The link  $L_i \subset \alpha_i \times D^2$  has two components. Consider the 2-spheres  $S_1, S_2$  formed by the cores of the handles  $H_1, H_2$  of  $C$  attached to

the components of  $L_i$ , capped off by the null-homotopies of the components of  $L$  in  $\alpha_i \times D^2$ . Due to the assumptions on the link, and since the handles  $H_1, H_2$  are disjoint, the intersection of  $S_1, S_2$  is non-trivial in  $\mathbb{Z}\pi_1(N)$ . This is a contradiction with the asphericity of  $N$ .

Consider a collection of elements  $\alpha = \{\alpha_1, \dots, \alpha_k\}$  in  $\pi_1 K$  such that the quotient of  $\pi_1$  by the normal closure of  $\alpha$  is isomorphic to  $\bar{\pi}$ . Represent  $\alpha$  by embedded curves in  $\partial N$  and consider  $M = N \cup_\alpha 2$ - handles where the handles are attached to Bing doubles of the cores of  $\alpha_i \times D^2 \subset \partial N$ .  $\square$

Examples of 4-manifolds  $M$  for which  $\lambda_{i,j}(M) \neq \lambda_{i,j+1}(M)$  are constructed in [5]. It is an interesting question whether there are manifolds for which vertical maps are not isomorphisms as well.

## 7. REPRESENTATIONS $Q, R, S$ .

Consider a set  $\mathcal{M} = \{m\}$  of generators of  $H_1(D^4 \setminus C)$  provided by the meridians to the handles of  $C$ . The elements of  $\mathcal{M}$  are in 1 – 1 correspondence with the leaves of  $T_C$ , and are parametrized by the multiindices  $I = (i_1, j_1, \dots, i_n, j_n)$  where  $n$  is the height of  $C$ , the indices  $i$  correspond to the branching of the planar surface stages, and the indices  $j$  correspond to the components of the attaching links  $L$ .

**Definition 7.1.** Consider the set  $X = \{x\}$  whose elements are in 1 – 1 correspondence with the elements of  $\mathcal{M}$ . Let  $R[C]$  denote the quotient of the free associative ring  $\mathbb{Z}\{X\}$  generated by  $X$  by the ideal generated by the monomials  $M = x_{I_1} \cdots x_{I_k}$  such that

- either  $M$  contains repeating variables, or
- $M$  contains variables  $x_I, x_{I'}$  whose first common ancestor in  $T_C$  is unmarked.

The second condition may be rephrased as follows: consider the multiindices  $I = (i_1, \dots, j_n), I' = (i'_1, \dots, j'_n)$ . Consider the first index where these sequences differ: if it is one of the  $j$ 's then any monomial containing  $x_I, x_{I'}$  is in the ideal. Consider the Magnus expansion  $M$ :

$$(7.1) \quad GM(C) \longleftarrow GM(F_{\mathcal{M}}) \longleftarrow F_{\mathcal{M}} \xrightarrow{M} \mathbb{Z}\{X\} \longrightarrow R[C]$$

**Proposition 7.2.** *The Magnus expansion induces a well-defined homomorphism, which abusing the notation is also denoted  $M$ :  $GM(F_{\mathcal{M}}) \longrightarrow R[C]$ .*

*Proof.* The kernel of  $F_{\mathcal{M}} \rightarrow GM(F_{\mathcal{M}})$  is normally generated by the relations (6.1). Note that every every term (besides 1) in the expansion  $M([m^f, m^g])$  has a repeating variable:  $x$ . Similarly, every term in the expansion  $M([m_1^f, m_2^g])$  contains both variables  $x_1, x_2$ . Therefore the expansion of each relation is in the ideal defining  $R[C]$ .  $\square$

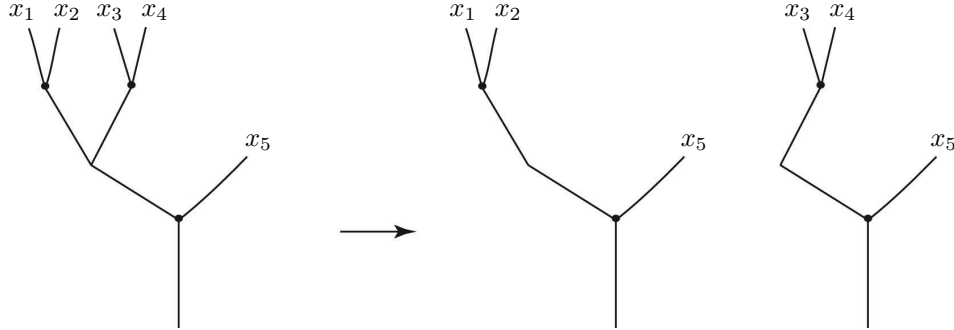


FIGURE 3.

**Definition 7.3.** Let  $v$  be a vertex of  $T_C$ . Assign to it an additive subgroup  $\tilde{R}_v \subset R[C]$ , as follows. Denote by  $T_v$  the subtree of  $T_C$  whose root is  $v$ ; let  $X_v$  denote all variables corresponding to the leaves of  $T_v$ .

At each unmarked vertex of  $T_v$  keep exactly one branch and erase the rest. Denote the resulting subtrees growing out of  $v$  by  $\{T_v^\alpha\}$ . Then by definition  $\tilde{R}_v$  is spanned by the monomials read off, clockwise, from all possible planar embeddings of  $T_v^\alpha$ , for all  $\alpha$ .

Suppose  $C$  has height  $n$  and assume all branches have uniform length (insert extra stages = annuli if necessary). Set  $\tilde{R}_k(C) = \bigoplus_v \tilde{R}_v \subset R[C]$ , where the summation is taken over all vertices  $v$  at height  $k$ . Denote  $\tilde{R}(C) = \tilde{R}_0(C) = \tilde{R}_r$  where  $r$  is the root of  $T_C$ .

For example, consider the  $f$ -cell in figure 2. Then there are two subtrees entering the computation of  $\tilde{R}(C)$ , shown in figure 3. There are a total of 8 planar embeddings of these subtrees, giving the monomials  $\{x_1x_2x_5, x_2x_1x_5, x_5x_1x_2, x_5x_2x_1, x_3x_4x_5, x_4x_3x_5, x_5x_1x_2, x_5x_2x_1, x_5x_3x_4, x_5x_4x_3\}$ . Note that some of the terms – for example  $x_1x_5x_2$  – do not appear since they do not arise from a subtree.

We will be interested in the subring  $1 + \tilde{R}(C)$  of  $R[C]$ . Note that it consists of (some of the) monomials of “maximal length”: if  $X_I$  is a monomial in  $\tilde{R}(C)$  then for any

variable  $x \in X$ , inserting  $x$  anywhere in  $X_I$  gives a trivial element of  $R[C]$ . The product in  $1 + \tilde{R}(C)$  is given by

$$(1 + \sum_I \alpha_I X_I)(1 + \sum_I \beta_I X_I) = 1 + \sum_I (\alpha_I + \beta_I) X_I.$$

**Definition 7.4.** For each vertex  $v$  of  $T_C$  consider the subring

$$S_v = 1 + \tilde{R}_v + \text{higher order terms}$$

of  $R[C]$ . By higher order terms we mean all terms of the form

$$T = f_1 x_1 f_2 x_2 \dots f_m x_m f_{m+1}$$

where the monomial  $x_1 \dots x_m$  (obtained from  $T$  by deleting the  $f$ 's) is in  $\tilde{R}_v$ , and at least one of the monomials  $f_1, \dots, f_{m+1} \in \mathbb{Z}[X]$  is not equal to 1. Similarly, set  $S_k = 1 + \tilde{R}_k + \text{higher order terms}$ . Observe that  $S_0(C) = 1 + \tilde{R}(C)$ : the monomials in  $\tilde{R}(C)$  already have maximal length, so there are no higher order terms.

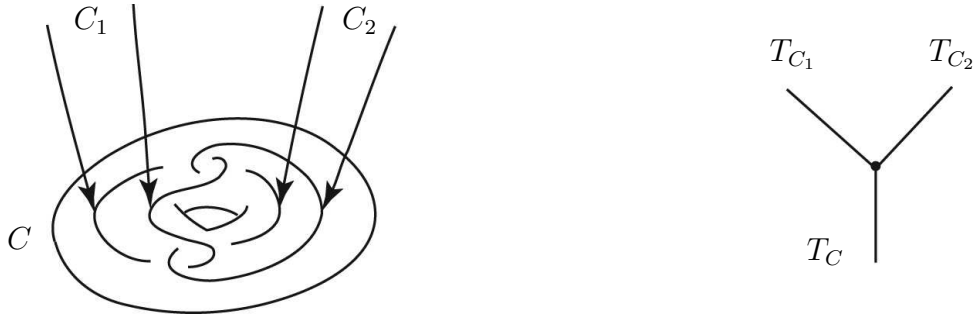


FIGURE 4. Raising the height: step 1.

7.5. It is useful to note an inductive construction of the representation  $S_0(C) = 1 + \tilde{R}(C)$ . An  $f$ -cell of height  $k$  is assembled from a planar surface  $P$ ,  $\partial P = \gamma \cup \alpha_1 \cup \dots \cup \alpha_n$ , and  $f$ -cells of height  $k - 1$  attached to the components of the links  $L_i$ ,  $L_i \subset \alpha_i \times D^2$ . This assembly may be decomposed into two steps. Step 1 (figure 4) corresponds to  $P = \text{annulus}$ , so there is just one link  $L$ . It follows from definitions 7.3, 7.4 that in this case  $\tilde{R}_v \cong (\tilde{R}_{v_1} \otimes \tilde{R}_{v_2}) \oplus (\tilde{R}_{v_2} \otimes \tilde{R}_{v_1})$ . Here the map  $\tilde{R}_{v_i} \otimes \tilde{R}_{v_j} \rightarrow \tilde{R}_v$  is defined on generators by  $X_i \otimes X_j \mapsto X_i \cdot X_j$  (the product of monomials). Step 2 (figure 5) combines the results of step 1 which are attached to an arbitrary planar surface. In this case  $\tilde{R}_v \cong \tilde{R}_{v_1} \oplus \tilde{R}_{v_2}$ .

The following lemma summarizes the basic properties of the representation  $S$ .

**Lemma 7.6.**

1. Let  $m$  be a meridian to a body surface of  $C$ , and let  $v$  be the corresponding vertex in  $T_C$ . Then there exists a word  $w \in F_{\mathcal{M}}$  representing it so that  $M(w) \in S_v$ .
2. In particular, let  $m_0$  denote a meridian to the bottom stage of  $C$  in  $D^4$  (for example, a meridian to  $\gamma$  in  $S^3$ .) Then there exists a word  $w_0$  representing it in  $F_{\mathcal{M}}$  such that  $M(w_0) \in S_0(C) = 1 + \tilde{R}(C)$ .

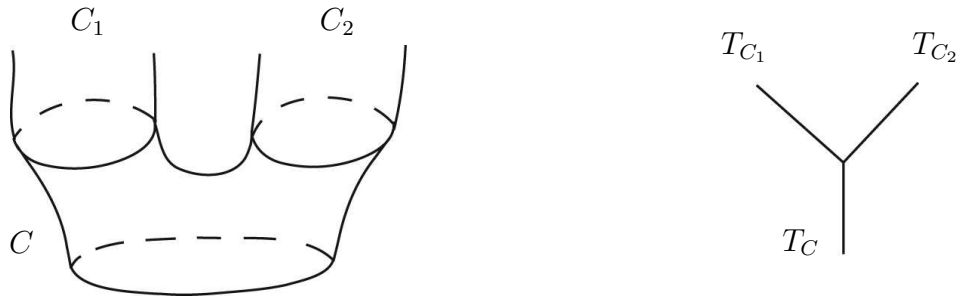


FIGURE 5. Raising the height: step 2.

*Proof.* The proof is inductive, moving from the handles down. If  $m$  is a meridian to a handle of  $C$  then  $M(v) = 1 + x$  and the statement is obviously true. Suppose the statement holds for the meridians to all body surfaces at height  $k + 1$ , and let  $m$  be a meridian to a surface  $P$  at height  $k$ . Note that the statement is independent of a choice of the meridians: if one of the meridians is replaced by a conjugate, the Magnus expansion still satisfies the condition. Denote, as usual,  $\partial P = \gamma \cup \alpha_1 \cup \dots \cup \alpha_n$ ; the surfaces at height  $k + 1$  are attached to  $P \times D^2$  along the links  $L_i$ ,  $L_i \subset \alpha_i \times D^2$ . For each  $i$ , the meridian  $m$  is conjugate to the curve  $\wedge_i$  (connected to the basepoint). Therefore for the inductive step it suffices to consider only step 1 of the height raising discussed above. In other words, one can assume that  $P$  is an annulus, and there is only one link  $L \subset \alpha \times D^2$ .

Consider the map  $\pi_1(\alpha \times D^2 \setminus L) \longrightarrow \pi_1(D^4 \setminus C)$ . The map is obtained by pushing  $\alpha \times D^2 \setminus$  (a thickening of  $L$ ) slightly into the complement of  $C$  in  $D^4$ . Let  $L = (l_1, \dots, l_n)$ ; denote the corresponding  $f$ -cells of height  $k - 1$  attached to them by  $C_1, \dots, C_n$  (see figure 4.) To distinguish them from the meridians to the handles of  $C$ , denote the meridians to the components of  $L$  in the solid torus by  $m'_1, \dots, m'_n$ , and let  $z_1, \dots, z_n$  be the corresponding variables for the Magnus expansion. Denote the longitude of the torus,  $\{p\} \times \partial D^2$ , by  $l$ , and the corresponding variable by  $y$ .

The meridians  $m'_j$  to the components of  $L$  may be viewed as meridians to the bottom surface stages of  $C_j$ . By the inductive assumption, there are preimages  $w_j$  of  $i_*(m'_j)$  in  $F_{\mathcal{M}}$  such that the Magnus expansion  $M(w_j)$ , composed with the projection to  $R[C]$ , is in  $S_{v_j} = 1 + \tilde{R}_{v_j} + \text{higher order terms}$ . In the following diagram, the map  $\phi$  between the free groups is defined on the generators by taking the preimage  $w_j$  of  $i_*(m'_j)$  in  $F_{\mathcal{M}}$ . Then  $\psi(z_j)$  is defined as  $M(\phi(m'_j)) - 1 = M(w_j) - 1$ .

$$\begin{array}{ccccc} \pi_1(\alpha \times D^2 \setminus L)/(\pi_1(\alpha \times D^2 \setminus L))^q & \longleftarrow & F_{m'_1, \dots, m'_n} & \xrightarrow{M'} & \mathbb{Z}\{z_1, \dots, z_n, y\} \\ & & \downarrow \phi & & \downarrow \psi \\ \pi_1(D^4 \setminus C)/(\pi_1(D^4 \setminus C))^q & \longleftarrow & F_{\mathcal{M}} & \xrightarrow{M} & \mathbb{Z}\{X\} \longrightarrow R[C] \\ & & \downarrow i_* & & \end{array}$$

Recall from the discussion preceding this lemma that

$$\tilde{R}_v \cong \oplus[\tilde{R}_{v_{i_1}} \otimes \dots \otimes \tilde{R}_{v_{i_n}}]$$

where the direct sum is taken over all permutations of  $\{1, \dots, n\}$ , and the inclusion  $\tilde{R}_{v_{i_1}} \otimes \dots \otimes \tilde{R}_{v_{i_n}} \rightarrow \tilde{R}_v$  is defined on the additive generators by multiplication of the monomials. The curve  $\wedge$  may be viewed as a meridian  $m$  to the bottom stage of  $C$ . Let  $w$  be a word representing  $\wedge$  in the free group. We will use the assumptions 3.7 on the links  $L$  in the definition of  $f$ -cells 4.1 (see also the accompanying discussion in section 3.6). In particular, every term with non-repeating variables in the expansion  $M'(w)$  contains each of the variables  $z_1, \dots, z_n$  (and in addition it may also contain  $y$ .) The expansion  $M(\phi(w))$  is obtained from  $M'(w)$  by replacing each  $z_i$  and  $y$  with  $\psi(z_i)$ ,  $\psi(y)$ . The proof is completed by the observation that

$$\psi(z_{i_1}) \dots \psi(z_{i_k}) \psi(y) \psi(z_{i_{k+1}}) \dots \psi(z_{i_n})$$

is an element of  $S_v$ , provided that for each  $j$ ,  $\psi(z_{i_j}) \in S_{v_{i_j}}$ . Note that we didn't use (or assert) the existence of an induced map of the ring with non-repeating variables  $R(z_1, \dots, z_n, y) \rightarrow R[C]$ . The expansion  $M'(w) \in \mathbb{Z}\{z_1, \dots, z_n, y\}$  may contain a *proper* subset of the variables  $\{z_1, \dots, z_n\}$ , provided that at least one of them, say  $z_i$ , is *repeated*. However by assumption  $\psi(z_i) \in S_{v_i}$ , so according to definition 7.3 every term of  $\psi(z_i)$  contains all of the variables associated to a subtree  $T_{v_i}^\alpha$ . Then to analyze  $\psi(z_i) \cdots \psi(z_i)$  consider the product of any two such terms. Either they correspond to the same tree  $T^\alpha$  and then the product contains repeated variables and so is trivial in  $R[C]$ , or they correspond to different subtrees  $T^\alpha, T^\beta$ , and then the product is again trivial in  $R[C]$ , by the second condition in definition 7.1.  $\square$

*Remark.* The statements in proposition 7.6 do not, in general, hold for any word  $w$  representing  $m$  in  $F_{\mathcal{M}}$ . For example, take a word  $w_0$ , given by the proposition, and assume the height of  $C$  is greater than 1. Another preimage of  $m_0$  in  $F_{\mathcal{M}}$  is given by  $w_0 \cdot$  (a relation of type  $(R_4)$ ). If the relation corresponds to a body surface at height  $> 1$  then the Magnus expansion of this element of  $F_{\mathcal{M}}$  is not in  $S_0$ .

7.7. Consider the Magnus expansion

$$\pi_1(D^4 \setminus C)/(\pi_1(D^4 \setminus C))^q \longleftarrow F_{\mathcal{M}} \xrightarrow{M} \mathbb{Z}[X] \longrightarrow R(C)$$

and define  $H_k(C) = \{g \in \pi_1(D^4 \setminus C)/(\pi_1(D^4 \setminus C))^q \mid \text{there exists } w \text{ representing } g \text{ in } F_{\mathcal{M}} \text{ with } M(w) \in S_k\}$ . These subgroups form a filtration

$$\pi_1(D^4 \setminus C)/(\pi_1(D^4 \setminus C))^q = G_{n+1}(C) \triangleright H_n(C) \triangleright \dots \triangleright H_1(C)$$

Lemma 7.6 implies that the subgroups  $G_k(C)$  of  $\pi_1(D^4 \setminus C)/(\pi_1(D^4 \setminus C))^q$ , normally generated by the meridians to the body surfaces of  $C$  at height  $k$  (see section 5.3), are (normal) subgroups of  $H_k(C)$ . Note that the analogous filtrations are defined for  $GM(C)$ .

To define invariants of  $f$ -cells in the next section, we need to fix a more specific subspace of  $\tilde{R}_v$ , for each  $v$ , containing precisely the monomials with non-trivial  $\bar{\mu}$ -invariants of the links  $\hat{L}$  in the definition of  $f$ -cells (see definition 4.1 and the discussion at the end of section 3.6.) The definition is similar to that of  $\tilde{R}_v$  but it involves only a specific order of the variables  $X$ .

**Definition 7.8.** Let  $v$  be a vertex of  $T_C$ . Consider the subtrees  $T_v^\alpha$  of  $T_C$  whose root is  $v$ , as in definition 7.3. Then  $Q_v$  is the additive subgroup of  $R[C]$  spanned by the monomials read off, clockwise, from the fixed planar embedding, defined in 4.3, of  $T_v^\alpha$ , for all  $\alpha$ . (Therefore  $Q_v \subset \tilde{R}_v$ .) Set  $Q_k(C) = \bigoplus_v Q_v \subset R[C]$ , where the summation is taken over all vertices  $v$  at height  $k$ . Also denote  $Q(C) = Q_0(C) = Q_r$  where  $r$  is the root of  $T_C$ .

In the example in figures 2, 3,  $Q(C)$  is spanned by the monomials  $x_1x_2x_5$ ,  $x_3x_4x_5$ . (Compare with the computation of  $\tilde{R}(C)$  in this example, following definition 7.3.)

We will also use an alternative, inductive, description of  $Q(C)$ , analogous to that of  $\tilde{R}(C)$  (see 7.5). For each leaf  $l$  of  $T_C$ , the corresponding  $Q_l$  is the subgroup ( $\cong \mathbb{Z}$ ) of  $R[C]$  spanned by  $x_l$ . Suppose  $Q_v$  is defined for vertices of  $T_C$  at height  $> k$ , and let  $v$  be an (unmarked) vertex at height  $k$ . Moving down the  $f$ -cell, from height  $k+1$  to height  $k$  may be decomposed into steps, figures 4, 5. The first step (corresponding to  $P = \text{annulus}$ ) gives  $Q_v \cong Q_{v_1} \otimes Q_{v_2}$ . The second step (figure 5)

gives  $Q_v \cong Q_{v_1} \oplus Q_{v_2}$ . To combine these two steps, denote  $\partial P = \gamma \cup \alpha_1 \cup \dots \cup \alpha_n$ ; surfaces at height  $k + 1$  are attached along the links  $L_i \subset \alpha_i \times D^2$ . Let  $I_i$  be the (ordered) index set for the components of  $L_i$ . Then

$$(7.2) \quad Q_v = \bigoplus_i \bigotimes_{j \in I_i} Q_{v_j}$$

*Remark.* The structure of  $Q(C)$  may be read off from the tree  $T_C$  associated to  $C$ : the “generators” correspond to the leaves of  $T_C$ ; then form a tensor product for each marked vertex of the tree and a direct sum for each unmarked vertex.

**7.9. The ring structure.** For each  $v$ ,  $S_v$  is a subring of  $R[C]$ . Consider  $1 + \tilde{R}_v$  as the quotient of  $S_v$  by the ideal generated by the higher order terms (see definition 7.4), and let  $p_1: S_v \rightarrow 1 + \tilde{R}_v$  denote the projection. Similarly,  $1 + Q_v$  is the quotient of  $1 + \tilde{R}_v$  by the ideal generated by all monomials which do not respect the fixed order of the variables,  $p_2: 1 + \tilde{R}_v \rightarrow 1 + Q_v$ . The product in  $1 + \tilde{R}_v$ ,  $1 + Q_v$  is given by

$$(1 + \sum_I \alpha_I X_I)(1 + \sum_I \beta_I X_I) = 1 + \sum_I (\alpha_I + \beta_I) X_I.$$

Let  $m$  be a meridian to the bottom stage of  $C$ , then by proposition 7.6 there exists a word  $w$  representing it in the free group whose Magnus expansion  $M(w)$  is an element of  $S(C)$ . Consider its image in  $1 + Q(C)$ :

$$(7.3) \quad p_2(p_1(M(w))) = 1 + \sum_I \alpha_i X_I$$

where the summation is over all subtrees with a prescribed planar embedding, as discussed above. The coefficients  $\alpha_I$  are well-defined with respect to the relations  $(R_1) - (R_3)$ . (That is, with respect to multiplying  $w$  by a conjugate of one of the relations  $(R_1) - (R_3)$ .) The next section introduces an invariant well-defined with respect to  $(R_4)$  as well.

## 8. AN INVARIANT OF FLEXIBLE CELLS.

The main purpose of this section is to prove the following statement.

**Lemma 8.1.** *Let  $K$  be a knot in  $S^3$ , and let  $(C, \gamma)$  be an  $f$ -cell. Then there is a homomorphism  $\phi: \pi_1(S^3 \setminus K) \rightarrow \mathbb{Z}$  with the following properties.*

1. *Let  $m$  be a meridian to  $K$  in  $S^3$ . Then  $\phi(m) \neq 0$ .*
2. *If  $f$  is a realization  $(C, \gamma) \rightarrow (D^4, S^3)$  with  $f(\gamma) = K$  then for any  $g \in \pi_1(S^3 \setminus K)$ ,  $\phi(g) \neq 0$  implies  $g \neq 1 \in \pi_1(D^4 \setminus C)/(\pi_1(D^4 \setminus C))^q$ .*

*Remark.* Since the homomorphism  $\phi$  factors through  $\pi_1(S^3 \setminus K)/(\pi_1(S^3 \setminus K))^q \cong \mathbb{Z}$ ,  $\phi$  is a multiple of the linking number, and the lemma may be rephrased as saying that if  $g$  has a non-trivial linking number with  $\gamma$  then  $g \neq 1 \in \pi_1(D^4 \setminus C)/(\pi_1(D^4 \setminus C))^q$ . The main content is in the proof of the lemma, which is generalized from knots to the setting of links in section 9.

**8.2. Notation.** Given  $g \in \pi_1(S^3 \setminus \gamma)/\pi_1(S^3 \setminus \gamma)^q$ , according to lemma 7.6 there is a word  $w$  representing it in the free group  $F_{\mathcal{M}}$  whose Magnus expansion  $M(w)$  is an element of the subring  $S(C)$  of  $R(C)$ . Denote by  $\overline{M}(w)$  the image of  $M(w)$  under the projection  $S(C) \rightarrow 1 + Q(C)$ , so  $\overline{M}(w) = p_2(p_1(M(w)))$  using the notation of (7.3).

**8.3. Definition of  $\Phi$  in the height = 1 case.** First consider the special case when the first stage planar surface  $P$  is a pair of pants,  $\partial P = \gamma \cup \alpha_1 \cup \alpha_2$ . We will use the notations introduced in the beginning of section 5.6, and we'll use the Magnus expansion (5.1). In particular, the set  $X$  of the variables corresponding to the meridians to the handles of  $C$  in  $D^4$  is divided into two subsets  $X_{I_1}, X_{I_2}$ , where the indices reflect the components of the links  $L_i \subset \alpha_i \times D^2$  that the handles are attached to.

Let  $Y_i$  be a monomial with non-repeating variables of maximal length in the variables  $X_{I_i}$ ,  $i = 1, 2$ , respecting the preferred order (see 4.3). Note that  $Q(C)$  in this case is 2-dimensional, spanned by the monomials  $Y_1, Y_2$ . Denoting by  $W_i$  a word representing the curve  $\wedge_i$  in the free group, given by the commutative diagram 5.2, note that  $\overline{M}(W_i) = \mu_i Y_i$ , where  $\mu_i \neq 0$ ,  $i = 1, 2$ .

**Proposition 8.4.** *Given an element  $g \in \pi_1(S^3 \setminus L)/\pi_1(S^3 \setminus L)^q$ , let  $w$  be a word representing it as in 8.2, and consider its expansion in  $1 + Q(C)$ :*

$$\overline{M}(w) = 1 + \alpha_1 Y_1 + \alpha_2 Y_2$$

for some  $\alpha_1, \alpha_2$ . Then  $\Phi(g) := \mu_2 \alpha_1 + \mu_1 \alpha_2$  is an invariant of  $g$ .

*Proof.* The coefficients  $\alpha_i$  are well-defined with respect to the relations  $(R_1) - (R_3)$ , see the discussion following equation (7.3). The relation  $(R_4)$  is given by  $W_1(W_2)^{-1}$ , and its expansion is of the form

$$\overline{M}(W_1(W_2)^{-1}) = 1 + \mu_1 Y_1 - \mu_2 Y_2.$$

Let  $w'$  be  $w$  multiplied by a conjugate of  $W_1(W_2)^{-1}$ ,  $\overline{M}(w') = 1 + \alpha'_1 Y_1 + \alpha'_2 Y_2$ . Then  $\alpha'_1 = \alpha_1 + \mu_1$ ,  $\alpha'_2 = \alpha_2 - \mu_2$ . Therefore  $\phi(w') = \phi(w)$ .  $\square$

Consider the *general height 1 case*:  $\partial P = \gamma \cup \alpha_1 \cup \dots \cup \alpha_k$ . As above, let  $Y_j$  be the preferred monomial in the variables  $X_{I_j}$ , and  $\overline{M}(W_j) = 1 + \mu_j Y_j$ ,  $\mu_j \neq 0$ ,  $j = 1, \dots, k$ . Define  $\mu'_j = \prod_{i \neq j} \mu_i$ . The proof of the following statement is a direct generalization of the proof in the pair of pants case.

**Proposition 8.5.** *Given an element  $g \in \pi_1(S^3 \setminus L)/\pi_1(S^3 \setminus L)^q$ , as in proposition 8.4 consider the expansion in  $1 + Q(C)$ :  $\overline{M}(w) = 1 + \sum_j \alpha_j Y_j$ . Then  $\Phi(g) := \sum_j \alpha_j \mu'_j$  is an invariant of  $g$ .*

*Remark.* There is, in fact, a collection of  $I_1! \cdots I_k!$  invariants  $\Phi$ , parametrized by the monomials in non-repeating variables  $X_{I_1}, \dots, X_{I_k}$ . We chose a specific  $\Phi$ , reflecting the non-trivial  $\bar{\mu}$ -invariants of the homotopically essential links  $\widehat{L}_j$ .

8.6. *Definition of the invariant  $\Phi$  in the general case.* The definition is inductive. Suppose the homomorphism  $\Phi: (Q(C), +) \rightarrow (\mathbb{Z}, +)$  is defined for  $f$ -cells of height  $< h$ , and let  $(C, \gamma)$  be an  $f$ -cell of height  $h$ .  $C$  is obtained from  $\overline{P} = P \times D^2$  by attaching  $f$ -cells  $\{C_j\}_{j \in I_i}$  of height  $h - 1$  to the components of links  $L_i$ ,  $L_i \subset \alpha_i \times D^2$ . Here  $I_i$  is the (ordered) index set for the components of  $L_i$ . As above, let  $\mu_i$  be the non-trivial  $\bar{\mu}$ -invariant of  $\widehat{L}_i$  in the expansion of  $\wedge_i$ , with the given order of the components of  $L_i$ . Let  $\Phi_j: Q(C_j) \rightarrow \mathbb{Z}$  denote the inductively defined invariant of  $C_j$ . Recall from (7.2) that

$$Q(C) := \bigoplus_i \bigotimes_{j \in I_i} Q(C_j).$$

Denoting  $\mu'_j = \prod_{i \neq j} \mu_i$ , define

$$\Phi: Q(C) \rightarrow \mathbb{Z} \text{ by } \Phi = \sum_i \mu'_i (\bigotimes_{j \in I_i} \Phi_j).$$

**Proposition 8.7.** *Given  $g \in \pi_1(S^3 \setminus K)/(\pi_1(S^3 \setminus K))^q$ , let  $w$  be a word representing it in the free group, as in 8.2. Then  $\Phi(\overline{M}(w))$  is well-defined, and will be denoted  $\phi(g)$ .*

*Proof.* The proof is inductive. The statement is true for  $f$ -cells of height 1 by proposition 8.5. Suppose the statement is true for  $f$ -cells of height  $< h$ , and let  $C$  be an  $f$ -cell of height  $h$ . Assembling  $C$  from  $f$ -cells of height  $h - 1$  will be separated into two steps: (1) attaching them to a link in a solid torus, and (2) attaching the results of step (1) to a (planar surface)  $\times D^2$ , see section 7.5 and figures 4, 5 in section 7.

*Step (1).* Consider an  $f$ -cell  $C$  of height  $h$  such that  $C = S^1 \times D^2 \times I \cup (C_1 \cup C_2)$  where the  $f$ -cells  $C_i$  have height  $h - 1$  and are attached along the components of

a link  $L = (l_1, l_2) \subset S^1 \times D^2 \times \{1\}$ . For simplicity of notation, we assume  $L$  has two components; the proof for a larger number of components goes through without any changes. Note that since the bottom stage surface of  $C$  is the annulus, there are no relations  $(R_4)$  at height 1. Given a relation  $r$  of type  $(R_4)$ , let  $I$  denote the ideal in  $R[C]$  generated by the Magnus expansion  $M(W) - 1$ , where  $W$  a word representing it. It suffices to prove that the intersection  $I \cap Q(C)$  is in the kernel of  $\Phi: Q(C) \rightarrow \mathbb{Z}$ . The representation  $Q = Q(C)$  decomposes as  $Q_1 \otimes Q_2$  where  $Q_i = Q(C_i)$ , and  $\Phi = \Phi_1 \otimes \Phi_2: Q \rightarrow \mathbb{Z}$ , so

$$\ker \Phi = (\ker \Phi_1) \otimes Q_2 + Q_1 \otimes (\ker \Phi_2).$$

Consider a relation  $r$  of type  $(R_4)$ . Since there are no relations of this type associated to the bottom stage surface,  $r$  corresponds to a body surface in either  $C_1$  or  $C_2$ , say in  $C_1$ .

First we impose an *additional assumption* that, in the context of definition 3.7, for each link  $L$  defining the  $f$ -cell  $C$  there is a word  $W$  representing  $\wedge$  in the free group such that  $W$  involves only the variables  $m_1, \dots, m_n$ , and not the longitude  $l$  of the solid torus. For example, this assumption is satisfied in the case  $L = (\text{iterated})$  Bing double. After giving a proof in this restricted setting, we show how the argument goes through in the general case. The assumption above implies that each relation  $r$  of type  $(R_4)$  has a word representing it in the free group, whose Magnus expansion is an element of either  $R[C_1]$  or  $R[C_2]$ .

Let  $r \in R[C_1] \subset R[C]$  be a relation, and denote by  $I_1$  and  $I$  the ideals generated by  $r$  in  $R[C_1]$ ,  $R[C]$  respectively. Observe that  $I \cap Q(C) = I \cap (Q(C_1) \otimes Q(C_2)) = (I \cap Q(C_1)) \otimes Q(C_2)$ . Since  $I_1 \subset \ker \Phi_1$ ,  $I \cap Q(C) \subset \ker \Phi$ , and the proof is complete.

Now consider the general case, i.e. we remove the extra assumption imposed in the paragraph above. The difference with that case is that even though  $r$  is a relation given by a surface in  $C_1$ , one cannot assume that  $r$  is an element of the subring  $R[C_1]$  of  $R[C]$ . However (see end of section 3.6)  $\wedge$  has a word representing it whose expansion is of the form  $1 + x_{i_1} \cdots x_{i_n} + \text{higher order terms}$ . That is, all *first non-vanishing terms* with non-repeating variables in its Magnus expansion are elements of  $R[C_1]$ . The proof is completed by the observation that only first non-vanishing terms contribute to  $I \cap Q(C)$ .

*Step (2)*, see figure 5. Now  $C$  equals  $\overline{P} = (P \times D^2) \cup C_1 \cup C_2$ , where the  $f$ -cells  $C_i$  have height  $h$  and whose bottom stage surfaces are annuli. For simplicity of notation we assume  $P$  is a pair of pants; the case of a planar surface with more boundary components is treated analogously. Denoting  $\partial P = \gamma \cup \alpha_1 \cup \alpha_2$ ,  $C_i$  is

attached along  $\alpha_i \times D_2$ ,  $i = 1, 2$ . In this case

$$R[C] \cong R[C_1] \oplus R[C_2], \quad Q(C) \cong Q(C_1) \oplus Q(C_2).$$

As above, given a relation  $r$  of type  $(R_4)$ , we need to show

$$I \cap Q(C) \subset \ker(\Phi: Q(C) \longrightarrow \mathbb{Z}).$$

We have  $\Phi = \mu_2\Phi_1 \oplus \mu_1\Phi_2$ . There are two cases to consider:  $r$  corresponds to a surface in  $C$  at height  $> 1$ , or it is a new relation corresponding to  $P$ . In the first case, one may assume  $r \in R[C_1]$ . Denote by  $I_1, I$  the ideals generated by  $r$  in  $R[C_1], R[C]$ . Then since  $R[C]$  is a direct sum of rings  $R[C_1] \oplus R[C_2]$ ,  $I = I_1 \subset R[C_1] \subset R[C]$ . Clearly then  $I \subset \ker(\Phi)$ .

Consider the second case:  $r$  is a new relation, corresponding to the bottom stage surface  $P$  of  $C$ . Consider  $\overline{P} \setminus (L_1 \cup L_2)$ . Denote the meridians to  $L_1$  by  $m'_1, \dots, m'_k$  and the meridians to  $L_2$  by  $m''_1, \dots, m''_l$ ; let  $\{x'_i\}, \{x''_j\}$  be the corresponding variables. Then  $r$  is of the form

$$r = 1 + \mu_1 x'_1 \cdots x'_k - \mu_2 x''_1 \cdots x''_l + \text{higher order terms.}$$

Consider the image of  $r$  in  $R[C]$ . Note that the first term  $\mu_1 x'_1 \cdots x'_k$  is in  $R[C_1]$ , the second term  $\mu_2 x''_1 \cdots x''_l$  is in  $R[C_2]$ , and in fact all higher order terms vanish in  $R[C]$ , since the first non-vanishing terms already have maximal length. Any element of  $R[C]$  of the form  $\mu_1 Y + \mu_2 Z$ , where  $Y \in R[C_1], Z \in R[C_2]$ , is in the kernel of  $\Phi$ . Therefore  $r \in \ker(\Phi)$ , and any other element in the ideal generated by  $r$  is longer and vanishes in  $R[C]$  (so in fact  $I = \{r\} \subset \ker(\Phi)$ .) This concludes the proof of proposition 8.7.  $\square$

Proposition 8.7 constructs a homomorphism  $\phi: \pi_1(S^3 \setminus K) \longrightarrow \mathbb{Z}$ , defined using the Magnus expansion  $GM(C) \longleftarrow F_{\mathcal{M}} \longrightarrow R[C]$ . In particular,  $\phi(g)$  is well-defined with respect to multiplication by an element of the relation subgroup in  $F_{\mathcal{M}}$ , so  $\phi(g) \neq 0$  implies  $g \neq 1 \in \pi_1(D^4 \setminus C)/(\pi_1(D^4 \setminus C))^q$ . Part (1) of lemma 8.1 is proved by inspection: at each surface stage  $P$  of  $C$ ,  $\partial P = \gamma_P \cup_i \alpha_i$ , the meridian to  $P$  is conjugate to the  $\wedge$ -curve corresponding to the solid torus  $\alpha_i \times D^2$ , for any given  $i$ . Applying the analysis at the end of section 3.6 inductively to the meridians to the surface stages of  $C$ , moving up from the meridian  $m$  to the bottom stage, one observes that there is a word  $w$  representing  $m$  in  $F_{\mathcal{M}}$  such that  $\overline{M}(w)$  is a generating monomial for  $Q(C)$ . Due to the tensor decompositions of  $Q(C)$  and  $\Phi$ ,  $\phi(m) = \Phi(\overline{M}(w)) \neq 0$ . This finishes the proof of lemma 8.1.  $\square$

## 9. APPLICATIONS TO LINK HOMOTOPY

In this section we show how the theory of flexible cells fits in the framework of Milnor's theory of link homotopy. We generalize the invariant  $\Phi$  defined in the previous section to a collection of  $f$ -cells to prove the following theorem. This result serves as a basis for the applications to the A-B slice problem [5].

**Theorem 9.1.** *If the components of a link  $L \subset S^3 = \partial D^4$  bound disjoint  $f$ -cells in  $D^4$  then  $L$  is homotopically trivial.*

The main new ingredient developed here is the invariant  $\Phi(C)$  of a collection of  $f$ -cells  $C$ , introduced in the course of the proof of theorem 9.1. This is a generalization of the invariant  $\phi$  of a single  $f$ -cell, defined in section 8.1. The crucial feature of  $\Phi$  is that it is *independent of an embedding*  $(C, \gamma) \hookrightarrow (D^4, \partial D^4)$  – a property important for applications in the A-B slice problem.

**Remark.** This theorem is a substantially stronger statement than the Link composition lemma [2], [6], see theorem 3.5 above. Recall the statement of that result: if both links  $\widehat{L}$  and  $\widehat{Q} = (l_1, \dots, l_{k+1})$  are homotopically essential in  $S^3$  then so is their composition  $L \cup \phi(Q)$  (where the map  $\phi$  “substitutes” the link  $Q$  in a neighborhood of the component  $l_{k+1}$ .) This follows from an application of theorem 9.1 to a simple example of an  $f$ -cell of height one: Suppose the link  $L \cup \phi(Q)$  is homotopically trivial, so its components bound disjoint maps of disks in  $D^4$ . Attach a collar  $S^1 \times D^2 \times I$  to a neighborhood  $S^1 \times D^2$  of  $l_{k+1}$ . Pushing the disks bounded by  $\phi(Q)$  into  $D^4$  slightly, so they are attached to the collar, one observes that the link  $\widehat{L}$  bounds disjoint maps of  $f$ -cells in  $D^4$ : the component  $l_{k+1}$  bounds an  $f$ -cell of height 1, and the components  $l_1, \dots, l_k$  bound maps of disks (which of course satisfy the definition of an  $f$ -cell of any height.) By theorem 9.1  $\widehat{L}$  is trivial. This contradiction concludes the proof of the link composition lemma.

*Proof of theorem 9.1.* Let  $L = (l_1, \dots, l_n)$  and suppose the components of  $L$  bound disjoint  $f$ -cells in  $D^4$ ,  $l_i = \partial C_i$ . Denote  $C = \cup_i C_i$ . Suppose  $L$  is homotopically essential, and without loss of generality one may assume  $L$  is almost homotopically trivial, so there is a well-defined and non-trivial  $\mu$ -invariant with non-repeating coefficients of length  $n$ . Order the components of  $L$  so that  $\mu_{1\dots n}(L) \neq 0$ .

The results of sections 5, 6, 7 and 8 generalize from the setting of a single  $f$ -cell as follows. Let  $\mathcal{M}_i$  denote a set of meridians to the handles of  $C_i$ . By Alexander duality  $H_1(D^4 \setminus C)$  is generated by  $\mathcal{M} = \cup_i \mathcal{M}_i$ . Denote the corresponding variables for the Magnus expansion by  $X_i$ ,  $X = \cup X_i$ . Again by Alexander duality, the

relations in  $\pi_1(D^4 \setminus C)/(\pi_1(D^4 \setminus C))^q$  are all of types  $(R_1) - (R_4)$  (see section 5.6), contributed by the  $f$ -cells  $C_i$ . Each relation of type  $(R_1) - (R_3)$  involves only variables in a single set  $\mathcal{M}_i$  (depending on the relation). The assumptions (see 3.6) on the links defining the  $f$ -cells imply that all first non-vanishing terms in the Magnus expansion of any relation of type  $(R_4)$  also involve the variables in a single  $X_i$ . Variables from other sets  $X_j$  may be present, but only in higher-order terms.

Define  $GM(C)$  as the free group  $F_{\mathcal{M}}$  modulo relations (6.1), where all of the meridians  $m, m_1, m_2$  involved in the commutators in (6.1) are elements of the same  $\mathcal{M}_i$ , for any given  $1 \leq i \leq n$ . Define  $R[C]$  as the quotient of  $\mathbb{Z}\{X\}$  by the ideal introduced in definition 7.1 where the variables  $x_I, x_{I'}$  are elements of  $X_i$  for the same  $i$ . Consider the Magnus expansion

$$(9.1) \quad GM(C) \longleftarrow GM(F_{\mathcal{M}}) \longleftarrow F_{\mathcal{M}} \xrightarrow{M} \mathbb{Z}\{X\} \longrightarrow R[C]$$

Following definitions 7.3, 7.4, 7.8, introduce  $\tilde{R}(C)$ ,  $S(C) = 1 + \tilde{R}(C)$ . Define  $Q(C)$  using the order on the components of  $L$  reflecting a non-trivial  $\mu$ -invariant (see above):

$$Q(C) = Q(C_1) \otimes \dots \otimes Q(C_n).$$

The proof of lemma 7.6 goes through, in particular given any element  $g \in \pi_1(S^3 \setminus L)/(\pi_1(S^3 \setminus L))^q$ , there is a word  $w_0$  representing it in  $F_{\mathcal{M}}$  such that  $M(w_0) \in S(C)$ . Denote by  $\overline{M}$  the composition of the Magnus expansion  $M$  with the projection  $S(C) \longrightarrow 1 + Q(C)$ . Denoting by  $\Phi_i$  the homomorphism  $Q(C_i) \longrightarrow \mathbb{Z}$  defined in 8.6, consider

$$\Phi = \otimes_i \Phi_i: Q(C) = \otimes_i Q(C_i) \longrightarrow \mathbb{Z}.$$

Given  $g \in \pi_1(S^3 \setminus L)/(\pi_1(S^3 \setminus L))^q$ ,  $\Phi(\overline{M}(w_0))$  is a well-defined integer. Moreover, if  $\Phi(\overline{M}(g)) \neq 0$ , then  $i_*(g) \neq 1 \in \pi_1(D^4 \setminus C)/(\pi_1(D^4 \setminus C))^q$ . Consider the commutative diagram

$$\begin{array}{ccc} \pi_1(S^3 \setminus L)/(\pi_1(S^3 \setminus L))^q & \xrightarrow{i_*} & \pi_1(D^4 \setminus C)/(\pi_1(D^4 \setminus C))^q \\ p_1 \uparrow & & p_2 \uparrow \\ F_{m_1, \dots, m_n} & \xrightarrow{\alpha} & F_{\mathcal{M}} = F_{\mathcal{M}_1, \dots, \mathcal{M}_n} \\ M_1 \downarrow & & M \downarrow \\ \mathbb{Z}\{x_1, \dots, x_n\} & \xrightarrow{\beta} & \mathbb{Z}\{X\} = \mathbb{Z}\{X_1, \dots, X_n\} \end{array}$$

Recall from the proof of lemma 8.1 at the end of section 8 that each meridian  $m_i$  has a word  $w_i$  representing it in  $F_{\mathcal{M}}$  such that  $\overline{M}(w_i)$  is a generating monomial

for  $Q(C_i)$ , and  $\Phi_i(\overline{M}(w_i)) \neq 0$ . In the diagram above  $\alpha$  is defined by setting  $\alpha(m_i) = w_i$ . Then  $\beta$  is given by  $\beta(1 + x_j) = M(\alpha(m_j))$ .

Since  $L$  is homotopically essential, there is a relation  $[m_i, l_i]$  in  $\pi_1(S^3 \setminus L)/(\pi_1(S^3 \setminus L))^q$  such that the Magnus expansion  $M_1$  of a word  $W$  representing it in  $F_{m_1, \dots, m_n}$  is of the form  $1 + \mu x_1 \cdots x_n + \dots$  where  $\mu \neq 0$ . However the projection of  $\beta(x_1 \cdots x_n)$  onto  $Q(C)$  is a product of generating monomials, one for each  $Q(C_i)$ , and it follows from the definition of  $\Phi$  that  $\Phi(\alpha(W)) \neq 0$ . Since  $\Phi(\overline{M}(w_0))$  is an invariant of  $g \in \pi_1(S^3 \setminus L)/(\pi_1(S^3 \setminus L))^q$ , where  $p_1(w_0) = g$ ,  $p_1(W) \neq 1 \in \pi_1(S^3 \setminus L)/(\pi_1(S^3 \setminus L))^q$ . But  $p_1(W) = [m_i, l_i]$  is a relation in that group. This contradiction concludes the proof of theorem 9.1.  $\square$

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