GRAPHS, LINKS, AND DUALITY ON SURFACES

VYACHESLAV KRUSHKAL

ABSTRACT. We introduce a polynomial invariant of graphs on surfaces, P_G, generalizing the classical Tutte polynomial. Poincaré duality on surfaces gives rise to a natural duality result for P_G, analogous to the duality for the Tutte polynomial of planar graphs. This property is important from the perspective of statistical mechanics, where Tutte polynomial is known as the partition function of the Potts model. For ribbon graphs, P_G specializes to the well-known Bollobás-Riordan polynomial, and in fact the two polynomials carry equivalent information in this context. Duality is also established for a multivariate version of the polynomial P_G. We then consider a $2-$variable version of the Jones polynomial for links in thickened surfaces, taking into account homological information on the surface. An analogue of Thistlethwaite's theorem is established for these generalized Jones and Tutte polynomials for virtual links.

1. INTRODUCTION

Tutte polynomial $T_G(X, Y)$ is a classical invariant in graph theory (see [29, 30, 1]), reflecting many important combinatorial properties of a graph G. For example, the chromatic polynomial, whose values at positive integer values of the parameter Q correspond to the number of colorings of G with Q colors, is a one-variable specialiazation of T_G. Tutte polynomial is also important in statistical mechanics, where it arises as the partition function of the Potts model, cf [26].

Two properties of Tutte polynomial are particularly important in these contexts: the contraction-deletion rule, and the duality

$$T_G(X, Y) = T_{G^*}(Y, X) \tag{1.1}$$

where G is a planar graph, and G^* is its dual. (The vertices of G^* correspond to the connected regions in the complement of G in the plane, and two vertices are connected by an edge in G^* whenever the two corresponding regions are adjacent.)

In this paper we introduce a $4-$variable polynomial, $P_{G,\Sigma}(X, Y, A, B)$, which is an invariant of a graph G embedded in a closed orientable surface Σ, which satisfies both the contraction-deletion rule and a duality relation analogous to (1.1). The variables X, Y play the same role as in the definition of the Tutte polynomial, while the additional variables A, B reflect the homological information of G in Σ. It follows

Supported in part by NSF grants DMS-0605280 and DMS-0729032.

that the Tutte polynomial is a specialization of P_G, where this extra information, reflecting the embedding $G \subset \Sigma$, is disregarded.

The main motivation for this work came from an attempt to understand the combinatorial structure underlying the Potts model on surfaces. As mentioned above, the partition function for the Potts model on the plane is given by the Tutte polynomial, while on surfaces essential loops are weighted differently from trivial loops (for references on the Potts model on surfaces, cf. [6, 8, 15, 24].) This leads to the introduction of additional variables, keeping track of the topological information of graphs on surfaces.

The intersection pairing and the Poincaré duality on an oriented closed surface Σ give rise to a symplectic structure on the first homology group $H_1(\Sigma)$. (The action of the mapping class group of the surface induces a representation of the symplectic group $Sp\,(2g, \mathbb{Z})$ on $H_1(\Sigma, \mathbb{Z})$, where g is the genus of the surface.) Given a subgroup V of $H_1(\Sigma)$, its "orthogonal complement" $V^{\perp}$ with respect to the intersection form is defined. Using this structure, one proves the duality relation

$$P_G\,(X, Y, A, B) \;=\; P_{G^*}\,(Y, X, B, A), \tag{1.2}$$

which may be viewed as a natural analogue of the duality (1.1) of the Tutte polynomial for planar graphs. For the dual graph G^* in (1.2) to be well-defined, it is natural to consider graphs G which are *cellulations* of Σ, that is, graphs such that each component of $\Sigma \smallsetminus G$ is a disk. Equivalently, such graphs may be viewed as orientable *ribbon graphs*, this point of view is presented in more detail in section 5.

For such graphs, there is a well-known $3-$variable polynomial defined by B. Bollobás and O. Riordan [2, 3]. We denote this graph by $BR_G(X, Y, Z)$ (its construction is recalled in section 5.) We show that this polynomial can be obtained as a specialization of P_G:

$$BR_G(X, Y, Z) \;=\; Y^g\, P_G(X-1, Y, YZ^2, Y^{-1}), \tag{1.3}$$

where g is the genus of the ribbon graph G. In fact the authors prove in [2, 3] that their polynomial is a universal invariant of ribbon graphs with respect to the contraction-deletion rule (we give a precise statement of this result in section 5.) Therefore in principle the two polynomials BR_G, P_G carry equivalent information about the ribbon graph G, although an expression of P_G in terms of BR_G does not seem to be as straightforward as (1.3). We note that the definition of the polynomial P_G could be normalized so the specialization to BR_G is obtained by simply setting one of the variables equal to 1 (see section 5). We chose a normalization making the duality statement (1.2) most natural.

Several authors have studied partial duality for the Bollobas-Riordan polynomial: [2] stated duality for a $1-$variable specialization, [10, 22] (see also [4, 11, 23]) proved duality for a $2-$variable specialization. These results may be recovered as a consequence of equations (1.2), (1.3), see sections 4 and 5.4; our result (1.2) is more general.

Note that a more general version of the polynomial P_G may be defined, with coefficients corresponding to subgroups of $H_1(\Sigma)$, see section 3. This polynomial also satisfies duality as a consequence of Poincaré duality on Σ. This more general polynomial may be used to distinguish different embeddings of a graph in Σ. (One may also generalize further and, avoiding the use of homology, consider *Tutte skein module* of a surface Σ: the vector space spanned by isotopy classes of graphs on Σ, modulo the contraction-deletion relation, see section 3.3. In this case the "polynomial" associated to a graph $G \subset \Sigma$ is the element of the skein module represented by G.) On the other hand, if one considers graphs on Σ up to the action of the diffeomorphism group of Σ (or if one studies ribbon graphs), then the relevant invariant is the finite-variable polynomial P_G, discussed above.

A *multivariate* version of the Tutte polynomial, where the edges of a graph are weighted, is important in the analysis of the Potts model [26]. We define its generalization, a multivariate version of the polynomial P_G, and establish a duality analogous to (1.2) in section 7. (A mutlivariate version of the Bollobás-Riordan polynomial, and its partial duality have been considered in [31].)

In section 6 a version of the Kauffman bracket and of the Jones polynomial on surfaces is considered, taking into account homological information on the surface. In particular, using the interpretation of a virtual link as an "irreducible" embedding of a link into a surface [20], this defines a generalization of the Jones polynomial for virtual links. For example, the Jones polynomial $J_L(t, Z)$ acquires a new variable Z which, in the state-sum expression, keeps track of the rank of the subgroup of the first homology group $H_1(\Sigma)$ of the surface represented by a resolution of the link diagram on the surface.

If a link L has an alternating diagram on Σ, the diagram may be checkerboard colored, and there is a graph G (Tait graph) associated to it. In this context we show (Theorem 6.1) that the generalized Kauffman bracket (and the Jones polynomial $J_L(q, Z)$) is a specialization of the polynomial P_G, generalizing the well-known relation between the Jones polynomial of a link in $3-$space and the Tutte polynomial associated to its planar projection due to Thistlethwaite [27]. The analogue of Thistlethwaite's relation for the classical Kauffman bracket of virtual links was proved in [5]. Theorem 6.1 generalizes these results to the polynomial J_L with the extra homological parameter Z. This relation between the generalized Jones polynomial $J_L(q, Z)$ and the polynomial P_G of the associated graph does not seem to have an immediately obvious analogue in terms of BR_G.

Section 2 recalls basic notions of symplectic linear algebra, which provides a convenient setting for the definition of P_G. The Tutte polynomial and the definition of the polynomial P_G, as well as a discussion of its basic properties, are given in section 3. Section 4 establishes the duality relation (1.2). We review the notion of a ribbon graph and the definition of the Bollobás-Riordan polynomial, and we establish the relation (1.3) in section 5. Section 5.4 shows that our duality result (1.2) implies the previously known partial duality relations for the Bollobás-Riordan polynomial. Section 6 defines the relevant versions of the Jones polynomial and of the Kauffman bracket and establishes a relationship between them and the polynomial P_G. Finally, section 7 discusses a multivariate version of the polynomial P_G and the corresponding duality relation.

Acknowledgements. This work is related to an ongoing project with Paul Fendley [12], [13] relating TQFTs, graph polynomials, and algebraic and combinatorial properties of models of statistical mechanics. I would like to thank Paul for many discussions that motivated the results in this paper.

2. Symplectic linear algebra

This section recalls a number of basic facts and introduces certain notation in the symplectic linear algebra setting which will be useful for the definition of the graph polynomial in section 3. Let Σ be a (not necessarily connected) orientable surface, and consider the intersection pairing

$$w\colon\; H_1(\Sigma,\mathbb{R})\times H_1(\Sigma,\mathbb{R})\longrightarrow \mathbb{R}.$$

The invariants considered below will not depend on the orientation. Poincaré duality implies that the bilinear form w is non-degenerate, in other words it is a symplectic form on the vector space $H_1(\Sigma,\mathbb{R})$. A note on the homology coefficients: the invariants below may be defined using either $\mathbb{Z}$ or $\mathbb{R}$, and these coefficients will be used interchangeably.

Let H be a graph embedded in the surface Σ, and let $i\colon H\hookrightarrow\Sigma$ denote the embedding. Denote

$$V=V(H)=\text{image}\,(i_*\colon\; H_1(H;\mathbb{R})\longrightarrow H_1(\Sigma;\mathbb{R})).$$

The "symplectic orthogonal complement" of V is defined by

$$V^{\perp}=V^{\perp}(H)\;=\;\{u\in H_1(\Sigma,\mathbb{R})|\,\forall v\in V(H),\,w(u,v)=0\}.$$

Consider the invariants $s(H)$, $s^{\perp}(H)$ of a graph H on Σ:

$$s(H):=\text{rank}(V/(V\cap V^{\perp})),\;\; s^{\perp}(H):=\text{rank}(V^{\perp}/(V\cap V^{\perp})). \tag{2.1}$$

Said differently, $s(H)$ is the dimension of a maximal symplectic subspace of V (with respect to the symplectic form w on $H_1(\Sigma,\mathbb{R})$), and similarly $s^{\perp}(H)$ is the dimension of a maximal symplectic subspace in $V^{\perp}$. Moreover, $s(H)$ is twice the genus of the surface obtained as a regular neighborhood of the graph H in Σ, and $s^{\perp}(H)$ is twice the genus of the surface obtained by removing a regular neighborhood of H from Σ. Also it will be useful to consider

$$l(H) := \operatorname{rank}(V \cap V^{\perp}), \tag{2.2}$$

Note that $V \cap V^{\perp}$ is a Lagrangian subspace of V. Finally, define

$$k(H) := \operatorname{rank}(\ker(i_*\colon\ H_1(H;\mathbb{R}) \longrightarrow H_1(\Sigma;\mathbb{R}))). \tag{2.3}$$

For example, for the graph H on the surface of genus 3, consisting of a single vertex and 3 edges, shown on the left in figure 2, $s(H) = s^{\perp}(H) = 2$, $l(H) = 1$, $k(H) = 0$. Note the identities relating these invariants for any graph $H \subset \Sigma$:

$$s(H) + s^{\perp}(H) + 2l(H) \ = \ 2g, \ \ k(H) + l(H) + s(H) \ = \ \operatorname{rank}(H_1(H)), \tag{2.4}$$

where g denotes the genus of Σ.

3. Tutte polynomial and graphs on surfaces

We will consider the following normalization of the Tutte polynomial of a graph G:

$$T_G(X,Y) = \sum_{H \subset G} X^{c(H)-c(G)}\, Y^{n(H)}. \tag{3.1}$$

The summation is taken over all spanning subgraphs H of G, that is the vertex set of H coincides with the vertex set of G. Therefore the sum contains 2^e terms, where e is the number of edges of G. In (3.1) $c(H)$ denotes the number of connected components of the graph H, and $n(H)$ is the *nullity* of H, defined as the rank of the first homology group $H_1(H)$ of H.

Now suppose G is a graph embedded in a surface Σ, let $i\colon G \longrightarrow \Sigma$ denote the embedding. Consider a collection of formal variables corresponding to the subgroups of $H_1(\Sigma)$. Given a subgroup $V < H_1(\Sigma)$, let $[V]$ denote the corresponding variable associated to it. Define

$$\widetilde{P}_{G,\Sigma}(X,Y) \;=\; \sum_{H\subset G} [i_*(H_1(H))]\,(X)^{c(H)-c(G)}\,Y^{k(H)}. \tag{3.2}$$

Here $[i_*(H_1(H))]$ is the formal variable associated to the subgroup equal to the image of $H_1(H)$ in $H_1(\Sigma)$ under the homomorphism i_* induced by inclusion; $k(H)$ is defined in (2.3). Therefore $\widetilde{P}_{G,\Sigma}$ may be viewed as a polynomial in X, Y with coefficients corresponding to the subgroups of $H_1(\Sigma)$. This polynomial may be used to distinguish different embeddings of a graph G into Σ.

However if two graphs G, G' in Σ are considered equivalent whenever there is a diffeomorphism taking G to G', one needs to define a polynomial invariant in terms of quantities which are invariant under the action of the mapping class group. We now introduce the polynomial $P_{G,\Sigma}$ which is the main object of study in this paper:

$$P_{G,\Sigma}(X,Y,A,B) \;=\; \sum_{H\subset G} X^{c(H)-c(G)}\,Y^{k(H)}\,A^{s(H)/2}\,B^{s^{\perp}(H)/2} \tag{3.3}$$

A surface Σ will usually be fixed, and the subscript Σ will be omitted from the notation. The invariants s, $s^{\perp}$ associated to a graph $H\subset\Sigma$ are defined in (2.1). Some elementary properties of this polynomial are summarized in the following statement.

Lemma 3.1.
(1) If e is an edge of G which is neither a loop nor a bridge, then $P_G = P_{G\smallsetminus e} + P_{G/e}$.
(2) If e is a bridge in G, then $P_G = (1+X)\,P_{G/e}$.
(3) If e is a loop in G which is trivial in $H_1(\Sigma)$, then $P_G = (1+Y)\,P_{G\smallsetminus e}$.

Remark. Note if G_1, G_2 are disjoint graphs in Σ, it is *not* true in general that $P_{G_1\cup G_2} = P_g\,P_{G'}$, see for example figure 1. (A similar comment applies to the case when G_1, G_2 in Σ are disjoint except for a single vertex v.) This is quite different form the case of the classical Tutte polynomial. However, the polynomial P_G is multiplicative with respect to disjoint unions in the context of ribbon graphs, see lemma 5.3.

Proof. The proof of lemma 3.1 is similar to the proof of the corresponding statements for the Tutte polynomial. To prove (1), consider an edge e which is neither a loop nor a bridge. Since e is not a loop, the sum (3.3) splits into two parts $P_G = S_1 + S_2$. S_1 consists of the terms with H containing the edge e, and S_2 consists of the terms with H not containing e. In the first case, the embedding $H\subset\Sigma$ is homotopic to the embedding $H/e\subset\Sigma$, and all of the invariants $c, k, s, s^{\perp}$ of H coincide with those of H/e. Therefore, $S_1 = P_{G/e}$. The terms in S_2 are in $1-1$ correspondence with

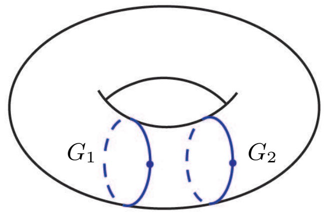

FIGURE 1. In general the polynomial P_G is not multiplicative with respect to disjoint unions: in this example,

$$P_{G_1} = P_{G_2} = 1 + B,\ P_{G_1 \cup G_2} = 2 + B + Y.$$

Note that P_G is multiplicative for *ribbon* graphs, see section 5.

the terms in $P_{G\smallsetminus e}$. Moreover, since e is not a bridge, $c(G) = c(G \smallsetminus e)$. It follows that $S_2 = P_{G\smallsetminus e}$.

To prove (2), suppose e is a bridge in G. Again the sum (3.3) splits:

$$P_G = \sum_{H\subset(G\smallsetminus e)} X^{c(H)-c(G)}\, Y^{k(H)}\, A^{s(H)/2}\, B^{s^{\perp}(H)/2} + \sum_{H\subset(G/e)} X^{c(H)-c(G)}\, Y^{k(H)}\, A^{s(H)/2}\, B^{s^{\perp}(H)/2}$$

More precisely, the subgraphs H parametrizing the second sum are all subgraphs of H containing e. Contracting e leaves each term in the second sum unchanged, and moreover the second sum is precisely the expansion of $P_{G/e}$.

There is a $1-1$ correspondence between the subgraphs H (not containing e) of G parametrizing the first sum and the subgraphs parametrizing the second sum. Given $H \subset G$, $e \notin H$, this correspondence associates to it the subgraph $\widetilde{H} \subset G/e$ obtained by identifying the two endpoints of e in H. Since e is a bridge, the homological invariants $k, s, s^{\perp}$ of H are identical to those of $\widetilde{H}$. However $c(H) - c(G) = c(\widetilde{H}) - c(G/e) + 1$. Therefore each term in the first sum equals the corresponding term in the expansion of $P_{G/e}$ times X. This concludes the proof of (2). The proof of (3) is analogous, noting that removing a loop e, which is homologically trivial on the surface, from a subgraph H decreases $k(H)$ by 1 and leaves other exponents in the expansion (3.3) unchanged. □

Lemma 3.2. *The Tutte polynomial (3.1) is a specialization of* P_G*:*

$$T_G(X,Y) \;=\; Y^g\, P_{G,\Sigma}(X,Y,Y,Y^{-1}),$$

where g *is the genus of the surface* Σ*.*

Remark. The polynomial P_G can be normalized to make the relation with the Tutte polynomial easier to state. For example, if one chose the exponent of Y in (3.3) to be $n(H) = \operatorname{rank} H_1(H)$ rather than $k(H)$, $T_G(X,Y)$ would be the specialization of the resulting polynomial obtained simply by setting $A = B = 1$. We chose the convention (3.3) to have a natural expression of duality (1.2), proved in lemma 4.1 below.

To prove lemma 3.2, recall from (2.4) that $s^{\perp}(H) = 2g - s(H) - 2l(H)$. The terms in the expansion (3.3) of P_G are of the form $X^{c(H)-c(G)}\ Y^{k(H)}\ A^{s(H)/2}\ B^{s^{\perp}(H)/2}$. Substituting $A = Y$, $B = Y^{-1}$, one gets $X^{c(H)-c(G)} Y^{g+k(H)+s(H)+l(H)}$. Recalling from (2.4) that $k(H) + l(H) + s(H) = n(H) = \mathrm{rank}\,(H_1(H))$, one gets the corresponding term in the expansion of the Tutte polynomial (3.1). □

3.3. **Tutte skein module.** One may generalize the polynomial P_G further and, avoiding the use of homology, consider *Tutte skein module* of a surface Σ: the vector space spanned by isotopy classes of graphs on Σ, modulo relations (1)-(3) in lemma 3.1. For example, the contraction-deletion relation states that $G = G \smallsetminus e + G/e$, where the three graphs $G, G \smallsetminus e, G/e$ are viewed as vectors in the skein module. In this case the "polynomial" associated to a graph $G \subset \Sigma$ is the element of the skein module represented by G. There is an expansion, analogous to (3.2), where each term in the expansion is an element of the skein module, and to get the polynomial $\widetilde{P}_{G,\Sigma}$ one applies homology to that expansion.

Note that a relative version of this skein module, specialized to $Y = 0$, in the rectangle – the *chromatic algebra* – was considered in [12, 13]. See also remark 6 following the statement of theorem 6.1 below concerning the relation between the Tutte skein module of Σ and the Kauffman skein module of $\Sigma \times I$.

4. Duality

The purpose of this section is to prove a duality result for the polynomial P_G defined in (3.3), which is analogous to the duality $T_G(X, Y) = T_{G^*}(Y, X)$ satisfied by the Tutte polynomial of planar graphs. The following result applies to *cellulations* of surfaces: graphs $G \subset \Sigma$ such that each connected component of $\Sigma \smallsetminus G$ is a disk. This is a natural condition guaranteeing that the dual G^* is well-defined. Equivalently, one may view G as a *ribbon graph*, see section 5.

Theorem 4.1. *Suppose G is a cellulation of a closed orientable surface Σ (equivalently, let G be an oriented ribbon graph.) Then the polynomial invariants of G and its dual G^* are related by*

$$P_G\,(X, Y, A, B) \;=\; P_{G^*}\,(Y, X, B, A) \tag{4.1}$$

Remark. The theorem follows from a corresponding statement, established in the proof below, about the polynomial $\widetilde{P}$ defined in (3.2). Specifically, $\widetilde{P}_{G^*}(Y, X)$ is obtained from $\widetilde{P}_G(X, Y)$ by replacing each coefficient $[V]$ (formally corresponding to a subgroup of $H_1(\Sigma)$) with its symplectic orthogonal complement $[V^{\perp}]$.

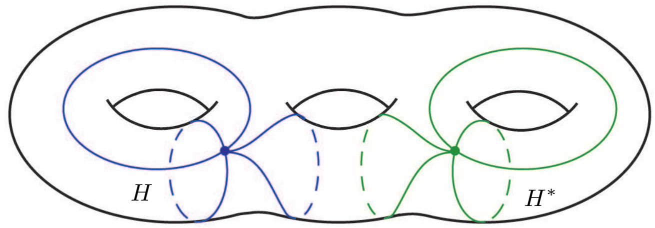

FIGURE 2. A subgraph H of a cellulation G of the genus 3 surface Σ, and the corresponding subgraph H^* of the dual cellulation G^*.

Proof. Consider the expansions (3.3) of both sides in the statement of the theorem:

$$P_G(X,Y,A,B) \;=\; \sum_{H\subset G} X^{c(H)-c(G)}\, Y^{k(H)}\, A^{s(H)/2}\, B^{s^{\perp}(H)/2} \tag{4.2}$$

$$P_{G^*}(Y,X,B,A) \;=\; \sum_{H^*\subset G^*} Y^{c(H^*)-c(G^*)}\, X^{k(H^*)}\, B^{s(H^*)/2}\, A^{s^{\perp}(H^*)/2}. \tag{4.3}$$

For each subgraph $H\subset G$, consider the subgraph $H^*\subset G^*$ whose edges are precisely all those edges of G^* which are disjoint from the edges of H. The theorem follows from the claim that the term corresponding to H in the expansion of P_G equals the term corresponding to H^* in the expansion of P_{G^*} above.

The simplest example of a cellulation G of a surface Σ of genus g is a graph consisting of a single vertex and $2g$ edges which are loops representing a symplectic basis of $H_1(\Sigma)$. Then its dual is the graph G^* also with a single vertex and $2g$ loops. Figure 2 shows the surface of genus 3 and a subgraph $H\subset G$ formed by 3 edges on the left in the figure. In this case H^* also consists of 3 edges as illustrated on the right in the figure.

The proof of the claim above follows from the observation

$$V(H^*)\cong V(H)^{\perp}. \tag{4.4}$$

To establish (4.4), first note that H^* is a retract of $\Sigma\smallsetminus H$. Indeed, the two cellulations G, G^* give rise to dual handle decompositions of the surface Σ. In the handle decomposition corresponding to G, the $0-$handles are disk neighborhoods of the vertices of G, the $1-$handles are regular neighborhoods of the edges of G, the $2-$handles correspond to the $2-$cells $\Sigma\smallsetminus G$. Let $\mathcal{H}$ (respectively $\mathcal{H}^*$) denote the union of the $0-$ and $1-$handles corresponding to H (respectively H^*). Note that H is a retract of $\mathcal{H}$, H^* is a retract of $\mathcal{H}^*$. If H is the entire graph G, H^* consists of all vertices of G^* and no edges, certainly in this case H^* is a retract of $\Sigma\smallsetminus H$. Removing one edge from H at a time, observe that the effect on the $H-$handle decomposition is removing a $1-$handle, while the effect on the dual handle decomposition is adding the co-core of the removed $1-$handle. To summarize, Σ is the union of $\mathcal{H}$, $\mathcal{H}^*$ along their boundary, so $\mathcal{H} = \Sigma \smallsetminus \mathcal{H}^* \cong \Sigma \smallsetminus H^*$, and so H is the retract of $\Sigma \smallsetminus H^*$.

To prove (4.4), consider the cellular $1-$chains on Σ (cf [14]). The fact that G and G^* give rise to dual handle decompositions is useful for computing the intersection pairing of $H_1(\Sigma)$: given $a, b \in H_1(\Sigma)$, a may be represented by a $1-$cycle in G, b by a $1-$cycle in G^*. The edges of G^* are transverse to the edges of G, and the intersection number $a \cdot b$ may be computed as the sum of the signed intersections of the corresponding edges.

Note that if e is a bridge in H, then the subgroup $V(H)$ represented by H in $H_1(\Sigma)$ does not change if e is removed from H: $V(H) = V(H \smallsetminus e)$. Similarly, if e is a bridge in H and e^* is the edge dual to e, then $V(H^*) = V(H^* \smallsetminus e^*)$. Therefore one may assume that H does not have bridges. Let $c = \sum_i \alpha_i c_i$ be a $1-$cycle representing an element in $V(H)^\perp$, where c_i are oriented edges in G^*. The intersection of c with any class in $V(H)$ is zero, and since H does not have bridges, this means that the intersection of c with every edge of H is zero. (Given an edge e in H which is not a bridge, there is a cycle representing a non-trivial class in $H_1(H)$ which is a linear combination of oriented edges in H, in which e enters with coefficient 1. If the intersection of c with e were not equal to zero, the intersection of c with this cycle is also non-zero, a contradiction.) This means that the sum $\sum_i \alpha_i c_i$ does not contain any edges dual to the edges of H. Therefore c is a $1-$cycle in H^*. This proves that $V(H)^\perp \subset V(H^*)$. The converse inclusion $V(H^*) \subset V(H)^\perp$ is clearly true, therefore $V(H^*) \cong V(H)^\perp$.

It follows that $s(H) = s^\perp(H^*)$, $s^\perp(H) = s(H^*)$, and one checks that $c(H^*) - c(G^*) = k(H)$, and similarly $c(H) - c(G) = k(H^*)$. This shows that the terms corresponding to H, H^* in (4.2, 4.3) are equal, concluding the proof of theorem 4.1. □

Note that partial duality results for the Bollobás-Riordan polynomial have been previously obtained by several authors. We discuss these results and show that they may be derived as a consequence of our theorem 4.1 in section 5.4 below.

5. Ribbon graphs and the Bollobás-Riordan polynomial

A *ribbon graph* is a pair (G, S) where G is a graph embedded in a surface S such that the embedding $G \hookrightarrow S$ is a homotopy equivalence. It is convenient to consider the surface S with a handle decomposition corresponding to the graph G: the 0-handles are disk neighborhoods of the vertices of G, and the 1-handles correspond to regular neighborhoods of the edges. (Other terms: cyclic graphs, fat graphs are also sometimes used in the literature to describe ribbon graphs.) G is an *orientable* ribbon graph is S is an orientable surface. Given a ribbon graph (G, S), one obtains a closed surface Σ by attaching a disk to S along each boundary component. Therefore a ribbon graph may be viewed as a *cellulation* of a closed surface Σ, i.e. a graph G embedded in Σ such that each component of the complement $\Sigma \setminus G$ is a disk. Conversely, given a cellulation G of Σ, one has a ribbon graph structure (G, S) where S is a regular neighborhood of G in Σ. We will use the notions of a ribbon graph and of a cellulation interchangeably. Note that given a cellulation $G \subset \Sigma$, its dual cellulation $G^* \subset \Sigma$ is well-defined.

Consider the Bollobás-Riordan polynomial of ribbon graphs [2, 3] (in this paper we only consider *orientable* ribbon graphs, therefore there are three, rather than four, variables): given a ribbon graph (G, S),

$$(5.1) \qquad BR_{G,S}(X,Y,Z) \;=\; \sum_{H \subset G} (X-1)^{r(G)-r(H)}\, y^{n(H)}\, Z^{c(H)-bc(H)+n(H)}.$$

The summation is taken over all spanning subgraphs H of G, and moreover each H inherits the ribbon structure from that of G: the relevant surface is obtained as the union of all 0-handles and just those 1-handles which correspond to the edges of H. To explain the notation in this definition, let $v(H), e(H)$ denote the number of vertices, respectively edges, of H, and let $c(H)$ be the number of connected components. Then $r(H) = v(G) - c(H)$, $n(H) = e(H) - r(H)$, and $bc(H)$ is the number of boundary components of the surface S. Note that $n(H)$ equals the rank of the first homology group $H_1(H)$, and the exponent of Z, $c(H) - bc(H) + n(H)$, equals $2g(H)$, twice the genus of the surface underlying the ribbon graph H. To simplify the notation, we will often omit the reference to the surface S and denote the graph by BR_G.

Lemma 5.1. *The Bollobás-Riordan polynomial of a ribbon graph may be obtained as a specialization of the polynomial P_G:*

$$(5.2) \qquad BR_{G,S}(X,Y,Z) \;=\; Y^g\, P_{G,\Sigma}(X-1, Y, YZ^2, Y^{-1}),$$

where Σ is the closed surface obtained by attaching a disk to S along each boundary component, and g is the genus of Σ.

The proof consists of showing that the corresponding terms in the expansions (3.3), (5.1) are equal. Indeed, substituting $A = YZ^2$, $B = Y^{-1}$ in (3.3) gives summands of the form

$$(X-1)^{c(H)-c(G)}\, Y^{k(H)+s(H)/2-s^{\perp}(H)/2}\, Z^{s(H)}.$$

Recalling from the identities (2.4) that $s^{\perp}(H) = 2g - s(H) - 2l(H)$ and $k(H) + l(H) + s(H) = n(H)$, note that these summands are equal to Y^g times the corresponding terms in (5.1). □

To discuss the relation between the polynomial P_G and the Bollobás-Riordan polynomial further, recall that the polynomial BR_G satisfies the following universality property. Let $\mathcal{G}$ denote the set of isomorphism classes [3] of connected ribbon graphs. Define the maps C_{ij} from $\mathcal{G}$ to $\mathbb{Z}[X]$ by $BR = \sum_{i,j} C_{ij} Y^i Z^j$. Further, given a commutative ring R and an element $x \in R$, $C_{ij}(x)$ will denote the map from $\mathcal{G}$ to R obtained by composing C_{ij} with the ring homomorphism $\mathbb{Z}[X] \longrightarrow R$ mapping X to x.

Theorem 5.2. [2, 3] *Let R be a commutative ring and $x \in R$ and $\phi\colon \mathcal{G} \longrightarrow R$ a map satisfying*
(1) $\phi(G) = \phi(G/e) + \phi(G \smallsetminus e)$ if e is ordinary, and
(2) $\phi(G) = x\, \phi(G/e)$ if e is a bridge.

Then there are elements $\lambda_{ij} \in R$, $0 \le j \le i$, such that

$$\phi \;=\; \sum_{i,j} \lambda_{ij} C_{ij}(x).$$

The polynomial P_G satisfies the properties (1), (2) in this theorem, therefore it follows that the coefficients of P_G may be expressed as linear combinations of the coefficients of BR_G. Note that the exponents $s(H), s^{\perp}(H)$ in the definition (3.3) of P_G are independent invariants of H – indeed, this is the main difference in the definitions of the two polynomials: each term in the expansion (5.1) of BR_G is defined in terms of the invariants of a ribbon subgraph H, while the terms in the expansion (3.3) involve the invariants associated to the embedding of H into the original fixed surface Σ. Therefore it does not seem likely that there is a straightforward expression for P_G in terms of BR_G similar to that in lemma 5.1, however it would be interesting to find an explicit expression.

Returning to the properties of the polynomial P_G, observe that the multiplicativity for disjoint unions and for one-point unions holds in the context of ribbon graphs (compare with the remark after lemma 3.1):

Lemma 5.3. *Properties (1)–(3) in lemma 3.1 hold for ribbon graphs G. In addition, for disjoint ribbon graphs G_1, G_2,*

(4) $P_{G_1 \sqcup G_2} \;=\; P_{G_1 \vee G_2} \;=\; P_G \cdot P_{G'}$.

Here by the polynomial P_G of a ribbon graph (G, S) we mean $P_{G,\Sigma}$ where as above Σ is the closed surface associated to S. For example, the closed surface associated to the graphs G_1, G_2 with the ribbon structure inherited from their embedding into the torus in figure 1 is the 2-sphere (and the surface associated to $G_1 \cup G_2$ is the disjoint union of two spheres), and not the torus. This illustrates the difference between the validity of the property (4) for ribbon graphs, but not in general for graphs on surfaces as in figure 1.

Proof. The proof of (1)–(3) is identical to that in lemma 3.1. To prove (4) for the disjoint union $G_1 \sqcup G_2$, consider subgraphs $H_1 \subset G_1$, $H_2 \subset G_2$ and let V_i denote the image of $H_1(H_i)$ in $H_1(\Sigma)$, $i = 1, 2$. Since the surface associated to $G_1 \sqcup G_2$ is the disjoint union of surfaces associated to G_1 and G_2, one has $k(H_1 \cup H_2) = k(H_1) + k(H_2), s(H_1 \cup H_2) = s(H_1) + s(H_2)$, and $s^{\perp}(H_1 \cup H_2) = s^{\perp}(H_1) + s^{\perp}(H_2)$. The proof for the one-vertex union $G_1 \vee G_2$ is directly analogous. □

5.4. **Prior results on duality of the Bollobás-Riordan polynomial.** Several authors have established partial duality of the Bollobás-Riordan polynomial. For example, [3] notes that

$$BR_G(1+t, t, t^{-1}) \;=\; BR_{G^*}(1+t, t, t^{-1}). \tag{5.3}$$

By lemma 5.1, $BR_G(1+t, t, t^{-1}) = Y^g\, P_G(t, t, t^{-1}, t^{-1})$, therefore (5.3) is a consequence of theorem 4.1. More generally, it is shown in [10, 22] (see also [4, 11, 23]) that there is duality for a 2−variable specialization:

$$BR_G(1+X, Y, (XY)^{-1/2}) \;=\; (X^{-1}Y)^g\, BR_{G^*}(1+Y, X, (XY)^{-1/2}) \tag{5.4}$$

Observe that according to lemma 5.1,

$$\begin{aligned} BR_G(1+X, Y, (XY)^{-1/2}) \;&=\; Y^g\; P_G(X, Y, X^{-1}, Y^{-1}), \\ BR_{G^*}(1+Y, X, (XY)^{-1/2}) \;&=\; X^g\; P_{G^*}(Y, X, Y^{-1}, X^{-1}). \end{aligned}$$

Therefore (5.4) may also be viewed as a consequence of theorem 4.1. It would be interesting to understand the full duality relation (4.1) in terms of the Bollobás-Riordan polynomial, since as discussed above, the polynomials P_G, BR_G carry equivalent information about a ribbon graph G.

6. Generalized Kauffman bracket and Jones polynomial of links on surfaces

Various relations between the Tutte polynomial and link polynomials are well known, for example see [27, 16], and more recently such relations have been established for link polynomials and the Bollobás-Riordan polynomial of associated graphs on surfaces, cf. [5, 7, 22]. In this section we consider a 2-variable generalization of the Jones polynomial of links in (surfaces)$\times I$, and more generally a 4−variable Kauffman bracket of link diagrams on a surface, and we establish an analogue of Thistlethwaite's theorem [27] relating these polynomials for alternating links in $\Sigma \times I$ and the polynomial P_G of the associated Tait graph on the surface Σ. Using the interpretation of virtual links as "irreducible" embeddings of classical links into surfaces [20], these results apply to virtual links.

Let L be a link embedded in $\Sigma \times I$, where Σ is a closed orientable surface. Consider a projection D of L onto the surface. By general position D is a diagram with a finite number of crossings. Each crossing may be resolved as shown in figure 3. Given a diagram D with n crossings, consider the set $\mathcal{S}$ of its 2^n resolutions. Each resolution $S \in \mathcal{S}$ is a disjoint collection of closed curves embedded in Σ. Denote by $\alpha(S)$ the number of resolutions of type (1) that were used to create it, by $\beta(S)$ the number of resolutions of type (2), and let $c(S)$ be the number of components of S. Consider the inclusion map $i\colon\, S \subset \Sigma$, and denote

$$k(S) \;=\; \text{rank}\,(\,\ker\{i_*\colon\; H_1(S) \longrightarrow H_1(\Sigma)\}).$$

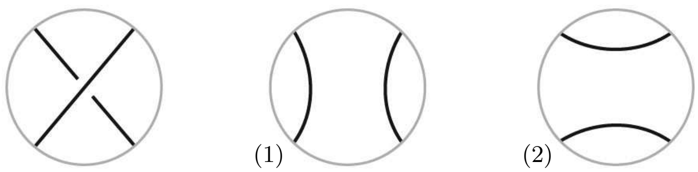

Figure 3. Resolutions of a crossing.

Generalizing the classical definition of the Kauffman bracket (cf [19, 1]), consider

$$\widetilde{K}_L(A,B,d) \;=\; \sum_{S\in\mathcal{S}} [i_*(H_1(S))]\, A^{\alpha(S)}\, B^{\beta(S)}\, d^{k(S)} \tag{6.1}$$

Here $[i_*(H_1(S))]$ denotes a formal variable associated to the subgroup $i_*(H_1(S))$ of $H_1(\Sigma)$. This definition is closely related to (more precisely, it may be viewed as a specialization of) the *surface bracket polynomial*, defined in the context of virtual links in [9, 21]. Two diagrams in Σ, representing isotopic embeddings of a link L in $\Sigma \times I$, are related by the usual Reidemeister moves, and the usual specialization

$$\widetilde{J}_L(t) \;=\; (-1)^{w(L)}\, t^{3w(L)/4}\, \widetilde{K}_D(t^{-1/4}, t^{1/4}, -t^{1/2} - t^{-1/2}), \tag{6.2}$$

where $w(L)$ is the writhe of L, is an invariant of an embedded oriented link $L \subset \Sigma\times I$. The polynomial $\widetilde{J}_L(t)$ with coefficients corresponding to subgroups of $H_1(\Sigma)$ may be used to distinguish non-isotopic links in $\Sigma \times I$ (also see remark 6 following theorem 6.1 below.) However if one is interested in studying links up to to the action of the diffeomorphisms of Σ, or in studying virtual links, a relevant invariant is the following finite-variable specialization. Denoting the rank of $i_*(H_1(S))$ by $r(S)$, define

$$K_D(A,B,d,Z) \;=\; \sum_{S\in\mathcal{S}} A^{\alpha(S)}\, B^{\beta(S)}\, d^{k(S)}\, Z^{r(S)}, \tag{6.3}$$

and the corresponding version of the Jones polynomial:

$$J_L(t,Z) \;=\; (-1)^{w(L)}\, t^{3w(L)/4}\, K_D(t^{-1/4}, t^{1/4}, -t^{1/2} - t^{-1/2}, Z). \tag{6.4}$$

Note that all of the polynomials considered here may be defined for *virtual* links [18], using their "irreducible" embeddings into surfaces [20]. Since $k(S)+r(S)$ equals the number $c(S)$ of components of S, it follows that for a virtual link L, the invariant K_D defined above specializes to the usual Kauffman bracket by setting $Z = d$:

$$[L](A,B,d) \;=\; d^{-1}\, K_L(A,B,d,d).$$

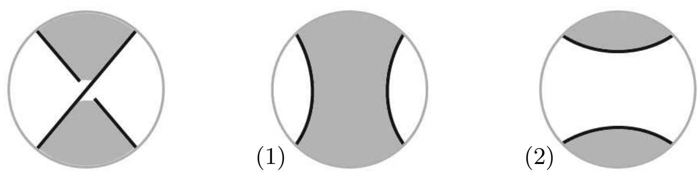

FIGURE 4. Checkerboard coloring near a crossing of an alternating diagram.

We now turn to the analogue for links on surfaces of Thistlethwaite's theorem [27] relating the Jones polynomial $J_L(t)$ of an alternating link L in S^3 to the specialization $T_G(-t,-t^{-1})$ of the Tutte polynomial of an associated Tait graph. Suppose L is a link in $\Sigma \times I$ which has an alternating diagram D on Σ. Then this diagram may be checkerboard-colored, as shown near each crossing on the left in figure 4. The associated Tait graph is the graph $G_D \subset \Sigma$ whose vertices correspond to the shaded regions of the diagram, and two vertices are connected by an edge whenever the corresponding shaded regions meet at a crossing (an example of an alternating link on the torus and the corresponding Tait graph are shown in figure 5 - compare with the example in [5].) The Tait graph is a well-defined graph $G \subset \Sigma$ if each component in the complement of a link diagram D is a disk; this condition holds for virtual links due to the irreducibility of their embedding into $\Sigma \times I$.

Theorem 6.1. *The generalized Kauffman bracket (6.3) of an alternating link diagram D on a surface Σ may be obtained from the polynomial P_G, defined by (3.3), of the associated Tait graph G as follows:*

$$K_D(A,B,d,Z) \;=\; A^{g+v(G)-c(G)}\, B^{-g+n(G)}\, d^{c(G)}\, Z^g\, P_G\left(\frac{Bd}{A}, \frac{Ad}{B}, \frac{A}{BZ}, \frac{B}{AZ}\right). \tag{6.5}$$

In particular, substituting $A = t^{-1/4}, B = t^{1/4}, d = -t^{1/2} - t^{-1/2}$ as in (6.4) yields an expression for the 2-variable Jones polynomials $J_L(t,Z)$ in terms of the polynomial P_G of the associated Tait graph.

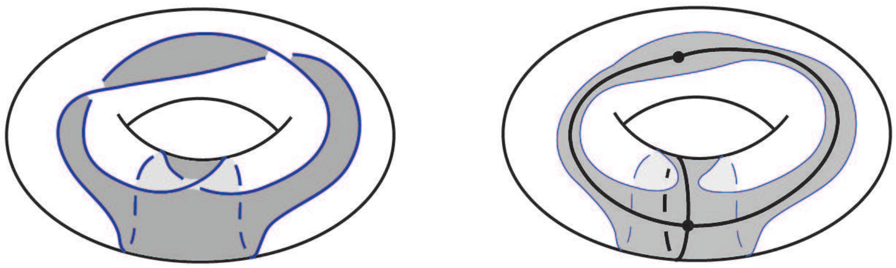

FIGURE 5. An alternating link diagram (left) and its Tait graph (right) on the torus.

Remarks.

1. Recall that $\widetilde{K}_L(A, B, d)$, defined in (6.1), is a polynomial in A, B, d with coefficients corresponding to subgroups $V < H_1(\Sigma)$. The equation (6.5) follows from a more general relation, which can be deduced from the proof of theorem 6.1, between the polynomials $\widetilde{K}_L$ and $\widetilde{P}_G(X, Y)$ (defined in (3.2)). In particular, each coefficient $[V]$, $V < H_1(\Sigma)$, of $\widetilde{P}_G$ is replaced with $V \cap V^{\perp}$ to get the corresponding coefficient of $\widetilde{K}_L$.

2. It would be interesting to establish a relation, analogous to (6.5) between these generalized versions of the Kauffman bracket and Jones polynomial and the Bollobás-Riordan polynomial. In principle, such a relationship follows from theorem 6.1 (see the discussion following theorem 5.2), but an explicit formula does not seem to be as straightforward as (6.5).

3. Theorem 6.1 asserts that the generalized Kauffman bracket $K_D(A, B, d, Z)$ may be obtained as a specialization of the polynomial $P_G(D)$. It would be interesting to find out whether K_D and P_G (or K_D and the Bollobás-Riordan polynomial BR_G) in fact determine each other.

4. Suppose D is an alternating link diagram (associated to a link $L \subset \Sigma \times I$) on an orientable surface Σ. Switching each crossing, one gets an alternating link diagram D' whose checkerboard coloring is precisely that of D with the colors switched on each face. Therefore the associated graphs G, G' are duals of each other. (To make this statement precise, it is convenient to consider virtual links, so the embedding $L \subset \Sigma \times I$ is "irreducible" [20], and then the Tait graphs G, G' are cellulations.) In this context theorem 6.1 gives a different perspective on the duality relation (4.1) for P_G.

5. Adapting the proof in [5], one may establish a generalization of theorem 6.1 from alternating diagrams to checkerboard-colored diagrams, using a *signed* version of the polynomial P_G. The proof uses the observation [17] that any such link diagram on a surface can be made alternating by switching some of the crossings, and then follows the idea [5] of labeling by -1 each edge of the Tait graph where a switch has been made.

6. One may generalize the correspondence between the Jones polynomial and P_G to *skein modules.* (This is a further generalization from the polynomials $\widetilde{K}_L$ and $\widetilde{P}_G$ whose coefficients are subgroups of $H_1(\Sigma)$.) Specifically, one may consider the isotopy classes of graphs on Σ modulo the contraction-deletion relation, see section 3.3, and the skein module of links modulo the Kauffman skein relation in figure 3, cf. [25, 28]. The author would like to thank Józef Przytycki for pointing out this perspective on the problem.

Proof of theorem 6.1. The terms in the expansions (3.3), (6.3) are in $1-1$ correspondence. Specifically, for each spanning subgraph $H\subset G(D)$ parametrizing the sum (3.3), consider the corresponding resolution $S(H)$: each crossing of the diagram D is resolved as in figure 4, where the resolution (1) is used if the corresponding edge is included in H, and the resolution (2) is used otherwise. Observing the effect of the resolutions on the shaded regions in figure 4, note that the collection of embedded curves $S(H)\subset\Sigma$ is the boundary of a regular neighborhood of H in Σ. Moreover, the number $\alpha(S)$ of resolutions of type (1) is precisely the number $e(H)$ of edges of H, and $\beta(S)$ equals $e(G)-e(H)$.

Observe

$$\alpha(S(H))=e(H)=v(H)-c(H)+n(H),$$

$$\beta(S(H))=e(G)-e(H)=n(G)-n(H)+c(H)-c(G).$$

Also note that since S is the boundary of a regular neighborhood of H, $r(S)=l(H)$, and $k(S)=c(H)+k(H)$. Therefore the summands $A^{\alpha(S)}\,B^{\beta(S)}\,d^{k(S)}\,Z^{r(S)}$ in (6.3) may be rewritten as

$$A^{v(H)-c(H)+n(H)}\,B^{n(G)-n(H)+c(H)-c(G)}\,d^{c(H)+k(H)}\,Z^{l(H)}.$$

Substituting the required variables, the summands in the expansion (3.3) of P_G are of the form

$$\left(\frac{Bd}{A}\right)^{c(H)-c(G)}\left(\frac{Ad}{B}\right)^{k(H)}\left(\frac{A}{BZ}\right)^{s(H)/2}\left(\frac{B}{AZ}\right)^{s^{\perp}(H)/2}.$$

The proof is completed by using the relations (2.4) to identify the exponents of A,B,d,Z on the two sides of (6.5). ☐

7. A multivariate graph polynomial

We conclude the paper by pointing out a multivariate version of the polynomial P_G, and observing the corresponding duality relation. (Note that a multivariate version of the Bollobás-Riordan polynomial, and a duality for a certain specialization have been established in [31].) Let G be a graph on a surface Σ, and let

$$\mathbf{v} \;=\; \{v_e\}_{e\in E(G)}$$

be a collection of commuting indeterminates associated to the edges of G. Following the notation used in (3.3), consider

$$\overline{P}_G(q,\mathbf{v},A,B) \;=\; \sum_{H\subset G} q^{c(H)}\, A^{s(H)/2}\, B^{s^{\perp}(H)/2} \prod_{e\in E(H)} v_e \tag{7.1}$$

Clearly, the "usual" multivariate Tutte polynomial Z_G [26] is a specialization of $\overline{P}_G$:

$$Z_G(q,\mathbf{v}) \;=\; \overline{P}_{G,\Sigma}(q,\mathbf{v},1,1),$$

The relation to the polynomial $P_G(X,Y,A,B)$ defined in (3.3) is given by

$$P_G(X,Y,A,B) \;=\; X^{-c(G)}\, Y^{-g-v(G)}\, \overline{P}_G(XY,\, Y,\, A/Y,\, BY),$$

where as usual $c(G)$ denotes the number of connected components of the graph G, $v(G)$ is the number of vertices of G, and g is the genus of the surface Σ. That is, to get the polynomial P_G, one sets in the multivariate version $\overline{P}_G$ all edge weights v_e equal to Y, and $q = XY$. The analogue of the duality (4.1) for the multivariate polynomial $\overline{P}_G$ is as follows.

Lemma 7.1. *Let G be a cellulation of a surface Σ (or equivalently a ribbon graph), and let G^* denote its dual. Then*

$$\overline{P}_{G^*}(q,\mathbf{v},A,B) \;=\; q^{-g+c(G^*)-v(G)}\Big(\prod_{e\in E(G)} v_e\Big)\, \overline{P}_G(q,q/\mathbf{v},B/q,Aq). \tag{7.2}$$

As the notation indicates, the edge weights of G^* in the formula on the right-hand side are given by $\{q/v_e\}_{e\in E}$. Using the relation $c(H) = v(H) - e(H) + n(H)$, note that the expansion of the polynomial $\overline{P}_G$ may be rewritten as

$$\overline{P}_G(q,\mathbf{v},A,B) \;=\; q^{v(G)} \sum_{H\subset G} q^{n(H)}\, A^{s(H)/2}\, B^{s^{\perp}(H)/2} \prod_{e\in E(H)} \frac{v_e}{q} \tag{7.3}$$

The proof of lemma 7.1 consists of identifying the terms in the expansions of the two sides, following the lines of the proof of theorem 4.1.

References

[1] B. Bollobás, *Modern graph theory*, Springer, 1998.

[2] B. Bollobás and O. Riordan, *A polynomial invariant of graphs on orientable surfaces*, Proc. London Math. Soc. (3) 83 (2001), 513–531.

[3] B. Bollobás and O. Riordan, *A polynomial of graphs on surfaces*, Math. Ann. 323 (2002), 81–96.

[4] S. Chmutov, *Generalized duality for graphs on surfaces and the signed Bollobas-Riordan polynomial*, arXiv:0711.3490

[5] S. Chmutov and I. Pak, *The Kauffman bracket of virtual links and the Bollobs-Riordan polynomial*, arXiv:math/0609012

[6] R. Costa-Santos and B.M. McCoy, *Dimers and the critical Ising model on lattices of genus* >1, Nuclear Phys. B 623 (2002), 439–473.

[7] O.T. Dasbach, D. Futer, E. Kalfagianni, X.-S. Lin, and N.W. Stoltzfus, *The Jones polynomial and graphs on surfaces*, J. Comb. Theory, Series B 98 (2008), 384-399.

[8] P. Di Francesco, H. Saleur and J.-B. Zuber, *Critical Ising correlation functions in the plane and on the torus*, Nuclear Phys. B 290 (1987), 527–581.

[9] H. Dye and L.H. Kauffman, *Minimal surface representations of virtual knots and links*, Algebr. Geom. Topol. 5 (2005) 509-535.

[10] J. A. Ellis-Monaghan and I. Sarmiento, *A Duality Relation for the Topological Tutte Polynomial*, AMS Eastern Section Meeting Special Session on Graph and Matroid Invariants, Bard College, 10/9/05. http://academics.smcvt.edu/jellis-monaghan

[11] J. A. Ellis-Monaghan, I. Sarmiento, *A recipe theorem for the topological Tutte polynomial of Bollobas and Riordan*, arXiv:0903.2643.

[12] P. Fendley and V. Krushkal, *Tutte chromatic identities from the Temperley-Lieb algebra*, Geometry & Topology 13 (2009) 709-741 [arXiv:0711.0016].

[13] P. Fendley and V. Krushkal, *Link invariants, the chromatic polynomial and the Potts model*, arXiv:0806.3484.

[14] A. Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002.

[15] J.L. Jacobsen and H. Saleur, *Conformal boundary loop models*, Nucl. Phys. B 788 (2008), 137-166.

[16] F. Jaeger, *Tutte polynomials and link polynomials*, Proc. Amer. Math. Soc., 103 (1988), 647-654.

[17] N. Kamada, *On the Jones polynomials of checkerboard colorable virtual knots*, Osaka J. Math. 39 (2002) 325-333.

[18] L.H. Kauffman, *Virtual Knot Theory*, European J. Comb. (1999) Vol. 20, 663-690.

[19] L.H. Kauffman, *State models and the Jones polynomial*, Topology 26 (1987), 395–407.

[20] G. Kuperberg, *What is a virtual link?*, Algebr. Geom. Topol. 3 (2003) 587-591.

[21] V.O. Manturov, *Kauffman-like polynomial and curves in 2-surfaces*, J. Knot Theory Ramifications 12 (2003), 1131-1144.

[22] I. Moffatt, *Knot invariants and the Bollobas-Riordan polynomial of embedded graphs*, arXiv:math/0605466.

[23] I. Moffatt, *Partial duality and Bollobas and Riordan's ribbon graph polynomial*, arXiv:0809.3014

[24] A. Morin-Duchesne and Y. Saint-Aubin, *Critical exponents for the homology of Fortuin-Kasteleyn clusters on a torus*, arXiv:0812.2925.

[25] J. Przytycki, *Skein modules of* 3*-manifolds*, Bull. Polish Acad. Sci. Math. 39 (1991), 91–100.

[26] A.D. Sokal, *The multivariate Tutte polynomial (alias Potts model) for graphs and matroids*, in "Surveys in Combinatorics, 2005", ed. Bridget S. Webb (Cambridge University Press, 2005), 173-226.

[27] Thistlethwaite, *A spanning tree expansion for the Jones polynomial*, Topology 26 (1987), 297-309.

[28] V. Turaev, *The Conway and Kauffman modules of a solid torus* Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 167 (1988), 79–89; translation in J. Soviet Math. 52 (1990), 2799–2805.
[29] W.T. Tutte, *A ring in graph theory*, Proc. Cambridge Phil. Society 43 (1947), 26-40.
[30] W.T. Tutte, *A contribution to the theory of chromatic polynomials*, Canadian J. Math. 6 (1954), 80-91.
[31] F. Vignes-Tourneret, *The multivariate signed Bollobas-Riordan polynomial*, arXiv:0811.1584

Department of Mathematics, University of Virginia, Charlottesville, VA 22904

E-mail address: krushkal@virginia.edu