SUBALGEBRAS OF QUASI-HEREDITARY ALGEBRAS ARISING FROM ALGEBRAIC AND QUANTUM GROUPS

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Quasi-hereditary algebras arise in several natural contexts. For example, their module categories often appear in connection with the representation theory of algebraic groups over fields of positive characteristic or of quantum groups at a root of unity. This point of view has been exploited by various authors, e. g., [CPS3–8], [G2], etc. Classical Schur algebras attached to the general linear groups GL_n (see [G1]) as well as their quantized versions associated to the quantum general linear groups (see [DD, PW], etc.) provide typical examples of quasi-hereditary algebras arising in this way. A second importance source of quasi-hereditary algebras comes from geometry, in the study of perverse sheaves. For example, it is proved in [PS1, PS2] that the perverse sheaf category on a suitably nice stratified topological space X is the module category for a quasi-hereditary algebra.

The first example of a Borel subalgebra of a quasi-hereditary algebra was given by Green [G2], though he did not give a general definition of the concept. Later, König [K1] introduced the notion of an *exact* Borel subalgebra, which did not fit Green's example, but did apply for the category $\mathcal{O}$ associated to a simple complex Lie algebra. The second author of this paper introduced in [K1; appendix] a further notion, which synthesized these two cases[1], but still does not obviously capture other interesting examples, such as given in [Dy] and [DR].

The notion of a *Borel subalgebra*, introduced in Section 2, does seem to capture all the known examples, and it is sufficiently rich to have an interesting theory. The original notion of [K1; appendix] is recast here as a *homological Borel subalgebra*,

[1] Unfortunately, the two cases do not appear as closely related as [K1; appendix] suggests. The second author wishes to acknowledge here that his proof of [K1; appendix, Thm. F] on the existence of exact Borel subalgebras is incorrect, and no repair seems likely. The construction does, however, lead in that context to Borel subalgebras having only "dominant" weights; see §2. One can also choose a homological (or even excellent) Borel subalgebra in an appropriately constructed generalized Schur (or q-Schur algebra — see Section 7), but one is *not* guaranteed an exact Borel subalgebra, as claimed in [K1; appendix, Thm. E].

and several new and stronger variations are proposed. All these variations mirror to some extent properties of Borel subgroups of algebraic groups, discovered by van der Kallen and Polo [vdK]. We also demonstrate that König's notion of an exact Borel subalgebra is interesting in the context of infinitesimal groups and quantum groups at a root of unity.

A key ingredient in our point of view centers on a categorical approach. Thus, we first define various *Borel categories* associated to a highest weight category (i. e., the module category of a quasi-hereditary algebra A). Then we consider natural constructions, such as passing to endomorphism algebras, which give rise to subalgebras of quasi-hereditary algebras. In some cases, this process is interesting and non-trivial. The systematic use of the categorical approach has the additional advantage of making some of the discussion invariant under Morita equivalence.

Our paper is organized as follows. In Section 1, we review some preliminary material. Section 2 takes up the idea of a *Borel category* (and a Borel subalgebra B) associated to a highest weight category (and its associated quasi-hereditary algebra A). The axioms imply that tensor induction $A\otimes_B -$ of some irreducible B-modules give standard modules for A, while the other irreducible modules induce to the zero module. Some general constructions and a triangular decomposition are also discussed.

Section 3 continues the discussion of Section 2 by considering the existence of Borel subalgebras for highest weight categories having a graded Kazhdan-Lusztig theory (in the sense of [CPS6]). The main result here is an improvement in Theorem 3.5 of an unpublished result due to the first two authors and E. Cline (which was itself inspired by an earlier variation by Dyer [Dy]): Quasi-hereditary algebras A, even basic algebras, with a Kazhdan-Lusztig theory [CPS6] always have natural subalgebras B that are, under a mild additional assumption (given as Hypothesis (1.1)), always Borel subalgebras. Moreover, we point out that, conversely, the existence of a suitably strong Borel subalgebra implies the Lusztig characteristic p conjecture. (See this discussion in (3.9).) This result provides a substitute for a (still unpublished) theory announced by König [K4] in the exact Borel subalgebra case.

In Section 4, we define homological, exact, and excellent Borel categories. We also consider the question of realizing these categories in terms of subalgebras of quasi-hereditary algebras. This leads to the notion of an exact, homological, and excellent Borel subalgebra of a quasi-hereditary algebra. As mentioned above, exact Borel subalgebras have been previously defined by König [K1], and studied by him in a number of papers [K2, K3]. "Homological" Borel subalgebras or categories capture "Kempf's vanishing theorem" for reductive groups, while the "excellent" theory abstracts from even stronger properties in Schubert theory. A pleasant consequence (given in §4) of the axioms for an excellent Borel category is that they guarantee fairly quickly the existence of an associated subalgebra. This gives a new and easier proof of a result of Woodcock [W].

In general, the *existence* of these various types of Borel subalgebras is difficult to establish for many important classes of quasi-hereditary algebras. The remainder of the paper takes up this existence question. Thus, in Section 5, we prove that exact Borel subalgebras exist in the context of the algebras arising from the representation theory of infinitesimal group schemes and quantum enveloping algebras. Although our methods are different here than König's [K1], the results mirror a similar result of his for the category $\mathcal{O}$ associated to a complex simple Lie algebra.[2]

Section 6 develops some necessary material, largely due to van der Kallen [vdK] in the algebraic group case, for quantum enveloping algebras, which are then applied in Section 7 to prove that the quasi-hereditary algebras associated to quantum groups at a root of unity have excellent Borel subalgebras. (Of course, as a special case, we obtain similar results in the classical algebraic group case.) Finally, Section 8 contains a few final remarks concerning Borel subalgebras in the case of q-Schur algebras.

This paper suggests several interesting questions. First, we raise here the question of the existence of excellent Borel subalgebras for some of the quasi-hereditary algebras which play an important role in the non-describing characteristic representation theory of finite groups of Lie type in type different from type A. One speculates that such a theory would require some kind of "Schubert theory" for these algebras (along the lines of [vdK], e. g., do there exist suitable "Joseph modules"?) Even in type A, there is no *combinatorial* proof that the usual Borel subalgebras of Schur or q-Schur algebras are excellent, or even homological, though these properties do hold. Second, for those quasi-hereditary algebras connected to perverse sheaves, it might be highly interesting to interpret geometrically the existence of various types of Borel subalgebras.

1. Preliminaries

Throughout this paper, k is a fixed field, which for convenience is algebraically closed. We will work with various abelian categories $\mathcal{C}$ over k. We assume that there is given a set $\Lambda \equiv \Lambda(\mathcal{C})$ — the set of *weights* of $\mathcal{C}$ — indexing the distinct isomorphism classes of simple objects of $\mathcal{C}$. For $\lambda \in \Lambda$, write $L(\lambda)$ for a fixed representative from the corresponding isomorphism class of irreducible objects. The category $\mathcal{C}$ is generally assumed to have both enough injective and enough projective objects. Let $I(\lambda)$ (resp., $P(\lambda)$) denote the injective hull (resp., projective cover) of $L(\lambda)$. There will often be several categories in play at any one time, so that we write $L(\mathcal{C},\lambda)$, $P(\mathcal{C},\lambda)$, etc. in place of $L(\lambda)$, $P(\lambda)$, etc. when $\mathcal{C}$ needs to be mentioned.

As is well-known, if $\mathcal{C}$ is a k-finite category having finitely many simple objects, then $\mathcal{C} \cong A$-mod for a (non-uniquely) determined finite dimensional algebra A over k. In the case $\mathcal{C} = A$-mod (finite dimensional, left modules) or mod-A (finite dimensional, right modules) for a k-algebra A, we may write $L(A,\lambda)$, $P(A,\lambda)$, etc. in place of $L(\lambda)$, $P(\lambda)$, etc.

[2]Actually, our methods can be adapted to give a more conceptual approach to the category $\mathcal{O}$ situation.

We will consider *highest weight categories* $\mathcal{C}$ over k, in the sense of [CPS3]. In this case, Λ comes equipped with a poset structure $\leq$. *We will always assume that $\mathcal{C}$ is k-finite with finite weight poset Λ unless explicitly stated otherwise.*[3] If $\mathcal{C}$ is a highest weight category with weight poset $(\Lambda, \leq)$, let $\Delta(\lambda)$ (resp., $\nabla(\lambda)$) denote the standard (resp., costandard) object associated to λ. Thus, $L(\lambda)$ is isomorphic to the head (resp., socle) of $\Delta(\lambda)$ (resp., $\nabla(\lambda)$). Given $\lambda \in \Lambda$, $\Delta(\lambda)$ (resp., $\nabla(\lambda)$) can be characterized as the largest quotient (resp., subobject) of the projective cover $P(\lambda)$ (resp., injective envelope $I(\lambda)$) of $L(\lambda)$ having composition factors isomorphic to $L(\mu)$ with $\mu \leq \lambda$. Also, $P(\lambda)$ (resp., $I(\lambda)$) has a decreasing (resp., increasing) filtration with top (resp., bottom) section isomorphic to $\Delta(\lambda)$ (resp., $\nabla(\lambda)$) and with lower (resp., higher) sections isomorphic to $\Delta(\mu)$ (resp., $\nabla(\mu)$) for some $\mu > \lambda$.

A module category $\mathcal{C} = A$-mod for a finite dimensional algebra is a highest weight category (relative to some poset structure on Λ) if and only if A is quasi-hereditary [CPS3; (3.6)]. Given the quasi-hereditary algebra A, the standard and costandard objects in $\mathcal{C} = A$-mod depend on the chosen poset structure $\leq$ on Λ. When this fact needs to be emphasized we may write $\Delta(A, \leq, \lambda)$, $\nabla(A, \leq, \lambda)$ for these objects as a way of indicating their dependence on $\leq$. We will also sometimes say that "A is quasi-hereditary (with poset $(\Lambda, \leq)$)" to mean that $\leq$ is a given poset structure on the weight set Λ of A (i. e., of A-mod) with respect to which A-mod has the structure of a highest weight category.

An ideal Γ in the weight poset Λ of a highest weight category $\mathcal{C}$ determines a full subcategory $\mathcal{C}[\Gamma]$ consisting of objects having composition factors $L(\gamma)$, $\gamma \in \Gamma$. Then $\mathcal{C}[\Gamma]$ is again a highest weight category with weight poset Γ. For $\gamma \in \Gamma$, $\nabla(\mathcal{C}[\Gamma], \gamma) \cong \nabla(\mathcal{C}, \gamma)$ and $\Delta(\mathcal{C}[\Gamma], \gamma) \cong \Delta(\mathcal{C}, \gamma)$. The inclusion functor $i_*: \mathcal{C}[\Gamma] \to \mathcal{C}$ admits both a left adjoint i^* and a right adjoint $i^!$ satisfying $i^* i_* \cong i^! i_* \cong \mathrm{id}_{\mathcal{C}[\Gamma]}$. For $M, N \in \mathrm{Ob}(\mathcal{C}[\Gamma])$, we always have an equality $\mathrm{Ext}^\bullet_{\mathcal{C}[\Gamma]}(M, N) \cong \mathrm{Ext}^\bullet_{\mathcal{C}}(i_* M, i_* N)$; see [CPS3; §3]. A very useful consequence (often used without explicit mention) is that if λ is maximal in Γ, then $\Delta(\lambda)$ (resp., $\nabla(\lambda)$), when regarded as an object in $\mathcal{C}[\Gamma]$, becomes projective (resp., injective), i. e., $\Delta(\mathcal{C}, \lambda) \cong P(\mathcal{C}[\Gamma], \lambda)$ and $\nabla(\mathcal{C}, \lambda) \cong I(\mathcal{C}[\Gamma], \lambda)$ (allowing a slight abuse of notation).

If $\Omega \subset \Lambda$ is a coideal, the quotient category $\mathcal{C}(\Omega)$ of $\mathcal{C}$ by the Serre subcategory $\mathcal{C}[\Lambda \backslash \Omega]$ is again a highest weight category with weight poset Ω. If $j^*: \mathcal{C} \to \mathcal{C}(\Omega)$ denotes the quotient functor, then j^* carries $L(\mathcal{C}, \omega), \nabla(\mathcal{C}, \omega), \Delta(\mathcal{C}, \omega), I(\mathcal{C}, \omega)$, and $P(\mathcal{C}, \omega)$ to their corresponding objects in $\mathcal{C}(\Omega)$ for all $\omega \in \Omega$. Also, j^* admits both a left adjoint $j_!$ and a right adjoint j_* giving sections for j^*, in the sense that $j^* j_! \cong j^* j_* \cong \mathrm{id}_{\mathcal{C}(\Omega)}$.[4]

Fix a highest weight category $\mathcal{C}$ having weight poset $(\Lambda, \leq)$. We can consider the

[3] For many definitions and constructions, the finiteness of Λ is assumed only for convenience. In § 5, we consider some k-finite highest weight categories with infinite weight posets, and we are able, effectively, to use the same definition.

[4] A general discussion of quotient categories can be found in [F; §15]. In this paper, we consider — except briefly in §4 — categories $\mathcal{C}$ equivalent to the category A-mod of finite dimensional modules for a finite dimensional algebra. The relevant quotient categories all arise in the following way: Let $\Gamma \subseteq \Lambda$ be a set of irreducible A-modules, and let $\mathcal{S} = \mathcal{C}[\Gamma]$ be the full

various poset structures $\leq'$ on Λ relative to which $\mathcal{C}$ is a highest weight category having the same ∇-objects and Δ-objects as defined by our original partial ordering $\leq$. Among these partial orderings there is a unique minimal partial ordering $\leq_{\rm min}$, which is generated by the preorder $\leq_{\rm pre}$ on Λ obtained by putting $\lambda \leq_{\rm pre} \mu$ if and only if $L(\lambda)$ is a composition factor of $\nabla(\mu)$ or of $\Delta(\mu)$. The multiplicity equalities $[P(\lambda) : \Delta(\mu)] = [\nabla(\mu) : L(\lambda)]$ and $[I(\lambda) : \nabla(\mu)] = [\Delta(\mu) : L(\lambda)]$ given in [CPS3; (3.11)] show that each $P(\lambda)$ (resp., $I(\lambda)$) has a filtration in which the $\Delta(\mu)$-sections (resp., $\nabla(\mu)$-sections) are correctly ordered relative to $\leq_{\rm min}$.

In practice, it is often possible to generate $\leq_{\rm min}$ from the preorder $\leq'_{\rm pre}$ defined by putting $\mu \leq'_{\rm pre} \lambda$ provided the multiplicity $[\nabla(\lambda) : L(\mu)]$ of $L(\mu)$ as a composition factor of $\nabla(\lambda)$ is not zero. Obviously, this occurs when the following condition holds:

(1.1) Hypothesis. *For any* $\lambda, \mu \in \Lambda$, *if* $[\Delta(\lambda) : L(\mu)] \neq 0$, *then* $[\nabla(\lambda) : L(\mu)] \neq 0$.

Of course, the "dual" Hypothesis $(1.1)^{\rm op}$, obtained by reversing the roles of the Δ and ∇-objects in (1.1), holds. However, (1.1) will usually be sufficient for our purposes. Both (1.1) and $(1.1)^{\rm op}$ hold provided the category $\mathcal{C}$ has a strong duality in the sense of [CPS4], so that any $\Delta(\lambda)$ has the same image in the Grothendieck group of $\mathcal{C}$ as $\nabla(\lambda)$. Another important situation in which (1.1) holds often occurs when A is realized as $A = \widetilde{A} \otimes_{\mathcal{O}} k$, where $\widetilde{A}$ is an integral $\mathcal{O}$-quasi-hereditary algebra [CPS5] for a (local) commutative ring $\mathcal{O}$ with A_K split semisimple over the quotient field K of $\mathcal{O}$. For the argument, see the discussion of "Brauer theory" in [DPS1,2].[5]

Recall that a k-finite category $\mathcal{C}$ is (left) *directed* with respect to a poset structure $\leq$ on its weight set Λ provided

$$\mathrm{Ext}^1_{\mathcal{C}}(L(\lambda), L(\mu)) \neq 0 \Longrightarrow \lambda < \mu.$$

If $\mathcal{C}$ is directed by $\leq$, it is a highest weight category with respect to the poset $(\Lambda, \leq)$ with standard objects $\Delta(\lambda) \cong L(\lambda)$ and costandard objects $\nabla(\lambda) \cong I(\lambda)$. Any Serre subcategory $\mathcal{S}$ of a directed category $\mathcal{C}$ is directed by restricting $\leq$ to the weight set $\Lambda' \subset \Lambda$ of $\mathcal{S}$. Also, the quotient category $\mathcal{C}/\mathcal{S}$ is directed by restricting $\leq$ to the weight set $\Lambda \backslash \Lambda'$ of $\mathcal{C}/\mathcal{S}$.

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subcategory of all A-modules having composition factors $L(\gamma)$, for $\gamma \in$, . Let $e : A \to A$ be the idempotent projection onto $\bigoplus_{\lambda \notin \Gamma} P(A, \lambda)$. Then the quotient category $\mathcal{C}/\mathcal{S}$ identifies with eAe-mod with quotient functor $j^* : \mathcal{C} \to \mathcal{C}/\mathcal{S}$ given by $M \mapsto eM$, $M \in \mathrm{Ob}(\mathcal{C})$. We have $j_! = A \otimes_{eAe} -$ and $j_* = \mathrm{Hom}_{eAe}(Ae, -)$. The set $\Lambda \backslash$, indexes the irreducible eAe-modules (which can be taken as $eL(A, \lambda)$, $\lambda \in \Lambda \backslash$,). If $\lambda \notin$, , it is easy to see that $j^* I(A, \lambda) \cong I(eAe, \lambda)$ and $j^* P(A, \lambda) \cong P(eAe, \lambda)$. See [F; (15.22)] which also develops the theory in a generality sufficient for the application in §4 below to highest weight categories with infinite posets.

[5] The hypothesis that $\mathcal{O}$ have Krull dim. ≤ 2 assumed in [DPS1]. We can instead use the fact that the $\widehat{\mathcal{O}}$-versions of $\Delta(\lambda)$, $\nabla(\lambda)$ are both $\widehat{\mathcal{O}}$-free lattices in the same irreducible $A_{\widehat{K}}$-module. Here $\widehat{\mathcal{O}}$ is the completion of $\mathcal{O}$, $\widehat{K}$ is the quotient field of $\widehat{\mathcal{O}}$. Passage to $\widehat{\mathcal{O}}$ is only required to assume that $\widehat{\mathcal{O}}$-versions of $\Delta(\lambda)$ and $\nabla(\lambda)$ exist.

Finally, recall that the ∇ and Δ-objects have simple homological characterizations. Namely, suppose that $M \in \mathrm{Ob}(\mathcal{C})$ and $\lambda \in \Lambda$. Then:

$$\text{(1.2)} \qquad \dim \mathrm{Ext}^n_{\mathcal{C}}(M, \nabla(\nu)) = \delta_{n,0}\delta_{\lambda,\nu}, \ \forall\, n \geq 0, \ \nu \in \Lambda \iff M \cong \Delta(\lambda).$$

The " $\Longrightarrow$ " implication is trivial: Let $M = \Delta(\lambda)$. Replacing $\mathcal{C}$ by $\mathcal{C}[,]$ for an ideal , containing both λ and ν and in which λ or μ is maximal, we can assume that $\Delta(\lambda)$ is projective or $I(\nu)$ is injective. Thus, the right hand side of (1.2) holds for positive n, while it trivially holds for $n = 0$. Conversely, the vanishing of $\mathrm{Ext}^1_{\mathcal{C}}(M, \nabla(\nu))$ for all $\nu \in \Lambda$ is well-known to imply that M has a filtration with sections of the form $\Delta(\lambda)$. Then from the $n = 0$ case, we obtain that $M \cong \Delta(\lambda)$ for some λ. (A simple direct proof for the " $\Longrightarrow$ " direction can be based on elementary (1.4) below, together with a dimension shift argument.) The evident dual statement characterizes ∇-objects.

If $\mathcal{C}$ is a highest weight category, let $D^b(\mathcal{C})$ be the corresponding bounded derived category. (We can usually work with $D^b(\mathcal{C})$, rather than $D^-(\mathcal{C})$ or $D^+(\mathcal{C})$ because $\mathcal{C}$ has finite global dimension.) A variation on (1.2) remains valid for some objects in $D^b(\mathcal{C})$. Let $D^{b,\leq 0}(\mathcal{C})$ (resp., $D^{b,\geq 0}(\mathcal{C})$) denote the full subcategory of $D^b(\mathcal{C})$ having objects isomorphic (in the sense of derived categories) to complexes concentrated in non-positive (resp., non-negative) degrees. We have the following lemma.

(1.3) Lemma. (1) *If $X \in \mathrm{Ob}(D^{b,\leq 0}(\mathcal{C}))$ and $\mathrm{Hom}^n_{D^b(\mathcal{C})}(X, \nabla(\nu)) = 0$, for all $\nu \in \Lambda$ and all $n > 0$, then X is isomorphic to an object in $\mathcal{C}$ which has a filtration with sections of the form $\Delta(\lambda)$, $\lambda \in \Lambda$.*

(2) *If $X \in \mathrm{Ob}(D^{b,\geq 0}(\mathcal{C}))$ and $\mathrm{Hom}^n_{D^b(\mathcal{C})}(\Delta(\nu), X) = 0$ for all $\nu \in \Lambda$ and all $n > 0$, then X is isomorphic to an object in $\mathcal{C}$ which has a filtration with sections of the form $\nabla(\lambda)$, $\lambda \in \Lambda$.*

Proof. We will only prove (1); a dual argument will establish (2). Suppose that $X \in \mathrm{Ob}(D^{b,\leq 0}(\mathcal{C}))$ is not isomorphic to an object in $\mathcal{C}$. A standard truncation argument implies that X is isomorphic to a complex $\cdots \to 0 \to M^{-n} \xrightarrow{d^{-n}} M^{-n+1} \to \cdots$ in which $n > 0$ and $\mathrm{Ker}\, d^{-n} \neq 0$. Thus, for some injective object I in $\mathcal{C}$, $\mathrm{Hom}^n_{D^b(\mathcal{C})}(X, I) \neq 0$. Since I has a filtration with ∇-sections, this means that $\mathrm{Hom}^n_{D^b(\mathcal{C})}(X, \nabla(\nu)) \neq 0$ for some $\nu \in \Lambda$. Thus, M is isomorphic to an object in $\mathcal{C}$. Since $\mathrm{Hom}^\bullet_{D^b(\mathcal{C})} = \mathrm{Ext}^\bullet_{\mathcal{C}}$ for objects in $\mathcal{C}$, M has the required filtration (using using the case $n = 1$). $\square$

We conclude this section with the following useful result.

(1.4) Lemma. *Let $\lambda \in \Lambda$. Then:*

(1) *Any $M \in \mathrm{Ob}(\mathcal{C})$ is a nonzero homomorphic image of $\Delta(\lambda)$ if and only if $\dim \mathrm{Hom}_{\mathcal{C}}(M, \nabla(\mu)) = \delta_{\lambda,\mu}$ for all $\mu \in \Lambda$.*

(2) *Any $M \in \mathrm{Ob}(\mathcal{C})$ is a submodule of $\nabla(\lambda)$ if and only if $\dim \mathrm{Hom}_{\mathcal{C}}(\Delta(\mu), M) = \delta_{\lambda,\mu}$ for all $\mu \in \Lambda$.*

Proof. We will prove (1), and leave the dual argument for (2) to the reader. Clearly, any nonzero epimorphic image M of $\Delta(\lambda)$ satisfies the stated dimension condition. Conversely, suppose the dimension condition holds. If M has a composition factor $L(\nu)$ with $\nu \not\leq \lambda$, then there is a nonzero morphism $M \to I(\nu)$. But $I(\nu)$ has a filtration with bottom section $\nabla(\nu)$ and higher sections $\nabla(\omega)$, $\omega > \nu$. So $\mathrm{Hom}_{\mathcal{C}}(M, I(\nu)) = 0$, a contradiction. Thus, all the composition factors $L(\nu)$ of M satisfy $\nu \leq \lambda$. Since M clearly has lead $L(\lambda)$, it follows that M is an epimorphic image of $\Delta(\lambda)$, as required. □

2. Standard Borel categories and subalgebras

Let $\mathcal{C}$ be a fixed highest weight category with finite weight poset $(\Lambda^+, \leq^+)$. The "+"-notation is chosen because we view Λ^+ as analogous to a set of "dominant weights". In this and the next several sections, we consider various "Borel categories" for $\mathcal{C}$ (as well as associated algebras). This section considers the most general notion, which we call a (standard) Borel category associated to $\mathcal{C}$. As we shall see later, these Borel categories and their dual "costandard" variation occur commonly in algebraic and quantum groups (as well as in some Morita equivalent contexts). Although we are ultimately interested in algebras and their module categories, we obtain the best perspective and flexibility by setting up a categorical framework.

We begin with the following preliminary lemma.

(2.1) Lemma. *Let $\Psi : \mathcal{C} \to \mathcal{B}$ be an exact, additive functor from $\mathcal{C}$ to a highest weight category $\mathcal{B}$ with weight poset $(\Lambda, \leq)$. Assume that Λ^+ is a subset of Λ, and that $\leq^+$ is compatible with $\leq$ (i. e., $\lambda \leq^+ \mu \implies \lambda \leq \mu$ for all $\lambda, \mu \in \Lambda^+$). Also, assume that Ψ admits a left adjoint $\Psi_! : \mathcal{B} \to \mathcal{C}$. The following three statements are equivalent:*

(1) *For $\lambda \in \Lambda^+$, $\Psi\nabla(\mathcal{C}, \lambda)$ is isomorphic to a nonzero subobject of $\nabla(\mathcal{B}, \lambda)$.*

(2) *For $\mu \in \Lambda^+$, $\Psi_!\Delta(\mathcal{B}, \mu)$ is a nonzero homomorphic image of $\Delta(\mathcal{C}, \mu)$, and $\Psi_!\Delta(\mathcal{B}, \mu) = 0$ if $\mu \in \Lambda \setminus \Lambda^+$.*

(3) *For $\mu \in \Lambda$, we have*

$$\Psi_!\Delta(\mathcal{B}, \mu) \cong \begin{cases} \Delta(\mathcal{C}, \mu), & \mu \in \Lambda^+; \\ 0, & \mu \notin \Lambda^+. \end{cases}$$

Proof. (1) $\iff$ (2): By adjointness, for $\lambda \in \Lambda^+$ and $\mu \in \Lambda$, we have

$$\mathrm{Hom}_{\mathcal{C}}(\Psi_!\Delta(\mathcal{B}, \mu), \nabla(\mathcal{C}, \lambda)) \cong \mathrm{Hom}_{\mathcal{B}}(\Delta(\mathcal{B}, \mu), \Psi\nabla(\mathcal{C}, \lambda)).$$

Therefore, by (1.4), (1) implies that $\dim \mathrm{Hom}_{\mathcal{C}}(\Psi_!\Delta(\mathcal{B}, \mu), \nabla(\mathcal{C}, \lambda)) = \delta_{\lambda\mu}$, and (2) follows. Conversely, (2) implies $\dim \mathrm{Hom}_{\mathcal{B}}(\Delta(\mathcal{B}, \mu), \Psi\nabla(\mathcal{C}, \lambda)) = \delta_{\lambda\mu}$, and (1) follows.

(3) $\Longrightarrow$ (2) is trivial.

(1) $\Longrightarrow$ (3): Since now (2) holds, by dimension consideration, it suffices to show that there is a surjective homomorphism $\Psi_! \Delta(\mathcal{B}, \lambda) \to \Delta(\mathcal{C}, \lambda)$, for all $\lambda \in \Lambda^+$. We make the following preliminary claim:

($*$) If $\lambda \in \Lambda^+$, then $\Psi L(\mathcal{C}, \lambda)$ has socle isomorphic to $L(\mathcal{B}, \lambda)$, while all other composition factors $L(\mathcal{B}, \nu)$ satisfy $\nu < \lambda$.

If $\lambda \in \Lambda^+$ is minimal (w.r.t. $\leq^+$), then $\nabla(\mathcal{C}, \lambda) \cong L(\mathcal{C}, \lambda)$, so that $\Psi L(\mathcal{C}, \lambda) \cong \Psi \nabla(\mathcal{C}, \lambda)$ is a nonzero subobject of $\nabla(\mathcal{B}, \lambda)$, and ($*$) holds in this case. If λ is not minimal, induction implies that ($*$) holds for any composition factor of $\nabla(\mathcal{C}, \lambda)/L(\mathcal{C}, \lambda)$. Therefore, since $0 \neq \Psi \nabla(\mathcal{C}, \lambda) \subseteq \nabla(\mathcal{B}, \lambda)$, $\Psi L(\mathcal{C}, \lambda)$ has socle $L(\mathcal{B}, \lambda)$. Of course, the other composition factors $L(\mathcal{B}, \nu)$ must satisfy $\mu < \lambda$. So ($*$) holds for all $\lambda \in \Lambda^+$.

In particular, ($*$), together with the exactness of Ψ, implies, for all $\lambda \in \Lambda^+$, that $L(\mathcal{B}, \lambda)$ occurs with multiplicity one as a composition factor of $\Psi \Delta(\mathcal{C}, \lambda)$, while all other composition factors $L(\mathcal{B}, \nu)$ satisfy $\nu < \lambda$. By highest weight theory for $\mathcal{B}$, there is a nonzero map $\Delta(\mathcal{B}, \lambda) \to \Psi \Delta(\mathcal{C}, \lambda)$ which remains nonzero upon composition with the natural surjection $\Psi \Delta(\mathcal{C}, \lambda) \to \Psi L(\mathcal{C}, \lambda)$. By adjointness, there is a nonzero map $\Psi_! \Delta(\mathcal{B}, \lambda) \to \Delta(\mathcal{C}, \lambda)$ remaining nonzero upon composition with $\Delta(\mathcal{C}, \lambda) \twoheadrightarrow L(\mathcal{C}, \lambda)$. Hence, $\Psi_! \Delta(\mathcal{B}, \lambda)$ maps surjectively onto $\Delta(\mathcal{C}, \lambda)$, as required. $\square$

(2.2) Definition. A **Borel category** associated to the highest weight category $\mathcal{C}$ is a highest weight category $\mathcal{B}$ with weight poset $(\Lambda, \leq)$, together with a exact, additive functor $\Psi : \mathcal{C} \to \mathcal{B}$ satisfying the following conditions:

(B0) The functor Ψ admits a left adjoint $\Psi_! : \mathcal{B} \to \mathcal{C}$.

(B1) $\Lambda^+ \subseteq \Lambda$ and $\leq^+$ is compatible with $\leq$.

(B2) For $\lambda \in \Lambda^+$, $\Psi \nabla(\mathcal{C}, \lambda)$ is isomorphic to a nonzero subobject of $\nabla(\mathcal{B}, \lambda)$.

(B3) For $\lambda \in \Lambda^+$, $\Delta(\mathcal{B}, \lambda) \cong L(\mathcal{B}, \lambda)$. There is a second poset structure $\leq'$ on Λ making $\mathcal{B}$ into a directed category.

(2.3) Remark. (1) Observe that in the above definition, (2.1) implies that condition (B2) can be replaced by the condition given in (2.1(3)). Also, observe that condition (B3) implies that we have $\Psi_! L(\mathcal{B}, \lambda) \cong \Delta(\mathcal{C}, \lambda)$ for $\lambda \in \Lambda^+$, and $\Psi_! L(\mathcal{B}, \lambda) = 0$ for $\lambda \in \Lambda \setminus \Lambda^+$ (because $\Psi_! \Delta(\mathcal{B}, \lambda) = 0$ and $\Psi_!$ is right exact).

(2) It is usually possible to assume $\leq' = \leq$. Note that, since the highest weight category structure on $\mathcal{B}$ defined by the poset structure $\leq'$ on Λ is directed, we have $\nabla(\mathcal{B}, \leq, \lambda) \subseteq \nabla(\mathcal{B}, \leq', \lambda)$. Thus, if Hypothesis (1.1) holds for $\mathcal{C}$, then condition (2.2(3)) implies that $\leq^+_{\min}$ is compatible with $\leq'$. After replacing $\leq^+$ with $\leq^+_{\min}$ (thus keeping the same standard and costandard modules in $\mathcal{C}$), the highest weight category structure on $\mathcal{B}$ defined by $\leq'$ also defines $\mathcal{B}$ as a Borel category for $\mathcal{C}$ (with the same Ψ). Therefore, we may assume $\leq = \leq'$ in this case. However, even when

Hypothesis (1.1) does hold, two distinct poset structures can arise naturally, so it is useful to allow some flexibility in (2.2).

(3) Assume the set-up in (2.2), but that $\mathcal{B}$ is directed by $(\Lambda, \leq)$. Then $\mathcal{B}$ can be replaced by any full abelian subcategory $\mathcal{B}'$ containing the image of $\Psi : \mathcal{C} \to \mathcal{B}$, all objects $L(\mathcal{B}, \lambda)$ for $\lambda \in \Lambda^+$, and having enough injectives. In this case, the poset Λ' for $\mathcal{B}'$ consists of the subposet of Λ corresponding to the irreducible objects in $\mathcal{B}'$.

We now reformulate the notion of a Borel category in terms of algebras. Suppose that $\Psi : \mathcal{C} \to \mathcal{B}$ defines a Borel category (though we make no special assumption about the compatibility of $\leq^+$ with $\leq'$). We wish to realize $\mathcal{C}$ and $\mathcal{B}$ as module categories A-mod and B-mod, respectively, so that Ψ is defined by an algebra homomorphism $\iota : B \to A$. (In this case, we think of the pair (B, ι) as a *prealgebra* of A — of course, in ideal situations, ι will be injective and so identify B with a subalgebra of A.)

More generally, if $\iota : B \to A$ is a homomorphism of finite dimensional algebras, let $\iota^* : A\text{-mod} \to B\text{-mod}$ be the pull-back functor (i. e., if M is an A-module, then $\iota^* M$ is the B-module obtained by making B act on M through the morphism ι). The functor ι^* admits a right adjoint $\iota_* = \mathrm{Hom}_B(A, -)$ and a left adjoint $\iota_! = A \otimes_B -$.

(2.4) Lemma. *Let $\Psi : \mathcal{C} \to \mathcal{B}$ define $\mathcal{B}$ as a Borel category in the sense of (2.2). Let P be a projective generator for $\mathcal{B}$, and set $B = \mathrm{End}_{\mathcal{B}}(P)^{\mathrm{op}}$, $A = \mathrm{End}_{\mathcal{C}}(\Psi_! P)^{\mathrm{op}}$. Let $\iota : B \to A$ be defined by $\iota(b) = \Psi_!(b)$. Then:*

(1) *$Q = \Psi_! P$ is a projective generator for $\mathcal{C}$.*

(2) *The algebras A and B are quasi-hereditary.*

(3) *Identifying $\mathcal{C}$ to A-mod by means of the equivalence $\mathrm{Hom}_{\mathcal{C}}(Q, -) : \mathcal{C} \to A\text{-mod}$ and identifying $\mathcal{B}$ to B-mod by means of the equivalence $\mathrm{Hom}_{\mathcal{B}}(P, -) : \mathcal{B} \to B\text{-mod}$, the functor Ψ identifies with ι^*, while $\Psi_!$ identifies with $\iota_!$.*

Proof. Let $\lambda \in \Lambda^+$. Applying $\Psi_!$ to the natural surjection $P(\mathcal{B}, \lambda) \to \Delta(\mathcal{B}, \lambda)$ yields a surjective map $\Psi_! P(\mathcal{B}, \lambda) \to \Psi_! \Delta(\mathcal{B}, \lambda)$. But $\Psi_! \Delta(\mathcal{B}, \lambda) \cong \Delta(\mathcal{C}, \lambda)$, while $\Psi_! P(\mathcal{C}, \lambda)$ is a projective object in $\mathcal{C}$ (because $\Psi_!$ has an exact right adjoint Ψ). Therefore, $Q = \Psi_! P$ maps surjectively onto $\Delta(\mathcal{C}, \lambda)$ and hence also maps onto $L(\mathcal{C}, \lambda)$. It follows that $P(\mathcal{C}, \lambda)$ is a direct summand of Q, for all $\lambda \in \Lambda^+$, and so Q is a projective generator for $\mathcal{C}$. This proves (1).

By Morita theory, $\mathrm{Hom}_{\mathcal{C}}(Q, -) : \mathcal{C} \to A\text{-mod}$ and $\mathrm{Hom}_{\mathcal{B}}(P, -) : \mathcal{B} \to B\text{-mod}$ are equivalences of categories, so that statement [CPS3; (4.6)] implies (2).

Now (3) is a direct verification; we give the argument for completeness. The equivalence $\mathcal{C} \xrightarrow{\sim} A\text{-mod}$ is given by $M \mapsto \mathrm{Hom}_{\mathcal{C}}(\Psi_! P, M)$ for $M \in \mathrm{Ob}(\mathcal{C})$. Similarly, $N \mapsto \mathrm{Hom}_{\mathcal{B}}(P, N)$, $N \in \mathrm{Ob}(\mathcal{B})$, defines an equivalence $\mathcal{B} \xrightarrow{\sim} B\text{-mod}$. Therefore, $\mathrm{Hom}_{\mathcal{C}}(\Psi_! P, M) \mapsto \mathrm{Hom}_{\mathcal{B}}(P, \Psi M)$, $M \in \mathrm{Ob}(\mathcal{C})$, defines an exact functor $j^*: A\text{-mod} \to B\text{-mod}$. We claim that if $\mathrm{Hom}_{\mathcal{C}}(\Psi_! P, M)$ is regarded as a B-module using the homomorphism $\iota: B \to A$, then the isomorphism φ_M: $\mathrm{Hom}_{\mathcal{C}}(\Psi_! P, M) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{B}}(P, \Psi M)$ is a B-module isomorphism. For $M \in \mathrm{Ob}(\mathcal{C})$ and $N \in \mathrm{Ob}(\mathcal{B})$, let $\tau_M: \Psi_!\Psi M \to M$ and $\sigma_N: N \to \Psi\Psi_! N$ be the adjunction maps. Then φ_M is given by $\varphi_M(f) = \Psi(f)\sigma_P$ for $f \in \mathrm{Hom}_{\mathcal{C}}(\Psi_! P, M)$. Also, φ_M has inverse

$$\Psi_M: \mathrm{Hom}_{\mathcal{B}}(P, \Psi M) \to \mathrm{Hom}_{\mathcal{C}}(\Psi_! P, M), \quad g \mapsto \tau_M \Psi_!(g).$$

To show, for $f \in \mathrm{Hom}_{\mathcal{C}}(\Psi_! P, M)$, that $\varphi_M(f\Psi_!(b)) = \varphi_M(f)b$, for all $b \in B$, it suffices to check that

$$\begin{array}{ccc} P & \xrightarrow{\;b\;} & P \\ {\scriptstyle\sigma_P}\downarrow & & \downarrow{\scriptstyle\sigma_P} \\ \Psi\Psi_! P & \xrightarrow[\Psi\Psi_!(b)]{} & \Psi\Psi_! P \end{array}$$

is commutative. Since $\tau_{\Psi_! P}\Psi_!(\sigma_P) = \mathrm{id}$, we have

$$\begin{aligned} \sigma_P b &= \varphi_{\Psi_! P}\Psi_{\Psi_! P}(\sigma_P b) = \varphi_{\Psi_! P}(\tau_{\Psi_! P}\Psi_!(\sigma_P)\Psi_!(b)) \\ &= \varphi_{\Psi_! P}(\Psi_!(b)) = \Psi\Psi_!(b)\sigma_P, \end{aligned}$$

as desired. Thus $j^* \cong \iota^*$, i. e., j^* is induced by pull-back through $\iota: B \to A$. The other assertions of (3) are clear. □

In practice, the morphism $\iota : B \to A$ defined in (2.4c) need not be an injection. However, assume that Hypothesis (1.1) holds for $\mathcal{C}$. Then, as discussed above, we can assume that $\leq = \leq'$. Then, using (2.3), we can replace B by its image $B' = \iota(B)$ in A and replace Λ by the subposet indexing the irreducible B'-modules. Thus, A-mod $\to B'$-mod defines a Borel category which is realized as the module category of a subalgebra of A. For this reason, the remainder of this section focuses on subalgebras, though more general prealgebras could easily be considered. Throughout the discussion, A will be a quasi-hereditary algebra (with poset $(\Lambda^+, \leq^+)$). Usually, B will be a quasi-hereditary subalgebra (with poset $(\Lambda, \leq)$). So as not to be too pedantic, we usually will omit explicit mention of the associated posets $(\Lambda^+, \leq^+)$ and $(\Lambda, \leq)$.

(2.5) Definition. Let A be a quasi-hereditary algebra (with poset $(\Lambda^+, \leq^+)$). Let B be a quasi-hereditary subalgebra (with weight poset $(\Lambda, \leq)$) of A, and let $\Psi = |_B : A$-mod $\to B$-mod be the natural restriction (or pull-back) functor induced by the inclusion map $\iota : B \to A$. Then we say that B is a (standard) **Borel subalgebra** of A provided Ψ defines B-mod as a Borel category associated to A-mod in the sense of (2.2). (Observe that the left adjoint $\Psi_!$ must identify with tensor induction $A \otimes_B -$.)

In the case $\Psi : A$-mod $\to B$-mod is the restriction functor, the condition (B0) in (2.2) is automatic: the tensor induction $A \otimes_B -$ provides a left adjoint to Ψ.

Therefore, to verify whether B is a Borel subalgebra of A, we need only check conditions (B1)–(B3) in (2.2) with $\mathcal{C} = A$-mod and $\mathcal{B} = B$-mod.

If $\preceq$ is a poset structure on the weight set Λ^+ of a quasi-hereditary algebra A, a *$\preceq$-adapted listing* of Λ^+ consists of a listing $\lambda_1, \cdots, \lambda_n$ of the set Λ^+ such that $\lambda_i \preceq \lambda_j$ implies $i \geq j$. In this case, there exists a defining sequence

$$(2.6) \qquad 0 = J_0 \subseteq J_1 \subseteq \cdots \subseteq J_n = A$$

of idempotent ideals of A (in the sense of [CPS3; §3]) such that the A/J_{i-1}-module J_i/J_{i-1} is a direct sum of copies of $\Delta(A, \lambda_i) = \Delta(A, \preceq, \lambda_i)$ (and is a projective left or right A/J_{i-1}-module). We call (2.6) a *$\preceq$-adapted* defining sequence of A. In case B is a Borel subalgebra of A, we say that (2.6) is $\leq$-adapted provided it is $\leq|_{\Lambda^+}$-adapted. (Here $(\Lambda, \leq)$ is the weight poset for B-mod in (2.5).) Because $\leq^+$ is compatible with $\leq$, any $\leq$-adapted defined sequence is also $\leq^+$-adapted.

With this terminology, we can establish the following result which provides some necessary and sufficient conditions for a subalgebra B of A to be a Borel subalgebra.

(2.7) Theorem. *Let A be a quasi-hereditary algebra (with weight poset $(\Lambda^+, \leq^+)$. Let $\{\lambda_1, \cdots, \lambda_n\}$ be a $\leq$-adapted listing of Λ^+, and $0 = J_0 \subseteq J_1 \subseteq \cdots \subseteq J_n = A$ be the corresponding $\leq$-adapted defining sequence of A. Then:*

(1) *Suppose B is a Borel subalgebra of A as per (2.5). For each i, let e_i be a primitive idempotent of B satisfying $e_i L(B, \lambda_i) \neq 0$. Then $e_i \in J_i$, $e_i \notin J_{i-1}$, and the following equalities hold:*

$$(2.7.1) \qquad e_i B + J_{i-1} = e_i A + J_{i-1},$$

$$(2.7.2) \qquad e_i B e_i + J_{i-1} = e_i A e_i + J_{i-1}.$$

(2) *Assume the notation in (1). Let $\leq'$ be the poset structure on Λ which directs B-mod, and assume that $\leq^+$ is compatible with $\leq'$. Then there is an isomorphism*

$$(2.7.3) \qquad Be_i/(Be_i \cap J_{i-1}) \cong (Be_i + J_{i-1})/J_{i-1} \xrightarrow{\sim} L(B, \lambda_i).$$

(3) *Conversely, let B be a directed subalgebra of A satisfying (2.7.1) for primitive idempotents $e_1, \cdots, e_n \in B$. Assume each $e_i \in J_i \backslash J_{i-1}$. Suppose that A satisfies Hypothesis (1.1), and that $\leq^+ = \leq^+_{\min}$. Let $\leq = \leq_{\min}$ direct B-mod. Then B (with poset $(\Lambda, \leq)$) is a Borel subalgebra of A (with poset $(\Lambda^+, \leq^+)$).*

Proof. We first prove (1). Fix i and let $e_i \in B$ be a primitive idempotent such that $e_i L(B, \lambda_i) \neq 0$. By $(*)$ in the proof of (2.1),

$$[\Psi L(A, \lambda_i) : L(B, \nu)] \neq 0 \implies \nu \leq \lambda_i.$$

Also, $L(B, \lambda_i)$ occurs as a multiplicity 1 composition factor in the socle of $\Psi L(A, \lambda_i)$. Thus, the isomorphism $\mathrm{Hom}_A(\Psi_! Be_i, L(A, \lambda_j)) \cong \mathrm{Hom}_B(Be_i, \Psi L(A, \lambda_j))$ shows that the head of $Ae_i = \Psi_! Be_i$ contains $L(A, \lambda_i)$ with multiplicity one, together with summands $L(A, \lambda_j)$ with $j < i$. Thus, $(A/J_i)e_i = 0$. In particular, $e_i \in J_i$. Also, $(A/J_{i-1})e_i$ has a simple head $L(A, \lambda_i)$, so the image of e_i in A/J_{i-1} is a primitive idempotent (thus, $e_i \notin J_{i-1}$).

It follows that $(A/J_{i-1})e_i \cong \Delta(A, \lambda_i)$, while $e_i(A/J_{i-1}) \cong \nabla(A, \lambda_i)^*$ (k-linear dual). By property (2.2(3)), the restriction $\Psi\nabla(A, \lambda_i) = \nabla(A, \lambda_i)|_B$ is isomorphic to a B-submodule of $\nabla(B, \lambda_i)$, and hence to a B-submodule of $I(B, \lambda_i) \cong (e_iB)^*$. So, $e_i(A/J_{i-1})$ is a right B-module homomorphic image of e_iB, and hence of

$$e_iB/(e_iB \cap J_{i-1}) \cong e_i((B + J_{i-1})/J_{i-1}).$$

However, $e_i((B + J_{i-1})/J_{i-1})$ is a B-submodule of $e_i(A/J_{i-1})$. Thus, (2.7.1) must hold, and (2.7.2) follows immediately by multiplying (2.7.1) on the right by e_i. This proves (1).

To prove (2), we need to verify that (2.7.3) holds under the assumption that $\leq^+$ is compatible with the poset structure $\leq'$ directing B-mod. The B-module $Be_i/(Be_i \cap J_{i-1})$ has head $L(B, \lambda_i) \cong \Delta(B, \lambda_i)$ and is isomorphic to a submodule of $(A/J_{i-1})e_i \cong \Delta(A, \lambda_i)|_B$. Because B-mod is directed, the composition factors $L(B, \nu)$ of $Be_i/(Be_i \cap J_{i-1})$ satisfy $\nu \geq' \lambda_i$. Also, $L(B, \lambda_i)$ has multiplicity one in Be_i, hence in $Be_i/(Be_i \cap J_{i-1})$. Fix a filtration of Be_i with top section $\Delta(B, \lambda_i) \cong L(B, \lambda_i)$ and with other sections of the form $\Delta(B, \nu)$, $\lambda < \nu \in \Lambda$. If $\Delta(B, \nu)$ is the first section in this filtration (counting from the bottom) with nonzero image $\bar{\Delta}(B, \nu)$ in $\Delta(A, \lambda_i)|_B$, $\Psi_! \bar{\Delta}(B, \nu)$ has nonzero image in $\Delta(A, \lambda_i)|_B$ (by adjointness), so $\Psi_! \Delta(B, \nu) \neq 0$ and $\nu \in \Lambda^+$. Therefore, $\nu \leq^+ \lambda_i$. Since $\leq^+$ is compatible with $\leq'$, we obtain that $\nu = \lambda_i$ and so (2.7.3) must hold. This completes the proof of (2).

Now assume the hypotheses of (3). We must verify that conditions (B1)–(B3) in (2.2) hold.

Because each $e_i \in B$ is primitive and B is directed, the algebra $e_iBe_i \cong k$. Thus, (2.7.2) implies that $e_i(A/J_{i-1})e_i \cong e_iBe_i \cong k$, so the image of e_i in A/J_{i-1} must also be primitive. Let $L(A, \lambda_i)$ be the corresponding irreducible A-module, i. e., $L(A, \lambda_i)$ is isomorphic to the head of $(A/J_{i-1})e_i$. We let $\lambda_i \in \Lambda$ also index the irreducible B-module corresponding to e_i, i. e., $e_iL(B, \lambda_i) \neq 0$. In this way, we regard Λ^+ as a subset of Λ. (Observe that the e_i are not equivalent, since the J_i are distinct.) Now condition (B3) holds, since we are assuming that $\leq = \leq_{\min}$ directs B, so that $\Delta(B, \lambda) \cong L(B, \lambda)$ even for all $\lambda \in \Lambda$.

Since $e_i \in J_i$, the module $e_i(A/J_{i-1}) = e_i(J_i/J_{i-1})$ is an indecomposable summand of the right A-module J_i/J_{i-1}. Thus, $e_i(A/J_{i-1}) \cong \nabla(A, \lambda_i)^*$. It follows from (2.7.1) again that $\Psi\nabla(A, \lambda_i)$ is a B-submodule of $(e_iB)^* \cong I(B, \lambda_i) = \nabla(B, \lambda_i)$. So (B2) holds.

It remains to prove, under the assumption of Hypothesis (1.1), that $\leq^+ = \leq^+_{\min}$ is compatible with $\leq$. But if $L(A, \mu)$ is a composition factor of $\nabla(A, \lambda)$, then, by (2.1), the socle $L(B, \mu)$ of $\Psi L(A, \mu)$ is a composition factor of $\Psi\nabla(A, \lambda)$, and hence of $\nabla(B, \lambda)$. Thus, $\mu \leq \lambda$, as required. □

(2.8) Remark. Suppose that in (2.7), it is assumed that the directed algebra B arises as a path algebra. (It is always naturally the image of a finite dimensional path algebra.) Then, under the hypotheses of (1), it is easy to see that the condition

$$e_iAe_j \not\subseteq J_j \implies e_jBe_i \neq 0 \tag{2.8.1}$$

holds for all i, j. Conversely, in this case, part (3) remains true if the condition that Hypothesis (1.1) holds is replaced by condition (2.7.1). We leave the easy details to the reader. Finally, we remark that (2.7) can be reformulated in terms of prealgebras. In that case, we could always assume that B is a path algebra. (Alternatively, even in the subalgebra case, one could replace the right hand side of (2.7.1) by the more complicated "path" condition: $\exists\ i_1, \cdots, i_n$ with $i_1 = j$, $i_n = i$ and $e_iBe_{i+1} \neq 0$ for all $i < n$.)

In the following corollary, we use the fact that if A is a quasi-hereditary algebra with weight poset $(\Lambda^+, \leq^+)$, then the opposite algebra $A^{\rm op}$ is also a quasi-hereditary algebra with weight poset $(\Lambda^+, \leq^+)$. If $\{J_\bullet\}$ is a $\leq^+$-adapted defining sequence for A, it remains a $\leq^+$-adapted defining sequence for $A^{\rm op}$.

(2.9) Corollary. *Let A be a qausi-hereditary algebra (with weight poset $(\Lambda^+, \leq^+)$). Suppose B and B^- are quasi-hereditary subalgebras of A which have the same weight poset $(\Lambda, \leq)$. Assume that B is a Borel subalgebra of A and that $(B^-)^{\rm op}$ is a Borel subalgebra of $A^{\rm op}$. Also, assume that $\{J_\bullet\}$ is a $\leq$-adapted defining sequence of A, and that there exist idempotents $e_i \in B \cap B^-$ which are primitive in both B and B^- and satisfy $e_iL(B, \lambda_i) \neq 0$, $e_iL((B^-)^{\rm op}, \lambda_i) \neq 0$.*

Then there is a "filtered triangular decomposition"

$$A = B^-B. \tag{2.9.1a}$$

More precisely, we have

$$A = \bigcup_i B^-e_iB. \tag{2.9.1b.}$$

Proof. By (2.7a), we have $e_iB + J_{i-1} = e_iA + J_{i-1}$ and $B^-e_i + J_{i-1} = Ae_i + J_{i-1}$ for all i. Thus,

$$B^-e_iB + J_{i-1} = Ae_iA + J_{i-1} \tag{2.9.2}$$

for all i. Since $J_0 = 0$ and $J_1 = Ae_1A$, (2.9.1b) follows immediately from (2.9.2). $\square$

(2.10) Remarks. (1) The above corollary is inspired by a similar result for Schur algebras proved by Green [G2]. In turn, Green's result was extended to q-Schur algebras in [PW; (11.6.1)]; see §8. Recently, Du and Rui [DR; (3.5], 5.5) have proved such a factorization for q-Schur^2-algebras. These algebras do not obviously arise from Lie theory, but are interesting for the representation theory of finite groups of Lie type in the non-describing characeristics. Dyer [Dy] proved a suggestive and similar factorization, in an entirely different context; but he also assumes a strong duality, so (2.9.1a) is again a consequence. (See §3.)

The following result establishes that any quasi-hereditary algebra A has a Borel subalgebra B. In the next section, we will show that, under suitable assumptions, nice properties of A imply that B can be chosen to have similar nice properties. In the situation below, we even obtain that the irreducible A-modules restrict to irreducible B-modules. Such Borel subalgebras are called *strong*; we will return to that notion below in §3.

(2.11) Theorem. *Let A be a quasi-hereditary algebra (with weight poset $(\Lambda^+, \leq^+)$). Assume that $\leq^+ = \leq^+_{\min}$ and that Hypothesis (1.1) holds. Then A has a Borel subalgebra B with same weight poset as A. The algebra B can be chosen so that $\Lambda = \Lambda^+$ and every irreducible A-module restricts to an irreducible B-module.*

Proof. First, assume that A is a basic algebra. Let $\lambda_1, \cdots, \lambda_n$ be a $\leq^+$-adopted listing of Λ^+, and form the associated $\leq^+$-adapted defining sequence (2.6) for A.

Let A_0 be a Wedderburn complement for $\mathrm{rad}(A)$ in A, and let $e_1, \cdots, e_n \in A_0$ be a corresponding complete set of primitive orthogonal idempotents, listed so that $\dim e_i L(A, \lambda_j) = \delta_{ij}$ for all i, j. Since A is basic, the e_i form a basis for the commutative algebra A_0. If $f_i = e_1 + \cdots + e_i$, then $J_i = Af_iA$.

The subalgebra

$$B = A_0 \oplus \sum_{i<j} e_i \,\mathrm{rad}(A) e_j$$

has weight set $\Lambda = \Lambda^+$. Since

$$\mathrm{Hom}_B(Be_i, Be_j) \cong e_i B e_j \cong \begin{cases} e_i \,\mathrm{rad}(A) e_j, & i < j, \\ k, & i = j, \\ 0, & i > j, \end{cases}$$

the composition factors $L(B, \lambda_i)$ of the radical of Be_j satisfy $i < j$. Hence, B-mod is directed by defining $\lambda_i \leq \lambda_j$ if and only if $i \geq j$. Replace $\leq$ by $\leq_{\min}$.

Now we verify that the equalities (2.7.1) hold. For any i, $(A/J_{i-1})e_i \cong \Delta(A, \lambda_i)$, so that $e_i(A/J_{i-1})e_i \cong \mathrm{Hom}_A(Ae_i, \Delta(A, \lambda_i)) \cong k$. Hence,

$$e_i A e_i \subseteq k e_i + J_{i-1}. \qquad (2.11.1)$$

Since

$$e_i B = k e_i + \sum_{j>i} e_i \,\mathrm{rad}(A) e_j,$$

we see that

$$\begin{aligned} e_iA &= ke_i + \sum_j e_i \operatorname{rad}(A)e_j \\ &= ke_i + \sum_{j>i} e_i \operatorname{rad}(A)e_j + \sum_{j\leq i} e_i \operatorname{rad}(A)e_j \\ &\equiv e_iB \mod J_{i-1}, \end{aligned}$$

by (2.11.1) and the fact that $e_j \in J_{i-1}$ for $j < i$. This proves that (2.7.1) holds. Hence, B is a Borel subalgebra of A. By construction, the irreducible A-modules restrict to irreducible B-modules.

Any algebra which is Morita equivalent to the basic algebra A has the form

$$A' = \operatorname{End}_A\left(\bigoplus (Ae_i)^{\oplus m_i}\right)^{\mathrm{op}}$$

for some sequence $m_1, \cdots, m_n$ of *positive* integers. (For example, when each $m_i = 1$, we recover A.) Then

$$B' = \operatorname{End}_B\left(\bigoplus (Be_i)^{\oplus m_i}\right)^{\mathrm{op}}$$

is a subalgebra of A' which is Morita equivalent to B. Thus, B' defines a Borel subalgebra of A', and the proof is complete. □

In the notation of (2.11), it follows similarly that the subalgebra

$$B^- = A_0 \oplus \sum_{i>j} e_i \operatorname{rad}(A)e_j$$

has the property that $(B^-)^{\mathrm{op}}$ is a Borel subalgebra of A^{op}. Observe that both B-mod and $(B^-)^{\mathrm{op}}$-mod are directed by defining $\lambda_i \leq \lambda_j$ if and only if $i \geq j$. Then $\leq^+$ is compatible with this poset structure. Trivially, we have $B^-B = A$.

We conclude this section with the following proposition. Although it was inspired by Theorem 2.7, it can, in fact, be proved directly.

(2.12) Proposition. *Suppose B is a Borel subalgebra (with weight poset $(\Lambda, \leq)$) of a quasi-hereditary algebra A (with weight poset $(\Lambda^+, \leq^+)$). Assume that Hypothesis 1.1 holds for A-mod. Then:*

(1) *Let $e \in B$ be an idempotent such that $eL(B, \lambda) \neq 0$ for each $\lambda \in \Lambda^+$. Then eAe is Morita equivalent to A, and eBe is a Borel subalgebra of eAe.*

(2) *There exists an idempotent $e \in B$ such that eBe is a Borel subalgebra (with weight poset $(\Lambda^+, \leq|_{\Lambda^+})$ of A (with the same weight poset $(\Lambda^+, \leq^+)$). We can even assume that eBe is basic.*

Proof. As discussed above (2.3), we may as well assume that $\leq^+ = \leq^+_{\min}$ and that the poset structure $\leq$ on Λ directs B-mod. We will prove (1); (2) is then an easy consequence from Morita theory.

Let $\mathcal{S}$ be the Serre subcategory of B-mod generated by the irreducible B-modules $L(B,\lambda)$ for which $eL(B,\lambda) \neq 0$. Then $B\text{-mod}/\mathcal{S} \cong eBe\text{-mod}$ (see footnote 4 above). Let $j^* : B\text{-mod} \to eBe\text{-mod}$ be the quotient functor. The functor $\Psi' = j^*\Psi : A\text{-mod} \to eBe\text{-mod}$ has left adjoint $\Psi'_j = \Psi_! j_!$, and it is easy to verify that the conditions (1)–(4) of (2.2) hold. $\square$

In the context of the above result, we will say that Borel category or subalgebra has *only dominant weights* provided $\Lambda = \Lambda^+$ as sets. We mention that this condition is necessary and sufficient for $\Psi_!$ to kill only zero objects. (Note that the functor Ψ never kills any nonzero object or map.) It can always be achieved by suitably choosing the idempotent e above. This provides a replacement for the flawed [K1; appendix, Thm. F].

3. Borel subalgebras and Kazhdan-Lusztig theory

In this section, we use the construction of Borel subalgebras given in Theorem 2.7 to obtain Borel subalgebras for an important class of quasi-hereditary algebras which arise in the representation theory of algebraic groups, quantum groups, etc. The main result, given in Theorem 3.5, is inspired by a similar (but slightly weaker) unpublished result of the first two authors and E. Cline — See Corollary 3.6. As another application, we deduce a result of Dyer [Dy], giving a "triangular" factorization of A. (Again, we prove a somewhat stronger result.)

Let $A = \bigoplus_{n\geq 0} A_n$ be a positively $\mathbb{Z}$-graded (finite dimensional) algebra over k, and assume that A_0 is semisimple. (In what follows, we will refer to such algebras simply as "graded" algebras.) Let $\mathcal{C}^{\text{gr}}$ be the category of finitely generated $\mathbb{Z}$-graded A-modules. If $M = \bigoplus_n M_n \in \text{Ob}(\mathcal{C}^{\text{gr}})$, then, for any integer i, the "twist" $M(i)$ is the graded A-module obtained from M by shifting the grading i steps to the right, e. g., $M(i)_n = M_{n-i}$.

Let Λ^+ be the set of weights of A-mod. Since $\text{rad}(A) = \bigoplus_{n>0} A_n$, we can regard any irreducible A-module $L(\lambda)$, $\lambda \in \Lambda^+$, as a graded A-module, concentrated in degree 0. Then the graded modules $L(\lambda)(i)$, $\lambda \in \Lambda^+$, $i \in \mathbb{Z}$, are representatives from the set of isomorphism classes of irreducible objects in $\mathcal{C}^{\text{gr}}$.

The reader should keep in mind the useful fact that, given $M, N \in \text{Ob}(\mathcal{C}^{\text{gr}})$, we have the isomorphism

$$\text{Ext}^\bullet_{\mathcal{C}}(M,N) \cong \bigoplus_{i\in\mathbb{Z}} \text{Ext}^\bullet_{\mathcal{C}^{\text{gr}}}(M, N(i)), \tag{3.1}$$

relating the "ungraded $\text{Ext}^\bullet$-groups" with the "graded $\text{Ext}^\bullet$-groups".

The graded algebra A is called *tightly graded* provided that A is isomorphic to the graded algebra $\text{gr}A = \bigoplus_i \text{rad}(A)^i/\text{rad}(A)^{i+1}$ obtained from the radical filtration of A. Equivalently, A_0 is semisimple and A is generated as an A_0-module by its term A_1 in degree 1. We will make use of the following very elementary result.

(3.2) Lemma. *Let A be a tightly graded algebra. Let L, L' be irreducible A-modules (viewed as graded A-modules concentrated in degree 0). Suppose $m, n \in \mathbb{Z}$ are integers such that $\mathrm{Ext}^1_{\mathcal{C}^{\mathrm{gr}}}(L'(m), L(n)) \neq 0$. Then $m + 1 = n$.*

Proof. After twisting, we can assume that $m = 0$. Let $0 \to L(n) \to E \to L' \to 0$ be a non-split graded extension of L' by $L(n)$. Take $0 \neq v \in E_0$ which maps to a nonzero element in $L' = L'_0$. Then $E = Av = A_0 v \oplus A_1 v \oplus \cdots$. Since A_1 generates A as an A_0 module, $A_1 v \neq 0$ and so the socle $L(n)$ of E must have degree $n = 1$. $\square$

The main result (3.5) below will assume that the following Hypothesis holds.

(3.3) Hypothesis. *Let $\mathcal{C} = A$-mod be a highest weight category with weight poset Λ^+. For any $\lambda, \mu \in \Lambda^+$, the "restriction map" (induced by the surjection $\Delta(\lambda) \twoheadrightarrow L(\lambda)$)*

(3.3.1) $$\mathrm{Ext}^1_A(L(\lambda), L(\mu)) \to \mathrm{Ext}^1_A(\Delta(\lambda), L(\mu)) \quad \textit{is surjective.}$$

(3.4) Remark. In order to motivate the above hypothesis we recall some results from [CPS5, CPS6, CPS8, CPS9]. Again, let $\mathcal{C}$ be a highest weight category with weight poset Λ^+. Let $\ell : \Lambda^+ \to \mathbb{Z}$ be a function (the "length function" on weights). We say that $\mathcal{C}$ has a Kazhdan-Lusztig theory (relative to ℓ) provided that, given $\lambda, \mu \in \Lambda^+$, the non-vanishing of either $\mathrm{Ext}^n_{\mathcal{C}}(\Delta(\lambda), L(\mu))$ or $\mathrm{Ext}^n_{\mathcal{C}}(L(\lambda), \nabla(\mu))$ implies that $\ell(\lambda) - \ell(\mu) \equiv n \pmod 2$.

In case $\mathcal{C}$ has a Kazhdan-Lusztig theory, the surjectivity of the maps (3.3.1) holds for all weights λ, μ [CPS6; (4.3)]. (An analogue even holds for all n.) In fact, in all the standard examples arising in the representation theory of algebraic groups, quantum groups, and the category $\mathcal{O}$, the surjectivity condition (3.3.1) essentially *implies* the existence of a Kazhdan-Lusztig theory; see [CPS6]. This fact will hold in situations in which there is a suitably rich supply of "Hecke operators."

Now suppose that $\mathcal{C} = A$-mod and that the algebra A is graded as above. Then $\mathcal{C}^{\mathrm{gr}}$ is a *graded* highest weight category in the sense of [CPS5]. In this setting, it is also possible to consider pairity conditions along the lines of those expressed by a Kazhdan-Lusztig theory for $\mathcal{C}$. Namely, we say $\mathcal{C}^{\mathrm{gr}}$ has a *graded* Kazhdan-Lusztig theory relative to the length function $\ell : \Lambda \to \mathbb{Z}$ provided the non-vanishing of either $\mathrm{Ext}^n_{\mathcal{C}^{\mathrm{gr}}}(\Delta(\nu), L(\lambda)(m))$ or $\mathrm{Ext}^n_{\mathcal{C}^{\mathrm{gr}}}(L(\lambda), \nabla(\nu)(m))$ implies that $m = n \equiv \ell(\lambda) - \ell(\nu)$ mod 2. From (3.1), it follows that if $\mathcal{C}^{\mathrm{gr}}$ has a graded Kazhdan-Lusztig theory, then the ungraded category $\mathcal{C}$ has a Kazhdan-Lusztig theory. Conversely, [CPS8; (3.9)] proves that A-mod has a graded Kazhdan-Lusztig theory if and only if it has a Kazhdan-Lusztig theory and A is a Koszul algebra.[6]

[6] A graded algebra A (with A_0 semisimple) is *Koszul* provided that for simple A_0-modules L, L', and $m, n, p \in \mathbb{Z}$, we have

$$\mathrm{Ext}^p_{\mathcal{C}^{\mathrm{gr}}}(L(m), L'(n)) \neq 0 \implies n - m = p.$$

Recall that a Borel subalgebra B of a quasi-hereditary algebra A is said to "have dominant weights only" provided that the weight set Λ of B equals the weight set Λ^+ of A. In the following result, we establish the existence of such a Borel subalgebra B when A is tightly graded and Hypothesis 3.3 holds. This result should be compared to Theorem 2.10. The motivation for the definition of B below (as well as in (2.10)) comes from [Dy].

(3.5) Theorem. *Let A be a basic quasi-hereditary algebra (with weight poset $(\Lambda^+, \leq^+ = \leq^+_{\min})$) which satisfies Hypothesis 1.1. Also, assume that A is tightly graded and that Hypothesis 3.3 holds. Then A has a tightly graded Borel subalgebra B (with weight poset $(\Lambda^+, \leq^+)$) such that $\leq = \leq' = \leq_{\min} = \leq^+$. In addition, we can require that*

$$\operatorname{Ext}^1_B(L(B,\lambda), L(B,\mu)) \cong \operatorname{Ext}^1_A(L(A,\lambda), L(A,\mu)) \tag{3.5.1}$$

for all $\mu >^+ \lambda$ in Λ^+.[7]

Proof. Let $\{e_\lambda\}$ be a complete set of primitive orthogonal idempotents in A, indexed by the set Λ^+. We can assume that each $e_\lambda \in A_0$, and that $1 = \sum e_\lambda$ since A is basic. Define B to be the subalgebra of A generated by A_0 and the subspaces $e_\lambda A_1 e_\mu$ for $\lambda >^+ \mu$.

The set Λ^+ also indexes the isomorphism classes of irreducible B-modules: the distinct irreducible A-modules restrict to give the distinct irreducible B-modules. Since

$$\dim \operatorname{Ext}^1_B(L(B,\lambda), L(B,\mu)) = \dim\, e_\lambda A_1 e_\mu = \dim \operatorname{Ext}^1_A(L(A,\lambda), L(A,\mu)),$$

B-mod is directed by $\leq^+$ and (3.5.1) holds. By construction, B is tightly graded. We put $\leq = \leq^+$ on $\Lambda = \Lambda^+$.

Pick a $\leq^+$-adapted defining sequence $\{J_\bullet\}$, cf. (2.6). We will use (2.7(3)) to show that B defines a Borel subalgebra of A. If the defining sequence corresponds to the listing $\lambda_1, \cdots, \lambda_n$ of Λ^+, then write $e_i = e_{\lambda_i}$. Certainly, $e_i \in J_i \backslash J_{i-1}$. We will show that (2.7.1) holds.

First, consider the case $i = 1$. We want to prove that

$$e_1 A \subseteq e_1 B. \tag{3.5.1}$$

Since $e_1 B \subseteq e_1 A$ automatically as right B-modules, we obtain (after taking linear duals) a surjection

$$(e_1 A)^* \cong \nabla(A, \lambda_1)|_B \twoheadrightarrow I(B, \lambda_1) \cong (e_1 B)^*. \tag{3.5.2}$$

Koszul algebras are tightly graded. In fact, tight grading uses only the $p = 1$ case; this provides a converse to (3.2).

[7] This means that the quiver of B is a "full directed subgraph" of the quiver of A, in the sense that that if $\lambda < \mu$, then the number of edges from node λ to node μ in the quiver of B equals the number of edges from λ to μ in the quiver of A.

Since $[\nabla(A,\lambda_1) : L(A,\lambda_1)] = 1$, we see that $[\nabla(A,\lambda_1)|_B : L(B,\lambda_1)] = 1$. Of course, $L(B,\lambda_1)$ occurs with multiplicity 1 in $I(B,\lambda_1)$ and is, in fact, the B-socle of $I(B,\lambda_1)$, so $L(B,\lambda_1)$ cannot be killed by the homomorphism (3.5.2). *If* $L(B,\lambda_1)$ *is* the B-socle of $\nabla(A,\lambda_1)$, then it follows that (3.5.2) is an isomorphism.

Since $\nabla(A,\lambda_1)$ is a graded A-module, it is a graded B-module. The B-socle of $\nabla(A,\lambda_1)$ is the B-submodule annihilated by the graded ideal $\mathrm{rad}(B)$, so it is graded, too.

Let $N \cong L(B,\mu)$ be in a homogeneous summand of the B-socle of $\nabla(A,\lambda_1)$ for some $\mu \neq \lambda_1$. Let $M = AN$ be the (graded) A-submodule of $\nabla(A,\lambda_1)$ generated by the one-dimensional subspace N. Because $\mathrm{rad}(A) = \bigoplus_{n>0} A_n$, $L(A,\mu)$ is the head of M. Let M' be a maximal A-quotient module of M all of whose composition factors $L(A,\tau)$ have the property that τ is not strictly greater than μ. Then there exists a (graded) A-module morphism $\Delta(A,\mu) \to M'$ covering the image of N in M' — hence, M' is a homomorphic image of $\Delta(A,\mu)$. Since $L(A,\lambda_1)$ lives in the socle of M, $M' \neq M$. Hence, for some $\nu >^+ \mu$, there is a commutative (push-out) diagram

$$\text{(3.5.4)} \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & M & \longrightarrow & M' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \sim\downarrow & & \\ 0 & \longrightarrow & L(A,\nu) & \longrightarrow & E' & \longrightarrow & M' & \longrightarrow & 0 \end{array}$$

of (graded) A-modules with exact rows and surjective vertical maps. (We are not specifying the grading degree of $L(A,\nu)$, though it has one.) Let $0 \to L(A,\nu) \to E \xrightarrow{f} \Delta(A,\mu) \to 0$ be the extension obtained by pulling E' back through the surjection $\Delta(A,\mu) \to M'$.

Apply the functor $\mathrm{Hom}_A(-, L(A,\nu))$ to the short exact sequence

$$0 \to \mathrm{rad}(\Delta(A,\mu)) \to \Delta(A,\mu) \to L(A,\mu) \to 0.$$

The resulting long exact sequence of $\mathrm{Ext}^{\bullet}_A$-groups and (3.3) imply that the extension

$$0 \to L(A,\nu) \to \mathrm{rad}(E) \to \mathrm{rad}(\Delta(A,\mu)) \to 0$$

(where $\mathrm{rad}(E) = f^{-1}\,\mathrm{rad}(\Delta(A,\mu)))$ in A-mod splits. Thus, $0 \to L(A,\nu) \to \mathrm{rad}(E') \to \mathrm{rad}(M') \to 0$ also splits, so that $\mathrm{rad}(E') = L(A,\nu) \bigoplus Z$, where the graded A-module Z maps isomorphically onto $\mathrm{rad}(M')$. Thus, the graded A-module E'/Z is a homomorphic image of M with exactly two composition factors: $L(A,\mu)$ in its head and $L(A,\nu)$ in its socle. Because $M = AN$, the image of N in E'/Z generates E'/Z. It follows that $0 \to L(A,\nu) \to E'/Z \to L(A,\mu) \to 0$ does not split in A-mod. Since A is tightly graded, (3.2) implies that the grade of $L(A,\nu)$ in E'/Z must be one less than the grade of $L(A,\mu)$. Hence, $e_\nu A_1 e_\mu E'/Z \neq 0$. But $e_\nu A_1 e_\mu \subseteq \mathrm{rad}(B)$ and $\mathrm{rad}(B)N = 0$, a contradiction.

We conclude that the B-socle of $\nabla(A,\lambda_1)|_B$ is isomorphic to $L(B,\lambda_1)$ and hence the surjection (3.5.2) is an isomorphism, as required to show that (3.5.1) holds. This proves that (2.7.1) holds for $i=1$. The algebra $\bar{A}=A/J_1$ is still tightly graded and satisfies Hypothesis 3.3 (by elementary recollement theory for highest weight categories). It contains the subalgebra $\bar{B}=B/(B\cap J_1)$ which is generated by $\bar{A}_0=A_0/(A_0\cap J_1)$ and the $e_\lambda\bar{A}_1e_\mu$ for $\lambda>\mu$ in $\Lambda^+\backslash\{\lambda_1\}$. The proof that (2.7.1) holds for all i now follows by induction on $\#\Lambda^+$. □

As a consequence, we have the following result, part (1) of which is an unpulished result of Cline, Parshall, and Scott:

(3.6) Corollary. *Assume that A-mod has a graded Kazhdan-Lusztig theory relative to a length funciton $\ell:\Lambda^+\to\mathbb{Z}$. Then:*

(1) *The quasi-hereditary algebra A has a directed, tightly graded subalgebra B satisfying (2.7.1) as well as the quiver relation (3.5.1).*

(2) *If A satisfies Hypothesis 1.1, then B above can be taken to be a Borel subalgebra.*

Proof. This follows immediately from the discussion in (3.4) and (3.5). □

We can next prove the following result, proved under weaker hypotheses (essentially those of (3.6)) by Dyer [Dy].

(3.7) Corollary. *Assume that A is a tightly graded quasi-hereditary algebra (with weight poset Λ^+ with $\leq^+=\leq^+_{\min}$) satisfying Hypothesis 3.3. There exist directed, tightly graded subalgebras B and B^- such that $A=B^-B$. If Hypothesis 1.1 holds, then B is a Borel subalgebra, and if the "opposite" of Hypothesis 1.1 holds, then B^- is an "opposite" Borel subalgebra.*

Proof. We define B as in (3.5), and take B^- to be the subalgebra of A generated by A_0 and the subspaces $e_\lambda A_1 e_\mu$ for $\lambda<^+\mu$ (in the notation of the proof of (3.5)). Then the proof of (2.9) applies to give the factorization $A=B^-B$. (Notice this uses only (2.7.1) and its "opposite".) The last assertion is clear. □

We will say that a Borel subalgebra B is **strong** if every irreducible A-module restricts to an irreducible B-module. In this case, B must have dominant weights only (i. e., $\Lambda^+=\Lambda$), since, as with any subalgebra, every irreducible B-module must occur as a composition factor of some irreducible A-module. (This is not automatic in any prealgebra or categorical concept. However, there is a corresponding categorical *strong* concept, the requirement on a Borel category that $\Psi L(\mathcal{C},\lambda)$ be irreducible for each irreducible object $L(\mathcal{C},\lambda)$ in $\mathcal{C}$, together with the requirement that $\mathcal{B}$ have dominant weights only.)

(3.8) Proposition. *Suppose A is a quasi-hereditary algebra with a strong Borel subalgebra B. Then*

$$\dim\operatorname{Ext}^1_A(\Delta(A,\lambda),L(A,\mu))\leq\dim\operatorname{Ext}^1_B(L(B,\lambda),L(B,\mu))$$

for all $\lambda, \mu \in \Lambda^+$.

Proof. Let

$$(3.8.1) \qquad P_1 \to P_0 \to L(B, \lambda) \to 0$$

be the first terms of a minimal projective resolution of $L(\mathcal{B}, \lambda)$. Thus,

$$\dim \mathrm{Hom}_B(P_1, L(B, \mu)) = \dim\ \mathrm{Ext}^1_B(L(B, \lambda), L(B, \mu)).$$

Applying $\Psi_!$ to (3.8.1) gives a projective presentation

$$\Psi_! P_1 \to \Psi_! P_0 \to \Delta(A, \lambda) \to 0$$

of $\Delta(A, \lambda) = \Psi_! L(B, \lambda)$. Consequently,

$$\dim\ \mathrm{Hom}_A(\Psi_! P_1, L(A, \mu)) \geq \dim\ \mathrm{Ext}^1_A(L(A, \lambda), L(A, \mu)).$$

Comparison with the equality above and using $\Psi L(A, \mu) \cong L(B, \mu)$ gives the desired result. □

(3.9) Remarks (Equivalence with the Lusztig conjecture)**.** In the representation theories of reductive groups over an algebraically closed field k of characteristic $p > 0$, quantum groups (or enveloping algebras) at a root of unity, and in the structure of the category $\mathcal{O}$ for a complex semisimple Lie algebra, there is a "Lusztig" (or "Kazhdan-Lusztig") conjecture predicting the characters of certain irreducible modules. As proved in [CPS6, CPS7], all these conjectures are equivalent to the existence of a Kazhdan-Lusztig theory for certain associated highest weight categories $\mathcal{C}$, using an appropriate length function $\ell : \Lambda^+ \to \mathbb{Z}$. In some of these cases, a Koszul structure exists on the underlying quasi-hereditary algebra A ($\mathcal{C} \cong A$-mod); e. g., $\mathcal{O}$ has a Koszul structure; see, for example, [PS2]. A Koszul structure likely holds in the infinitesimal setting of [CPS7], using the Koszul results in [AJS]. Thus, in these cases, (3.6) can be applied to show that the associated quasi-hereditary algebras A possess strong Borel subalgebras, which are tightly graded and satisfy the quiver relation (3.5.1). The fact that the truth of these conjectures imply the existence of a Kazhdan-Lusztig theory, show, using (3.6) and (3.5.1), that these strong Borel subalgebras also satisfy a parity condition:

$$(3.9.1) \quad \mathrm{Ext}^1_B(L(B, \lambda), L(B, \mu)) \neq 0 \implies \ell(\lambda) - \ell(\mu) \equiv 1 \pmod 2, \quad \forall \lambda, \mu \in \Lambda^+.$$

At the same time, if a quasi-hereditary algebra A has a strong Borel subalgebra satisfying the parity condition above (for some ℓ), (3.8) implies that following condition holds for A:

$$(3.9.2) \quad \mathrm{Ext}^1_A(\Delta(A, \lambda), L(A, \mu)) \neq 0 \implies \ell(\lambda) - \ell(\mu) \equiv 1 \pmod 2\ \forall \lambda, \mu \in \Lambda^+.$$

In the cases under consideration, the parity condition (3.9.2) implies the Lusztig conjecture or Kazhdan-Lusztig conjecture (see [CPS6, CPS9]).

In particular, we may conclude that, in the infinitesimal characteristic p case, the existence of a strong Borel subalgebra satisfying the parity condition (3.9.1) implies and is even equivalent to the Lusztig conjecture. A similar statement holds for the Kazhdan-Lusztig conjecture for $\mathcal{O}$, which is known to be true, but, as yet, has no algebraic proof.

A similar program (connecting Kazhdan-Lusztig theory with the existence of what are essentially strong Borel subalgebras with an exact functor $\Psi_!$) had been proposed by König [K4], but has apparently not been completely carried out. The above remarks provide a reasonable substitute theory.

4. Homological, excellent and exact Borel categories

We keep the notation of §2. Thus, let $\mathcal{C}$ be a fixed highest weight category with finite weight poset $(\Lambda^+, \leq^+)$. We have defined in §2, the idea of a Borel category $\mathcal{B} \to \mathcal{C}$. In this section, we present various strengthening of the conditions on a Borel category. As we will see in later sections, these categories do arise in the representation theory of algebraic and quantum groups.

It will often be convenient to work in the language of bounded derived categories. Thus, given a additive, right exact functor $\Psi : \mathcal{A} \to \mathcal{E}$ (of highest weight categories), there is a left derived functor $\mathbf{L}\Psi : D^b(\mathcal{A}) \to D^b(\mathcal{E})$ of bounded derived categories. If Ψ is left exact, then there is a right derived functor $\mathbf{R}\Psi : D^b(\mathcal{A}) \to D^b(\mathcal{E})$. If Ψ is exact, we usually just denote $\mathbf{L}\Psi$ and $\mathbf{R}\Psi$ simply by Ψ. The following result, taken from the [K1; appendix] will often be used. It complements (2.1).

(4.1) Lemma. *Let $\Psi : \mathcal{C} \to \mathcal{B}$ be an exact, additive functor having a left adjoint $\Psi_!$. Assume that $\mathcal{B}$ is a highest weight category with weight poset $(\Lambda, \leq)$ such that $\Lambda^+ \subseteq \Lambda$. The following statements are equivalent:*

(1) $\Psi\nabla(\mathcal{C}, \lambda) \cong \nabla(\mathcal{B}, \lambda)$ *for all* $\lambda \in \Lambda^+$;

(2) *For* $\omega \in \Lambda$,

$$\mathbf{L}\Psi_!\Delta(\mathcal{B}, \omega) \cong \begin{cases} \Delta(\mathcal{C}, \omega), & \omega \in \Lambda^+, \\ 0, & \omega \notin \Lambda^+. \end{cases}$$

Proof. It is known that $\mathbf{L}\Psi_!$ is a left adjoint to $\mathbf{R}\Psi$, the right derived functor of Ψ. Also, since Ψ is exact, we have $\Psi V \cong \mathbf{R}\Psi V$ for $V \in \mathrm{Ob}(\mathcal{C})$. Thus,

$$(4.1.1) \qquad \mathrm{Hom}^n_{D^b(\mathcal{C})}(\mathbf{L}\Psi_!\Delta(\mathcal{B}, \omega), \nabla(\mathcal{C}, \lambda)) \cong \mathrm{Hom}^n_{D^b(\mathcal{B})}(\Delta(\mathcal{B}, \omega), \Psi\nabla(\mathcal{C}, \lambda)).$$

Note that $\mathbf{L}\Psi_!\Delta(\mathcal{B}, \omega) \in \mathrm{Ob}(D^{b,\leq 0}(\mathcal{C}))$. Thus, the equivalence between (1) and (2) follows from (4.1.1) and (1.1) and its dual, together with (1.2). □

The following presents the basic idea of a homological Borel category for $\mathcal{C}$. It is a slightly more flexible notion of a similar concept for algebras introduced by the second author in [K1; appendix], with the "homological" adjective.

(4.2) Definition. Let $\Psi : \mathcal{C} \to \mathcal{B}$ define a Borel category, in the sense of (2.2). We say that $\mathcal{B}$ is a **homological Borel category** if for $\lambda \in \Lambda^+$, we have $\Psi\nabla(\mathcal{C}, \lambda) \cong \nabla(\mathcal{B}, \lambda)$. (Equivalently, by (4.1(2)), $\mathbf{L}\Psi_!\Delta(\mathcal{B}, \omega)$ is isomorphic to $\Delta(\mathcal{C}, \omega)$ if $\omega \in \Lambda^+$ and is zero otherwise.)

Since $\nabla(\mathcal{C}, \lambda) \cong L(\mathcal{C}, \lambda)$ for $\lambda \in \Lambda^+$ minimal, we see that $\Psi L(\mathcal{C}, \lambda) \cong \nabla(\mathcal{B}, \lambda)$ for such λ. From this, it follows by an easy induction argument that, given any $\lambda \in \Lambda^+$, $\Psi L(\mathcal{C}, \lambda)$ is a nonzero subobject of $\nabla(\mathcal{B}, \lambda)$. In particular, $\Psi L(\mathcal{C}, \lambda)$ has socle isomorphic to $L(\mathcal{B}, \lambda)$. Therefore, because Ψ is exact and any nonzero morphism $\Delta(\mathcal{C}, \lambda) \to \nabla(\mathcal{C}, \lambda)$ (which is unique up to nonzero scalar factor) has image isomorphic to $L(\mathcal{C}, \lambda)$, and hence gives a nonzero morphism $\Psi\Delta(\mathcal{C}, \lambda) \to \Psi\nabla(\mathcal{C}, \lambda)$ for all $\lambda \in \Lambda^+$.

(4.3) Remark. Let $\Psi : \mathcal{C} \to \mathcal{B}$ be an exact, additive functor of highest weight categories $\mathcal{C}$ and $\mathcal{B}$ having weight posets Λ^+ and Λ, respectively. Assume that Λ^+ is a subposet of Λ. Also assume that

$$\Psi\nabla(\mathcal{C}, \lambda) \cong \nabla(\mathcal{B}, \lambda), \quad \Psi\Delta(\mathcal{C}, \lambda) \cong \Delta(\mathcal{B}, \lambda) \quad \forall \lambda \in \Lambda^+.$$

(The second condition above is condition (4.2(3)), and the first condition is dual to it. The two conditions rarely occur together.) Then Λ^+ is an ideal of Λ with respect to the minimal partial ordering $\leq_{\min}$, and Ψ induces an equivalence $\mathcal{C} \cong \mathcal{B}[\Lambda^+]$. To see this, let $\tau \in \Lambda^+$. There is a unique (up to scalar) nonzero morphism $\varphi\colon \Delta(\mathcal{C}, \tau) \to \nabla(\mathcal{C}, \tau)$, and $\mathrm{Im}(\varphi)$ identifies with $L(\mathcal{C}, \tau)$. A similar statement holds for $\mathcal{B}$, so $\Psi L(\mathcal{C}, \tau) \cong L(\mathcal{B}, \tau)$ if $\Psi(\varphi) \neq 0$ and $\cong 0$ if $\Psi(\varphi) = 0$. Suppose $\mu \leq'_{\min} \lambda \in \Lambda^+$. By the definition of $\leq'_{\min}$, there exists a sequence $\mu = \nu_1, \cdots, \nu_t = \lambda$ in Λ such that, for $1 \leq i < t$, $L(\mathcal{B}, \nu_i)$ is a composition factor of $\nabla(\mathcal{B}, \nu_{i+1})$ or $\Delta(\mathcal{B}, \nu_{i+1})$. If $\nu_{i+1} \in \Lambda^+$, the exactness of Ψ now implies that $\nu_i \in \Lambda^+$. Thus, $\mu \in \Lambda^+$, so that Λ^+ is an ideal in $(\Lambda, \leq_{\min})$, and Ψ factors through $i_*\colon \mathcal{B}[\Lambda] \to \mathcal{B}$, giving an exact functor $\Psi\colon \mathcal{C} \to \mathcal{B}[\Lambda^+]$.

If $\lambda \in \Lambda^+$ is minimal, $\Psi L(\mathcal{C}, \lambda) = \Psi\Delta(\mathcal{C}, \lambda) \cong \Delta(\mathcal{B}, \lambda) \cong L(\mathcal{B}, \lambda)$. An easy induction shows that $\Psi L(\mathcal{C}, \lambda) \cong L(\mathcal{B}, \lambda)$ for all $\lambda \in \Lambda^+$. Thus, $\leq_{\min} \subset \leq'_{\min}$, and the result follows from the Comparison Theorem [PS1; § 5] (see also [CPS4; (1.5)]).

In the following definition, we present several strengthenings of the notion of a Borel category for $\mathcal{C}$. Note that each definition presents an independent concept. In particular, part (1) is equivalent to a definition by König [K1].

(4.4) Definition. Let $\Psi : \mathcal{C} \to \mathcal{B}$ define a homological Borel category for $\mathcal{C}$ as in Definition 4.2.

(1) We say that $\mathcal{B}$ is an **exact Borel category** for $\mathcal{C}$ if $\Lambda = \Lambda^+$;

(2) We say that $\mathcal{B}$ is an **excellent Borel category** for $\mathcal{C}$ provided Ψ has a right adjoint Ψ_* and there is a function $\mathfrak{f} : \Lambda \to \Lambda^+$ such that, for all $\lambda \in \Lambda$, we have

$$\text{(4.4.1)} \qquad \mathbf{R}\Psi_*\nabla(\mathcal{B}, \lambda) = \nabla(\mathcal{C}, \mathfrak{f}(\lambda)), \qquad \forall \lambda \in \Lambda$$

and such that the adjunction map

$$\Psi\Psi_*\nabla(\mathcal{B},\lambda) \to \nabla(\mathcal{B},\lambda) \qquad \text{is surjective.}$$

(3) We say that $\mathcal{B}$ is a **complete Borel category** provided that the adjunction functor $\mathbf{L}\Psi_!\Psi \to \mathrm{id}_{D^b(\mathcal{C})}$ is an isomorphism in the derived category $D^b(\mathcal{C})$. (In particular, $\Psi_!\Psi \to \mathrm{id}_{\mathcal{C}}$ is an isomorphism.)

(4.5) Proposition. *Assume that $\Psi : \mathcal{C} \to \mathcal{B}$ defines an excellent Borel category. Then:*

(1) *For $\lambda \in \Lambda^+$, $\mathfrak{f}(\lambda) = \lambda$.*

(2) *For $\lambda \in \Lambda^+$, $\Psi\Delta(\mathcal{C},\lambda)$ has a multiplicity free Δ-filtration in $\mathcal{B}$. Any $\mu \in \Lambda$ satisfying $\mathfrak{f}(\mu) = \lambda$ occurs precisely once as a section in a Δ-filtration of $\Psi\Delta(\mathcal{C},\lambda)$.*

(3) *Ψ defines a complete Borel category of $\mathcal{C}$.*

Proof. Let $\lambda \in \Lambda^+$. We have already argued that, for a general Borel category, we have $\mathrm{Hom}_{\mathcal{B}}(\Psi\Delta(\mathcal{C},\lambda), \Psi\nabla(\mathcal{C},\lambda)) \neq 0$. Hence,

$$\begin{aligned} 0 &\neq \mathrm{Hom}_{\mathcal{B}}(\Psi\Delta(\mathcal{C},\lambda), \Psi\nabla(\mathcal{C},\lambda) \\ &\cong \mathrm{Hom}_{\mathcal{B}}(\Psi\Delta(\mathcal{C},\lambda), \nabla(\mathcal{B},\lambda)) \\ &\cong \mathrm{Hom}_{\mathcal{C}}(\Delta(\mathcal{C},\lambda), \Psi_*\nabla(\mathcal{B},\lambda)) \\ &\cong \mathrm{Hom}_{\mathcal{C}}(\Delta(\mathcal{C},\lambda), \nabla(\mathcal{C},\mathfrak{f}(\lambda)). \end{aligned}$$

Therefore, by (1.1), $\lambda = \mathfrak{f}(\lambda)$, as required by (1).

For $\lambda \in \Lambda^+$, $\mu \in \Lambda$, we have, for any integer n,

$$\begin{aligned} \dim \mathrm{Hom}^n_{D^b(\mathcal{C})}(\Psi\Delta(\mathcal{C},\lambda), \nabla(\mathcal{B},\mu)) &= \dim \mathrm{Hom}^n_{D^b(\mathcal{B})}(\Delta(\mathcal{C},\lambda), \Psi_*\nabla(\mathcal{B},\mu)) \\ &= \dim \mathrm{Hom}^n_{D^b(\mathcal{B})}(\Delta(\mathcal{C},\lambda), \nabla(\mathcal{C},\mathfrak{f}(\mu)) \\ &= \delta_{n,0}\delta_{\lambda,\mathfrak{f}(\mu)}. \end{aligned}$$

From the $n = 1$ case, it follows that $\Psi\Delta(\mathcal{C},\lambda)$ has a Δ-filtration. On the other hand, if M has a Δ-filtration in the highest weight category $\mathcal{B}$, then

$$\dim \mathrm{Hom}_{\mathcal{B}}(M, \nabla(\mathcal{B},\mu))$$

calculates the multiplicity of $\Delta(\mathcal{B},\mu)$ as a section in any Δ-filtration of M. Therefore, we see that any $\mu \in \Lambda$ with $\mathfrak{f}(\mu) = \lambda$ occurs precisely once in a Δ-filtration of $\Psi\Delta(\mathcal{C},\lambda)$. This completes the proof of (2).

To prove (3), we show that the adjunction functor $\theta : \mathbf{L}\Psi_!\Psi \to \mathrm{id}_{D^b(\mathcal{C})}$ is an isomorphism. Because $\mathbf{L}\Psi_!\Psi$ is an exact functor on triangulated categories, a standard truncation argument shows that it suffices to prove that $\theta(L(\lambda))$ is an isomorphism for all $\lambda \in \Lambda^+$, where $L(\lambda)$ is regarded as a complex concentrated in degree 0. By (2), $\mathbf{L}\Psi_!\Psi\Delta(\mathcal{C},\lambda) \cong \Delta(\mathcal{C},\lambda)$, so that $\theta(\Delta(\mathcal{C},\lambda))$ is either an isomorphism or the zero map. However, $\theta(\Delta(\mathcal{C},\lambda))$ is the adjunction of the identity map $\Psi\Delta(\mathcal{C},\lambda) \to \Psi\Delta(\mathcal{C},\lambda)$. Hence, $\theta(M)$ is an isomorphism for $M = \Delta(\mathcal{C},\lambda)$, $\lambda \in \Lambda^+$. In particular, $\theta(L(\mathcal{C},\lambda))$ is an isomorphism if λ is minimal in Λ^+. Now consider a non-minimal $\lambda \in \Lambda^+$, and form the exact sequence $0 \to Q(\lambda) \to \Delta(\mathcal{C},\lambda) \to L(\mathcal{C},\lambda) \to 0$ in $\mathcal{C}$. By induction, $\theta(Q(\lambda))$ is an isomorphism, while $\theta(\Delta(\mathcal{C},\lambda))$ is an isomorphism. It follows that $\theta(L(\mathcal{C},\lambda))$ is an isomorphism. This proves (3). □

(4.6) Proposition. *Every exact Borel category of $\mathcal{C}$ is homological.*

Proof. This fact is obvious from our definitions.[8] □

The following result shows that when a highest weight category $\mathcal{C}$ has a complete Borel category $\mathcal{B}$, then we can view $\mathcal{C}$ as fully embedded in $\mathcal{B}$. Thus, the situation is completely analogous to the case for a reductive algebraic group G, where the category of rational G-modules can be viewed as fully embedded into the category of rational B-modules for a Borel subgroup B. (This depends on the completeness of the variety G/B, which explains our choice of terminology.) When $\mathcal{B}$ is an excellent Borel category, the third part of the result below gives a homological characterization of the strict image of $\mathcal{C}$ in $\mathcal{B}$. As we will see in §7, in the case of algebraic groups, we do have excellent Borel categories, so that this result gives a new characterization of the category of rational G-modules as a subcategory of the category of rational B-modules. In fact, our main results there show that this extends to the case of "generalized q-Schur algebras" associated to quantum groups at a root of unity.

(4.7) Proposition. *Suppose $\Psi : \mathcal{C} \to \mathcal{B}$ defines a complete Borel category of $\mathcal{C}$. Then:*

(1) *Ψ defines a category equivalence of $\mathcal{C}$ (resp., $D^b(\mathcal{C})$) onto its image in $\mathcal{B}$ (resp., $D^b(\mathcal{B})$).*

(2) *The triangulated category $D^b(\mathcal{C})$ is equivalent to the quotient of $D^b(\mathcal{B})$ by the épaisse subcategory consisting of all objects $N \in \mathrm{Ob}(D^b(\mathcal{B}))$ satisfying*

(4.7.1) $$\mathrm{Hom}^n_{D^b(\mathcal{B})}(N, \nabla(\mathcal{B},\lambda)) = 0 \ \forall n \in \mathbb{Z}, \ \ \forall \lambda \in \Lambda^+.$$

(3) *Assume that $\mathcal{B}$ is an excellent Borel category. The $M \in \mathrm{Ob}(D^b(\mathcal{B}))$ which are isomorphic to an object of $\Psi(D^b(\mathcal{C}))$ are precisely the objects satisfying*

(4.7.2) $$\mathrm{Hom}^n_{D^b(\mathcal{B})}(\Delta(\mathcal{B},\nu), M) = 0, \ \ \forall n \in \mathbb{Z}, \ \forall \nu \in \Lambda \backslash \Lambda^+.$$

[8] This result is perhaps more interesting from the (categorical version of) König's formulation [K1] of an exact Borel subalgebra, and it is also true and immediate with that interpretation.

In particular, if $M \in \mathrm{Ob}(\mathcal{B})$ *satisfies (4.7.2), then* M *lies (up to isomorphism) in the image of* $\mathcal{C}$ *under* Ψ. *(In this case, the reader can replace* $\mathrm{Hom}^n_{D^b(\mathcal{B})}$ *in (4.7.2) by* $\mathrm{Ext}^n_{\mathcal{B}}$.*)*

Proof. It is enough to prove both (1) and (3) in the derived category cases. But (1) follows since $\mathbf{L}\Psi_!\Psi \cong \mathrm{id}_{D^b(\mathcal{C})}$.

Next we prove (3). For any $M = \Psi N$, we have $\mathrm{Hom}_{D^b(\mathcal{B})}(\Delta(\mathcal{B},\nu),\Psi N) \cong \mathrm{Hom}^n_{D^b(\mathcal{C})}(\mathbf{L}\Psi_!\Delta(\mathcal{B},\nu),N) =$ for all $\nu \in \Lambda\backslash\Lambda^+$ by (4.1), (4.7), and the definitions. Conversely, assume that $M \in \mathrm{Ob}(D^b(\mathcal{C}))$ satisfies (4.7.2). We will show that the adjunction map $\Psi\mathbf{R}\Psi_*M \to M$ is an isomorphism. For this, it is enough to check that the induced map

$$\mathrm{Hom}_{D^b(\mathcal{B})}(X,\Psi\mathbf{R}\Psi_*M) \to \mathrm{Hom}_{D^b(\mathcal{C})}(X,M) \tag{4.7.3}$$

is an isomorphism for all $X \in \mathrm{Ob}(D^b(\mathcal{B}))$. But, because $\mathcal{B}$ is excellent, the derived category $D^b(\mathcal{B})$ is generated by the $\Psi\Delta(\mathcal{C},\lambda)$, $\lambda \in \Lambda^+$, and the $\Delta(\mathcal{B},\nu)$, $\nu \in \Lambda\backslash\Lambda^+$. Hence, (4.6.1) implies that to prove (4.7.3) is an isomorphism for all X, it suffices to check that, for all integers n and all $\lambda \in \Lambda^+$, that the map

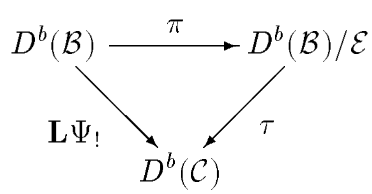
$$\mathrm{Hom}^n_{D^b(\mathcal{B})}(\Psi\Delta(\mathcal{C},\lambda),\Psi\mathbf{R}\Psi_*M) \to \mathrm{Hom}^n_{D^b(\mathcal{B})}(\Psi\Delta(\mathcal{C},\lambda),M) \tag{4.7.4}$$

is an isomorphism. However, since

$$\mathrm{Hom}^n_{D^b(\mathcal{B})}(\Psi\Delta(\mathcal{C},\lambda),M) \cong \mathrm{Hom}^n_{D^b(\mathcal{C})}(\Delta(\mathcal{C},\lambda),\mathbf{R}\Psi_*M)),$$

the map (4.7.4) has a right inverse defined by Ψ, and, taking the full embedding into account, it is therefore an isomorphism.

Finally, to prove (2), let $\mathcal{E}$ be the full subcategory of $D^b(\mathcal{B})$ consisting of all $N \in \mathrm{Ob}(D^b(\mathcal{B}))$ satisfying (4.7.1). Then $\mathcal{E}$ is clearly épaisse in the sense of Verdier [V], while $\mathbf{L}\Psi_!(\mathcal{E}) = 0$, by properties of the adjoint pair $(\mathbf{L}\Psi_1,\Psi)$ since any nonzero object in $D^b(\mathcal{C})$ has a nonzero morphism to some $\nabla(\mathcal{C},\lambda)[n]$. Thus, there is a commutative diagram



in which π is the quotient functor. Now $\pi\Psi$ provides a right inverse to τ. To show that it also gives a left inverse, it suffices to show that the adjunction morphism

$$X \to \Psi\mathbf{L}\Psi_!X \tag{4.7.5}$$

has mapping cone C lying in $\mathcal{E}$, for any $X \in \mathrm{Ob}(D^b(\mathcal{B}))$. However, if $\mathbf{L}\Psi_!$ is applied to the map in (4.7.5), it becomes an isomorphism. Thus, $\mathbf{L}\Psi_!C = 0$, so

$$\begin{aligned}\mathrm{Hom}^n_{D^b(\mathcal{B})}(C,\nabla(\mathcal{B},\lambda)) &\cong \mathrm{Hom}^n_{D^b(\mathcal{B})}(C,\Psi\nabla(\mathcal{C},\lambda)\\ &\cong \mathrm{Hom}^n_{D^b(\mathcal{C})}(\mathbf{L}\Psi_!C,\nabla(\mathcal{C},\lambda)) = 0\end{aligned}$$

for all $\lambda \in \Lambda^+$, $n \in \mathbb{Z}$. This completes the proof of (3). $\square$

The following propositions indicate some further properties of these various types of Borel categories.

(4.8) Proposition. *Suppose $\Psi : \mathcal{C} \to \mathcal{B}$ defines a homological Borel category. Then $\Psi\nabla(\mathcal{C}, \lambda)$, $\lambda \in \lambda^+$ fixed, is characterized in $\mathrm{Ob}(\mathcal{B})$ by the property that $\Psi\nabla(\mathcal{C}, \lambda)$ has $\mathcal{B}$-socle $L(\mathcal{B}, \lambda)$, has only $\mathcal{B}$-compostion factors $L(\mathcal{B}, \nu)$ with $\nu \leq \lambda$, and is maximal with these two properties (i. e., it is not a subobject of a larger $\mathcal{B}$-object with these properties).*

Proof. The indicated properties characterize $\nabla(\mathcal{B}, \lambda)$ as an object in $\mathcal{B}$. Thus, the result follows from the definitions. $\square$

(4.9) Proposition. *Assume that $\Psi : \mathcal{C} \to \mathcal{B}$ defines an exact and complete Borel category. Then Ψ is an equivalence of categories.*

Proof. Assume that $\mathcal{B}$ is both exact and complete. Because $\mathcal{B}$ is directed, $\Delta(\mathcal{B}, \lambda) \cong L(\mathcal{B}, \lambda)$ for all λ. Because $\mathbf{L}\Psi_!\Delta(\mathcal{B}, \lambda) \cong \Delta(\mathcal{C}, \lambda)$, it follows that the higher derived functors of $\Psi_!$ all vanish. Thus, $\Psi_!$ is an exact functor. Since $\Psi_!\nabla(\mathcal{B}, \lambda) \cong \Psi_!\Psi\nabla(\mathcal{C}, \lambda) \cong \nabla(\mathcal{C}, \lambda)$ by hypothesis, the result follows from (2.3). $\square$

(4.10) Proposition. *Suppose that $\Psi : \mathcal{C} \to \mathcal{B}$ defines an exact Borel category for $\mathcal{C}$. Let $\mathcal{C}'$ be a direct sum of blocks of $\mathcal{C}$. Then $\mathcal{C}'$ also has an exact Borel category.*

Proof. Let Λ' be the weight poset of $\mathcal{C}'$, so that Λ' is a subposet of Λ^+. Let $\pi : \mathcal{C} \to \mathcal{C}'$ be the natural projection functor. Form the the Serre subcategory $\mathcal{E}$ of $\mathcal{B}$ consisting of all objects which have composition factors $L(\mathcal{B}, \mu)$ for some $\mu \in \Lambda^+ \backslash \Lambda'$. Since $\mathbf{L}\Psi_! L(\mathcal{B}, \mu) \cong \Delta(\mathcal{C}, \mu)$ for all $\mu \in \Lambda$, it follows that $\pi\Psi_!(\mathcal{E}) = 0$. Thus, $\pi\Psi_! : \mathcal{B} \to \mathcal{C}'$ factors through a natural functor $\bar{\Psi}_! : \mathcal{B}/\mathcal{E} \to \mathcal{C}'$. If $p : \mathcal{B} \to \mathcal{B}/\mathcal{E}$ is the quotient functor, then $\bar{\Psi}_!$ is a left adjoint to the exact functor $\bar{\Psi} = p\Psi|_{\mathcal{C}'}$.

If $\lambda \notin \Lambda'$, then p maps the injective envelope $\nabla(\mathcal{B}, \lambda)$ of $L(\mathcal{B}, \lambda)$ to the injective envelope of $pL(\mathcal{B}, \lambda) = L(\mathcal{B}/\mathcal{E}, \lambda)$. It follows that $\bar{\Psi}\nabla(\mathcal{C}', \lambda) \cong \nabla(\mathcal{B}/\mathcal{E}, \lambda)$ for all such λ. Hence, $\bar{\Psi}$ defines $\mathcal{B}/\mathcal{E}$ as an exact Borel category for $\mathcal{C}'$. $\square$

Finally, we wish to study the issue of realizing $\mathcal{C}$ and $\mathcal{B}$ as module categories A-mod and $Bmod$ for finite dimensional (quasi-hereditary) algebras A and B, respectively, so that B is a *subalgebra* of A and Ψ is induced by the natural restriction of scalars. We will use the notation introduced in (2.4) above. In practice, the morphism $\iota : B \to A$ defined there need not be an injection (and so need not identify B with a subalgebra of A). As discussed in §2, one can replace B by its image B' in A and still obtain a Borel subalgebra of A, but the properties of this section may not hold for B'. However, there are two important cases in which B does identify with a subalgebra of A.

(4.11) Theorem. *Assume that $\mathcal{B}$ is a Borel category associated to the highest weight category $\mathcal{C}$. Let $B = \mathrm{End}_{\mathcal{B}}(P)^{\mathrm{op}}$ and $A = \mathrm{End}_{\mathcal{C}}(\Psi_! P)^{\mathrm{op}}$ for some projective* *generator P of $\mathcal{B}$. Then the map $\iota : B \to A$, obtained by putting $\iota(b) = \Psi_!(b)$, is an injective algebra homomorphism provided that $\mathcal{B}$ is either an exact or an excellent Borel category.*

Proof. By (2.4), we can assume that $\mathcal{B} = B$-mod and $\mathcal{C} = A$-mod. Suppose first that $\mathcal{B}$ is an exact Borel category for $\mathcal{C}$. Then $\Lambda^+ = \Lambda$. For $\lambda \in \Lambda$, $\Psi\nabla(A,\lambda) \cong \nabla(B,\lambda)$. Thus, if $b \in B$ is such that $\iota(b) = 0$, we have that b annihilates each $\nabla(B,\lambda)$. But B-mod is directed for the poset structure on Λ, so that, given $\lambda \in \Lambda$, $\nabla(B,\lambda) \cong I(B,\lambda)$. It follows that b annihilates each injective module $I(B,\lambda)$. Hence, b annihilates an injective generator for B-mod, so that $b = 0$, as required.

Now assume that $\mathcal{B}$ is an excellent Borel category for $\mathcal{C}$, and again let $b \in B$ lie in the kernel of ι. As in the exact case above, it is enough to show that b annihilates each injective module $I(B,\lambda)$, $\lambda \in \Lambda$. Because Ψ_* has an exact left adjoint, $\Psi_* I(B,\lambda)$ is an injective A-module. Since b annihilates $\Psi\Psi_* I(B,\lambda)$ by assumption, it suffices to show that $I(B,\lambda)$ is a homomorphic image of $\Psi\Psi_* I(B,\lambda)$. But for any weight $\mu \in \Lambda$, the adjunction map $\Psi\Psi_*\nabla(B,\mu) \to \nabla(B,\mu)$ is surjective, while the direct images $\mathbf{R}^n\Psi_*\nabla(B,\mu)$ vanish for $n > 0$. Since $I(B,\lambda)$ has a B-module filtration with sections of the form $\nabla(B,\mu)$, an easy induction argument establishes that $\Psi\Psi_* I(B,\lambda) \to I(B,\lambda)$ is surjective. Hence, $b = 0$. $\square$

The notion of an exact Borel subalgebra B of a quasi-hereditary algebra A has been defined by König [K1]. One requires that B be a directed algebra and that the induction functor $A \otimes_B -$ be exact and carry irreducible B-modules $L(B,\lambda)$ to standard modules $\Delta(A,\lambda)$. Also, it is assumed that the irreducible A- and B-modules are indexed by the same poset Λ.

We also define a subalgebra of a quasi-hereditary algebra A to be an *excellent Borel subalgebra* provided that B-mod is an excellent Borel category for the highest weight category A-mod (relative to some poset structure on its weight poset). The above theorem then says that if $\mathcal{B}$ is an exact (resp., excellent) Borel category for the highest weight category $\mathcal{C}$, then $\mathcal{C} \cong A$-mod for some quasi-hereditary algebra A having an exact (resp., excellent) Borel subalgebra B with $\mathcal{B} \cong B$-mod.

(4.12) Corollary. *Let $\mathcal{B}$ be an exact (resp., excellent) Borel category associated to a highest weight category $\mathcal{C}$. Assume that $\mathcal{B} \cong B$-mod and $\mathcal{C} \cong A$-mod with B an exact (resp., excellent) Borel subalgebra of A. Let J be the annihilator in A of all A-modules having composition factors $L(B,\gamma)$ for $\gamma \in$, . Then:*

(1) *Assume $\mathcal{B}$ is exact, and let , be an ideal in Λ. Then $\mathcal{C}[,] \cong A/J$-mod, and J is an idempotent ideal in A. We have $\mathcal{B}[,] \cong B/(J \cap B)$. Also, $\mathcal{B}[,]$ is an exact Borel category for $\mathcal{C}[,]$; thus, $B/(J \cap B)$ is an exact Borel subalgebra of A/J.*

(2) *Assume that $\mathcal{B}$ is exact, and let $,^+$ be an ideal in Λ^+. Let , be the ideal in Λ generated by , . Write $\mathcal{C}[,^+] \cong A/J$, and J is an idempotent ideal in A. Then $\mathcal{B}[,] \cong B/(J \cap B)$. Also, $\mathcal{B}[,]$ is an excellent Borel category for $\mathcal{C}[,^+]$; thus, $B/(B \cap J)$ is an excellent Borel subalgebra of A/J.*

Proof. Assume the hypotheses of (1). Let J' be the ideal in B consisting of all elements b annihilating all B-modules having composition factors of the form $L(B, \lambda)$, $\lambda \in$, . Then $\mathcal{B}[,] \cong B/J'$-mod. We need to prove that $J' = B \cap J$. The ideal J' (resp., J) can be characterized as the annihilator in B (resp., A) of all injective objects in $\mathcal{B}[,]$ (resp., $\mathcal{C}[,]$). Since B is an exact Borel subalgebra of A, any injective object in $\mathcal{C}[,]$, when regarded as a B-module, decomposes into a direct sum of various $\nabla(B, \gamma)$, $\gamma \in$, . Thus, $J' \subseteq B \cap J$. Conversely, $B \cap J$ annihilates any $\nabla(A, \gamma)$, $\gamma \in$, , so $B \cap J \subseteq J'$. The other assertions in (1) are now clear.

Now consider (2). First, we have , $\cap \Lambda^+ =$, $^+$. Next, if $\gamma \in$, $^+$, then $\Psi_! \Delta(B, \gamma) \cong \Delta(A, \gamma) \in \mathrm{Ob}(\mathcal{C}[, ^+])$. Since $L(B, \gamma)$ is a homomorphic image of $\Delta(B, \gamma)$ and $\Psi_!$ is right exact, it follows that $\Psi_! L(B, \gamma) \in \mathrm{Ob}(\mathcal{C}[, ^+])$ for all $\gamma \in$, . This implies that $\Psi_!$ maps the objects in $\mathcal{B}[,]$ to objects in $\mathcal{C}[, ^+]$. A similar argument, but working with the $\nabla(A, \gamma)$, $\gamma \in$, $^+$, shows that Ψ maps $\mathcal{C}[, ^+]$ to $\mathcal{B}[,]$. Similarly, if $\gamma \in$, , then $\gamma \le \gamma' \in$, $^+$, so that $\mathfrak{f}(\gamma) \le \mathfrak{f}(\gamma') = \gamma'$, implying that $\mathfrak{f}(\gamma) \in$, $^+$. It follows that $(\Psi_!, \Psi, \Psi_*)$ defines an adjoint triple from $\mathcal{C}[, ^+]$ to $\mathcal{B}[,]$ so that $\mathcal{B}[,]$ is an excellent Borel category associated to $\mathcal{C}[, ^]$, using the restriction of $\mathfrak{f}$ to , . The proof of (2) can now be completed as the proof of (4.11). $\square$

(4.13) Remark. Suppose that G is a connected algebraic group defined over an algebraically closed field k of positive characteristic p (for simplicity). The irreducible, rational G-modules are classified by the set Λ^+ of dominant weights for $G/\operatorname{rad}_u(G)$, the quotient of G by its unipotent radical $\operatorname{rad}_u(G)$, once a fixed maximal torus and Borel subgroup of G have been chosen. The set Λ^+ has a natural poset structure. Let $\mathrm{hy}(G)$ be the hyperalgebra of G (see [CPS2]).

We consider pairs (H, G) consisting of a semisimple, simply connected algebraic group G and a closed subgroup H which satisfy the following:

Property E: Let , $^+$ be a finite ideal in Λ^+ and consider the category $\mathcal{C}[, ^+]$ of finite dimensional rational G-modules with composition factors isomorphic to $L(\gamma)$, $\gamma \in$, $^+$. Let $\operatorname{ann}(\mathrm{hy}(G))$ be the annihilator of $\mathcal{C}[,]$ in $\mathrm{hy}(G)$. Define $\operatorname{ann}_\Gamma(\mathrm{hy}(H))$ to be the annihilator in $\mathrm{hy}(H)$ of the Serre subcategory of rational H-modules generated by $\mathcal{C}[,]$. Then $\mathrm{hy}(H)/\operatorname{ann}_\Gamma(\mathrm{hy}(H))$ is a subalgebra of $\mathrm{hy}(G)/\operatorname{ann}_\Gamma(\mathrm{hy}(G))$.

For example, if U denotes the unipotent radical of a Borel subgroup of G, then the pair (U, G) does *not* satisfy property E, since it clearly fails when , $^+ = \{0\}$. However, if H is a parabolic subgroup of G, then the pair (H, G) does satisfy property E. In case $H = B$, the result can be rephrased as follows (as has been essentially obtained by Woodcock [W; (3.3)]): For $\lambda \in X_+$, consider a function $f \in K[B]$ having the property that all weights μ in the B-module generated by f satisfy $\mu \le_{\mathrm{e}} \lambda$. Then f has the form $f = g|_B$ for some function $g \in K[G]$ such that all composition factors $L(\gamma)$ of the G-module generated by g satisfy $\gamma \le \lambda$.

Similar remarks hold in the case of quantum enveloping algebras. See §6 below.

5. Applications to infinitesimal algebraic and quantum groups

In the first part of this section, we assume that $k = \bar{\mathbb{F}}_p$, the algebraic closure of the prime field of characteristic $p > 0$. Let Φ be a root system with weight lattice X. Let G be a semisimple, simply connected algebraic group with root system Φ, defined and split over $\mathbb{F}_p$. Let $\mathrm{Fr}: G \to G$ be the Frobenius morphism. Fix a maximal split torus T and let B be a Borel subgroup containing T, corresponding to the *positive* roots. Then X identifies with the character group $X(T)$ of T.

For an integer $r > 0$, we form the group schemes $G_rT = (F^r)^{-1}(T)$ (the pull-back of T through the rth power of the Frobenius morphism) and $B_rT = (F^r|_B)^{-1}(T)$. Let $\mathcal{C}_r$ be the category of finite dimensional rational G_rT-modules. Similarly, let $\mathcal{B}_r$ be the category of finite dimensional rational B_rT-modules. Though they have infinite weight poset, both $\mathcal{C}_r$ and $\mathcal{B}_r$ are highest weight categories with poset X, given its ordinary partial ordering: $\lambda \le \mu \iff \mu - \lambda$ is a sum of positive roots. Also, $\mathcal{B}_r$ is directed by this partial ordering. (See [CPS3].) It is well-known that the restriction functor $j^*: \mathcal{C}_r \to \mathcal{B}_r$ has an exact left adjoint $j_!$ which satisfies the condition $j_! L(\mathcal{B}_r, \lambda) \cong \Delta(\mathcal{C}_r, \lambda)$ for all $\lambda \in X$. Thus, $\mathcal{B}_r$ satisfies all the axioms for a Borel category, except for the finiteness condition on the weight poset and that of $\mathcal{C}_r$. This may be taken as the definition of a Borel category in case $\mathcal{B}$ and $\mathcal{C}$ are both highest weight categories with infinite posets. In the next few paragraphs, we will demonstrate, that as might well be expected, this set-up leads to Borel categories for highest weight categories associated to $\mathcal{C}_r$ which have finite weight posets.

Now let , $\subset X$ be a finitely generated ideal, and let $\mathcal{C}_r^\Gamma = \mathcal{C}_r[,]$ and $\mathcal{B}_r^\Gamma = \mathcal{B}_r[,]$. Both $\mathcal{C}_r^\Gamma$ and $\mathcal{B}_r^\Gamma$ are highest weight categories with weight posets , , and the functor j^* restricts to define a functor $j^*: \mathcal{C}_r^\Gamma \to \mathcal{B}_r^\Gamma$. Then $j_!|_{\mathcal{B}_r}$ has image contained in $\mathcal{C}_r^\Gamma$, so that it provides a left adjoint (still denoted $j_!$) to $j^*: \mathcal{C}_r^\Gamma \to \mathcal{B}_r^\Gamma$. Clearly, $j_! L(\mathcal{B}_r^\Gamma, \lambda) \cong \Delta(\mathcal{C}_r^\Gamma, \lambda)$ for all $\lambda \in$, .

(5.1) Lemma. *Let $\Omega \subset$, be a finite coideal, and let $\mathcal{C}_r^\Gamma(\Omega)$ denote the quotient category $\mathcal{C}_r^\Gamma / \mathcal{C}_r^\Gamma[, \backslash \Omega]$. Then the functor $j^*: \mathcal{C}_r^\Gamma \to \mathcal{B}_r^\Gamma$ induces an exact functor $J^*: \mathcal{C}_r^\Gamma(\Omega) \to \mathcal{B}_r^\Gamma(\Omega)$ which has an exact left adjoint $J_!$. Furthermore, $\mathcal{B}_r^\Gamma(\Omega)$ is directed and $J_! L(\mathcal{B}_r^\Gamma(\Omega), \lambda) \cong \Delta(\mathcal{C}_r^\Gamma(\Omega), \lambda)$ for $\lambda \in \Omega$.*

Proof. Certainly, $\mathcal{B}_r^\Gamma(\Omega)$ is directed. It is clear that there is a (unique) commutative diagram

$$\begin{array}{ccc} \mathcal{C}_r^\Gamma & \xrightarrow{j^*} & \mathcal{B}_r^\Gamma \\ \pi^* \downarrow & & \downarrow \sigma^* \\ \mathcal{C}_r^\Gamma(\Omega) & \xrightarrow[J^*]{} & \mathcal{B}_r^\Gamma(\Omega) \end{array}$$

in which π^* and σ^* are the quotient functors. By [F; (15.23)] (and its evident dual versions), σ^* has both left and right adjoints $\sigma_!, \sigma_*$ which provide sections for σ^*. Similarly, π^* has both left and right adjoints $\pi_!$, π_*, providing sections for π^*.

Since $\pi^*\pi_* \cong \mathrm{id}_{\mathcal{C}_r^\Gamma(\Omega)}$, $J^* \cong J^*\pi^*\pi_* \cong \sigma^* j^* \pi_*$ has left adjoint $J_! = \pi^* j_! \sigma_!$. Clearly, $J_!$ is right exact. If $V \to W$ is a monomorphism in $\mathcal{B}_r^\Gamma(\Omega)$, by [F; (15.5(b))], there is an exact sequence $0 \to E \to \sigma_! V \to \sigma_! W$, where the weights of the composition factors of E do not lie in Ω. Now the exactness of $j_!$ and π^* and the fact that $\pi^* j_! E = 0$ imply that $J_! V \to J_! W$ is a monomorphism. Thus, $J_!$ is exact.

The directness of $\mathcal{B}_r^\Gamma(\Omega)$ implies there is an exact sequence

$$0 \to F(\lambda) \to \sigma_! L(\mathcal{B}_r^\Gamma(\Omega), \lambda) \to L(\mathcal{B}_r^\Gamma, \lambda) \to 0, \quad \forall\, \lambda \in \Omega,$$

in $\mathcal{B}_r^\Gamma$, where $\pi^* j_! F(\lambda) = 0$. It follows that

$$J_! L(\mathcal{B}_r^\Gamma(\Omega), \lambda) \cong \pi^* j_! L(\mathcal{B}_r^\Gamma, \lambda) \cong \pi^* \Delta(\mathcal{C}_r^\Gamma, \lambda) \cong \Delta(\mathcal{C}_r^\Gamma(\Omega), \lambda).$$

This establishes the lemma. □

Now we can prove the following result.

(5.2) Theorem. *Let , $\subset X$ be a finitely generated ideal, let $\Omega \subset$, be a finite coideal, and consider the quotient category $\mathcal{C}_r^\Gamma(\Omega) = \mathcal{C}_r^\Gamma/\mathcal{C}_r^\Gamma[, \backslash\Omega]$. There exists a quasi-hereditary algebra $A = A_\Omega$ such that A-mod $\cong \mathcal{C}_r(\Omega)$ and such that A has an exact Borel subalgebra.*

Proof. By the discussion above and (4.2(1)), $\mathcal{B}_r(\Omega)$ is a exact Borel category for the highest weight category $\mathcal{C}_r(\Omega)$. The result then follows from (4.2). □

Using (5.2) and (4.6), we obtain the following result which says that the blocks of the category $\mathcal{C}_r^\Gamma(\Omega)$ can be represented by quasi-hereditary algebras having exact Borel subalgebras.

(5.3) Corollary. *With the notation and assumptions in (5.2), let $\mathcal{D}$ be a direct summand of $\mathcal{C}_r^\Gamma(\Omega)$. Then there exists a quasi-hereditary algebra $A_\mathcal{D}$ such that $\mathcal{D} \cong A_\mathcal{D}$-mod and such that $A_\mathcal{D}$ has an exact Borel subalgebra.*

The above theorem and its corollary can be quantized to obtain similar results for "infinitesimal" quantum enveloping algebras. We only formulate the theorem and leave the corollary to the interested reader.

Now let k be an arbitrary algebraically closed field. For $0 \neq q \in k$, let $\mathfrak{U} = \mathfrak{U}_q$ be the quantized enveloping algebra with root system Φ and parameter q. Let $\mathfrak{H} = \mathfrak{H}_q$ and $\mathfrak{B} = \mathfrak{B}_q$ be the maximal toral subalgebra and the Borel subalgebra of $\mathfrak{U}$ corresponding to positive roots, respectively. Then X can be identified with the group of integral weights of type 1 for the algebra $\mathfrak{H}$.

Suppose that p is an odd prime. If Φ has a component of type G_2, we assume that $p > 3$. Let q be a primitive lth root of 1, where $l = p^e$ for some $e > 0$. (The restriction on l comes from [APW]: one might hope that the restriction can be weakened. In fact, in the dual situation for type A, l is only assumed to be an odd positive integer, see [PW].) Then there is a Frobenius homomorphism from $\mathfrak{U}$ to the ordinary universal enveloping algebra of the semisimple Lie algebra with root system Φ. Let $\mathbf{u}$ be the kernel the Frobenius homomorphism in the category of

Hopf algebras. We can now form the algebra $\mathbf{u}\cdot\mathfrak{H}$ and denote by $\mathcal{C}_{q,1}$ the category of $\mathbf{u}\cdot\mathfrak{H}$-modules which are integral and of type 1 as $\mathfrak{H}$-modules. Similarly, one can also form Borel analogue $\mathbf{b}$ of $\mathbf{u}$ and the algebra $\mathbf{b}\cdot\mathfrak{H}$. Let $\mathcal{B}_{q,1}$ the category of $\mathbf{b}\cdot\mathfrak{H}$-modules which are integral and of type 1 as $\mathfrak{H}$-modules. Then the categories $\mathcal{C}_{q,1}$ and $\mathcal{B}_{q,1}$ are highest categories with weight poset $(X,\le)$. Also, $\mathcal{B}_{q,1}$ is directed. For a ideal , $\subset X$, we also denote $\mathcal{C}^{\Gamma}_{q,1} = \mathcal{C}_{q,1}[,]$ and $\mathcal{B}^{\Gamma}_{q,1} = \mathcal{B}_{q,1}[,]$. We have the following theorem.

(5.4) Theorem. *Let , $\subset X$ be a finitely generated ideal, let $\Omega \subset$, be a finite coideal, and consider the quotient category $\mathcal{C}^{\Gamma}_{q,1}(\Omega) = \mathcal{C}^{\Gamma}_{q,1}/\mathcal{C}^{\Gamma}_{q,1}[, \backslash\Omega]$. There exists a quasi-hereditary algebra $A = A_\Omega$ such that A-mod $\cong \mathcal{C}_{q,1}(\Omega)$ and such that A has an exact Borel subalgebra.*

6. The excellent ordering

In this section, we take up the existence of excellent Borel subalgebras of various quasi-hereditary algebras associated to reductive groups and quantum groups and enveloping algebras. We begin with a modified definition of the excellent ordering.

As before, let Φ be a root system (and fix a set of positive roots). Let $X_+ \subset X$ the set of dominant weights, and W the Weyl group of Φ. For $\lambda \in X$, we denote by λ^+ the unique weight in $W\lambda \cap X_+$, and let $\lambda^- = w_0\lambda^+$, where w_0 is the longest element in W. The *excellent partial ordering* on X is defined by the rule that $\mu \le_{\mathrm{e}} \lambda$ if $\mu^+ < \lambda^+$ in the ordinary partial ordering or, if $\mu = z\lambda^-$ and $\lambda = w\lambda^-$ $(z, w \in W)$, then $z \le w$ in the Chevalley (Bruhat) ordering. For $\lambda,\mu \in X_+$, $\mu \le_{\mathrm{e}} \lambda \iff \mu \le \lambda$ (in the ordinary partial ordering).

This definition is given by the second author in [K1; appendix], and the original excellent ordering defined in [vdK] is a refinement of this. We will use the notation $\mathfrak{U} = \mathfrak{U}_q$ and $\mathfrak{B} = \mathfrak{B}_q$ as introduced before (6.4). (Thus, p, l and q are as described there.) As shown in [APW], the restriction functor $\mathrm{Res}^{\mathfrak{U}}_{\mathfrak{B}}$ from the category $\mathcal{C}_q$ of integral $\mathfrak{U}$-modules of type 1 to the category $\mathcal{B}_q$ of integral $\mathfrak{B}$-modules of type 1 has a right adjoint $\mathrm{Ind}^{\mathfrak{U}}_{\mathfrak{B}}$. The adjunction transformation $\mathrm{Res}^{\mathfrak{U}}_{\mathfrak{B}}\mathrm{Ind}^{\mathfrak{U}}_{\mathfrak{B}} \to \mathrm{id}$ induces the usual evaluation map. For an integral $\mathfrak{B}$-module M, $\mathrm{Ind}^{\mathfrak{U}}_{\mathfrak{B}}(M)$ is usually denoted by $H^0(M)$. We have also the right derived functors $H^j(-) = R^j\,\mathrm{Ind}^{\mathfrak{U}}_{\mathfrak{B}}(-)$. The category $\mathcal{C}_q$ is a highest weight category with weight poset X_+. It has ∇-objects $\nabla(\mathcal{C}_q,\lambda) = H^0(\lambda^-)$ and Δ-objects $\Delta(\mathcal{C}_q,\lambda) = H^0(-\lambda)^*$ for $\lambda \in X_+$.

For any simple root α, one has a minimal parabolic subalgebra $\mathfrak{P}_\alpha$ containing $\mathfrak{B}$. The induction functor $H^0_\alpha(-) = \mathrm{Ind}^{\mathfrak{P}_\alpha}_{\mathfrak{B}}(-)$ and its derived functors $H^j_\alpha(-)$ are defined similarly.

Let $\lambda = w\lambda^- \in X$, and let $w = s_1s_2\cdots s_r$ be a reduced expression of w, where $s_i = s_{\alpha_i}$ is the reflection with respect to a simple root α_i. Let $H^j_i(-) = H^j_{\alpha_i}(-)$. Then we define a $\mathfrak{B}$-module

$$N(\lambda) = H^0_1(H^0_2(\cdots H^0_r(\lambda^-)\cdots)).$$

The following proposition summarizes some facts about these modules.

(6.1) Proposition. (1) *The module $N(\lambda)$ depends only on λ; it is independent of the choice of w and the choice of the reduced expression of w.*

(2) *If $\lambda \in X_+$ (i. e., $w = w_0$), then $N(\lambda) = H^0(\lambda^-)|_{\mathfrak{B}}$.*

(3) *If $z \leq w$ are elements in W and $\mu = z\lambda^-$, $\lambda = w\lambda^-$, then the evaluation homomorphisms induce a surjective homomorphism $\varphi_{w,z}\colon N(\lambda) \to N(\mu)$.*

(4) *The weight space $N(\lambda)_\lambda$ is one-dimensional, and it serves as the $\mathfrak{B}$-socle of $N(\lambda)$.*

(5) *All weights μ of $N(\lambda)$ satisfy $\mu \leq_{\mathrm{e}} \lambda$; if $\lambda^+ = \mu^+$, then μ is a weight of $N(\lambda)$ iff $\mu \leq_{\mathrm{e}} \lambda$.*

(6) *$N(-\lambda)^*$ is the $\mathfrak{B}$-submodule of $\Delta(\mathcal{C}_q, \lambda^+)$ generated by a weight vector of weight λ.*

(7) *$H^0(N(\lambda)) \cong H^0(\lambda^-)$, and the evaluation map $H^0(N(\lambda)) \to N(\lambda)$ is surjective. Moreover, $H^i(N(\lambda)) = 0$ for $i > 0$.*

In order to prove this proposition and (6.3), we work in the set-up of [APW]. That is, let R be the local ring $\mathbb{Z}[v]_{\mathfrak{m}}$, where v is an indeterminate and $\mathfrak{m}$ is the maximal ideal generated by $v-1$ and an odd prime p. (As above, when dealing with a root system having a component of type G_2, assume $p > 3$.) Let k be the algebraic closure of the residue field $\mathbb{F}_p$ of R. Let q be a primitive lth root of 1, where $l = p^e > 1$. Thus, there is a ring homomorphism $R \to k$ sending v to q. If we define the quantized enveloping algebra $\mathfrak{U}_v$ and $\mathfrak{B}_v$ over the ring R instead over the field k, then the base change $R \to k$ allows us to compare the representation theory of $\mathfrak{U}_v$ and $\mathfrak{B}_v$ with the representation theory of the semisimple algebraic group G over k with root system Φ and its Borel subgroups, while the base change $R \to k$ gives a path from the representation theory of $\mathfrak{U}_v$ and $\mathfrak{B}_v$ to the representation theory of $\mathfrak{U}_q$ and $\mathfrak{B}_q$.

Proof of (6.1). As in [APW; § 5], define, for $\lambda \in X$, $N_R(\lambda)$ by working with $\mathfrak{U}_v$ instead of $\mathfrak{U}_q$. By [APW; § 3 & § 5], $N_R(\lambda)$ is a free R-module, and the base change $R \to k$ sends $N_R(\lambda)$ to $N(\lambda)$, while the base change $R \to k$ sends $N_R(\lambda)$ to $N_k(\lambda)$, the corresponding module defined for the corresponding algebraic group.

Now, to show (1)–(3) and (5), one may consider $N_R(\lambda)$ instead. For (1), see [APW; (5.9) & (5.10)]; for (2), see [APW; (5.6)]. For (3), [APW; (5.1)] gives a surjective homomorphism $\varphi_{w_0,z}\colon N(w_0\lambda^-) \to N(z\lambda^-)$ for any $z \in W$ (depending on a reduced expression for z). Then (3) follows from the fact that, for any $z \leq w$, we have $\varphi_{w_0,z} = \varphi_{w,z}\varphi_{w_0,w}$ for a suitable $\mathfrak{B}_q$-module homomorphism $\varphi_{w,z}\colon N(w\lambda^-) \to N(z\lambda^-)$. The truth of (5) follows from the that of the same statement for algebraic groups, see [vdK; (2.6)].

Since $N(\lambda)$ is a homomorphic image of $H^0(\lambda^-)$, by (2) and (3), the multiplicity of λ in $N(\lambda)$ is at most 1. Thus, to prove (4), it suffices to show that $\mathrm{soc}_{\mathfrak{B}}\, N(w\lambda^-)$ has no weight other than $w\lambda^-$ for $w \in W$. We proceed by induction on $\ell(w)$. It is trivial if $\ell(w) = 0$. Now let $w = s_\alpha w'$ with α simple and $\ell(w) = \ell(w') + 1$. Suppose that $\mu \neq w\lambda^-$ is a weight in $\mathrm{soc}_{\mathfrak{B}}\, N(w\lambda^-)$. Let V be the $\mathfrak{P}_\alpha$-submodule of $N(w\lambda^-)$ generated by a nonzero weight vector in $\operatorname{soc}_{\mathfrak{B}} N(w\lambda^-)$ of weight μ. Being a highest weight module for $\mathfrak{P}_\alpha$ with highest weight $\mu <_{\mathrm{e}} w\lambda^-$, V has no weight $w\lambda^-$, hence no weight $s_\alpha w\lambda^- = w'\lambda^-$. Because V is a $\mathfrak{P}_\alpha$-submodule of $N(w\lambda^-) = H^0_\alpha(N(w'\lambda^-))$, the evaluation map $N(w\lambda^-) \to H(w'\lambda^-)$ sends V to a nonzero submodule of $N(w'(\lambda^-))$, giving a weight vector in $\operatorname{soc}_{\mathfrak{B}} N(w'(\lambda^-))$ with weight other than $w'(\lambda^-)$. This is a contradiction, proving (4).

The statement (6) is the dual version of (4) (together with the existence of a surjective homomorphism $H^0(\lambda^-) \twoheadrightarrow N(\lambda)$ ensured by (2) and (3)).

To prove (7), let $w_0 = w'w = w''w'$. Let $w'' = s_1 s_2 \cdots s_r$, $w' = s_{r+1}s_{r+2}\cdots s_{r+t}$ and $w = s_{r+t+1}s_{r+t+2}\cdots s_{2r+t}$ be reduced expressions. By [APW; (5.8)], we have

$$\begin{aligned} H^0(N(\lambda)) &= H^0_1(H^0_2(\cdots H^0_{r+t}(N(\lambda))\cdots)) = H^0_1(H^0_2(\cdots H^0_{2r+t}(\lambda^-)\cdots)) \\ &= H^0_1(H^0_2(\cdots H^0_r(H^0(\lambda^-))\cdots)) \cong H^0(\lambda^-). \end{aligned}$$

Also, from this it is clear that the evaluation map identifies with the surjective map $N(w_0\lambda^-) \to N(\lambda)$ given in (3) (up to a $\mathfrak{B}_q$-module automorphism of $N(\lambda)$). The final assertion follows from the spectral sequence in [APW; (5.4)] and the fact (see [APW; (5.1)]) that

$$H^i_j(H^0_{j+1}(\cdots H^0_{r+t}(N(\lambda))\cdots)) = 0, \quad \forall\, i > 0 \text{ and } 1 \le j \le r+t.$$

The proof of (6.1) is complete. $\square$

Note that if we use the definition of the excellent ordering introduced in this paper instead of van der Kallen's definition in [vdK], the argument in [vdK] works well. This is made explicit in [W; Thm. 2.1 and remarks], who shows that all standard and costandard modules are the same. Thus, we have the following result on representations of the Borel subgroup B of the algebraic group G.

(6.2) Theorem ([vdK; (1.6)]). *Let $\mathcal{B}$ denote the category of rational B-modules. Then the category $\mathcal{B}$ is a highest weight category with respect to the poset $(X, \le_{\mathrm{e}})$. The ∇-objects are the $\nabla(\mathcal{B}, \lambda) = N(\lambda)$, $\lambda \in X$.*

Now it is not difficult to quantize Theorem (6.2).

(6.3) Theorem. *Let $0 \ne q \in k$. Assume that if q is a primitive lth root of unity, then $l = p^e > 1$ satisfies our conventions before (6.4). Then $\mathcal{B}_q$ is a highest weight category with respect to the poset $(X, \le_{\mathrm{e}})$. The ∇-objects are the $\nabla(\mathcal{B}_q, \lambda) = N(\lambda)$, $\lambda \in X$.*

Proof. The key to the result is to show any injective $\mathfrak{B}$-module has an *excellent filtration*, i. e., a filtration with $N(\lambda)$'s as sections. We work in the set-up given before the proof of (6.1) again. It is suffices to show that any injective $\mathfrak{B}_v$-module has an excellent filtration. Suppose $I(\lambda)$ be the injective hull of the one-dimensional $\mathfrak{B}_v$-module λ. By [APW; (6.3)], we one can choose a series of positive integers $0 < n_1 < n_2 < \cdots$ and a series of canonical $\mathfrak{B}_v$-injections

(6.3.1) $$H^0(-n_1\rho)\otimes(\lambda - n_1\rho) \hookrightarrow H^0(-n_2\rho)\otimes(\lambda - n_2\rho) \hookrightarrow \cdots \hookrightarrow I(\lambda)$$

such that $I(\lambda) = \bigcup_{i=1}^{\infty} H^0(-n_i\rho) \otimes (\lambda - n_i\rho)$. Also, by [APW; (6.17)], every $H^0(-n_i\rho) \otimes (\lambda - n_i\rho)$ has an excellent filtration. To see $I(\lambda)$ has an excellent filtration, it now suffices to show that one can choose a compatible set of excellent filtrations for these $H^0(-n_i\rho) \otimes (\lambda - n_i\rho)$'s. This will follow if we know that

$$Q_i = H^0(-n_{i+1}\rho) \otimes (\lambda - n_{i+1}\rho)/H^0(-n_i\rho) \otimes (\lambda - n_i\rho)$$

has an excellent filtration. Clearly, tensoring (6.3.1) over R with k gives the same series of injections for the corresponding algebraic group. Thus, the sequence

$$0 \to H^0(-n_i\rho) \otimes (\lambda - n_i\rho) \otimes_R k \to H^0(-n_{i+1}\rho) \otimes (\lambda - n_{i+1}\rho) \otimes_R k \to Q_i \otimes_R k \to 0$$

is exact. Also, tensoring with the fraction field of R, we have a similar exact sequence. It follows that Q_i is free (and of finite rank) over R (see [APW; (1.21)]), and that $Q_i \otimes_R k$ has an excellent filtration, by [vdK; (1.8)]. Therefore, by [APW; (5.16)], Q_i has an excellent filtration, as required.

Now, let

$$0 = I(\lambda)_0 \subset I(\lambda)_1 \subset I(\lambda)_2 \subset I(\lambda)_3 \subset \cdots \tag{6.3.2}$$

be an excellent filtration of $I(\lambda)$ with $I(\lambda)_i/I(\lambda)_{i-1} \cong N(\mu_i)$ for $i = 1, 2, \cdots$. Tensoring over R with k gives an excellent filtration of $I(\lambda) \otimes_R k$, the injective hull of the one-dimensional module λ for the Borel subgroup of the corresponding algebraic group. Since the category $\mathcal{B}$ of modules over the Borel subgroup B is a highest weight category with respect to the excellent ordering by (6.2), we obtain that $\mu_1 = \lambda$ and $\mu_i >_{\mathrm{e}} \lambda$ for all $i > 1$.

Finally, by base change $R \to k$, the above conclusions remain true. Therefore, the category $\mathcal{B}_q$ is a highest weight category with respect to the poset $(X, \leq_{\mathrm{e}})$. □

7. Generalized q-Schur algebras and their Borel subalgebras

In this and the following sections we will use the results in § 3 to give a proof of the existence of excellent Borel subalgebras for some algebras related to quantum enveloping algebras or quantum groups. We maintain the notation and conventions (on q, etc.) concerning the quantized enveloping algebras $\mathfrak{U}$, $\mathfrak{B}$, etc. from the previous two sections.

Let , $_+$ be a finite ideal of X_+. Then , $= W$, $_+$ is an ideal of $(X, \leq_{\mathrm{e}})$. Thus, we can form the highest weight categories $\mathcal{C}_q[, _+]$ and $\mathcal{B}_q[,]$. The latter is defined by using the poset $(, , \leq_{\mathrm{e}})$. However, it is also a highest weight category with poset $(, , \leq)$ — it is directed with respect to the partial ordering $\leq$. In [DS; §3 & §5], Du and Scott first defined the *generalized q-Schur algebra* $\mathfrak{U}\langle, _+\rangle$ associated to , $_+$. As constructed there, $\mathfrak{U}\langle, _+\rangle$ is a quotient algebra of $\mathfrak{U}$ with the property that there is an equivalence $\mathfrak{U}\langle, _+\rangle$-mod $\cong \mathcal{C}_q[, _+]$ of categories. Similarly, one can form a quotient algebra $\mathfrak{B}\langle, \rangle$ of $\mathfrak{B}$ with the property that $\mathfrak{B}\langle, \rangle$-mod $\cong \mathcal{B}_q[,]$, and one can show there is a natural algebra homomorphism $\mathfrak{B}\langle, \rangle \to \mathfrak{U}\langle, _+\rangle$.

Here we will give another concrete construction of the $\mathfrak{U}\langle\, ,\ _+\rangle$ and the Borel subalgebras $\mathfrak{B}\langle\, ,\ \rangle$ — the construction via coordinate algebras. Among other things, this construction provides an easy way to verify that, in type A, the classical q-Schur algebras and their Borel subalgebras are realized as generalized q-Schur algebras and their Borel subalgebras (see (7.1) below).

Recall that if C is a coalgebra and V is a right C-comodule, the *coefficient subcoalgebra* $\mathrm{Cf}(V)$ of V is, by definition, the linear span of the elements in the defining matrix of V. That is, if $\{v_j\}$ is a basis for V, and the structure map sends v_j to $\sum_j v_i \otimes f_{ij}$, then $\mathrm{Cf}(V)$ is the span of these f_{ij}'s. Observe that $\mathrm{Cf}(V)$ is a subcoalgebra of C.

Recall also that if D is a coalgebra spanned by the set Ξ of its group-like elements, then each $\xi \in \Xi$ spans an irreducible comodule over D, denoted by ξ again. In this way, Ξ indexes the set of isomorphism classes of irreducible D-comodules. Also, any D-comodule is completely reducible, i. e., it is a direct sum of irreducible D-comodules $\xi \in \Xi$.

(7.1) Lemma. *Let $\theta: C \to D$ be a surjective homomorphism of coalgebras with D spanned by the set Ξ of its group-like elements. Let* comod-C *be the category of right C-comodules. For* $\, \subset \Xi$*, let $\mathcal{O}_\Gamma C \in \mathrm{Ob}(\text{comod-}C)$ be the largest subcomodule of C whose restriction to D (via θ) is a direct sum of irreducible comodules indexed by elements in* , . *Then*

(1) $\mathrm{Cf}(\mathcal{O}_\Gamma C) = \mathcal{O}_\Gamma C$*;*

(2) *The category* comod-$\mathcal{O}_\Gamma C$ *is the full subcategory of the category* comod-C *consisting of objects whose restrictions to D are direct sums of irreducible comodules indexed by elements in* , .

Proof. Denote by Δ and ε the comultiplication and the counit of C. Choose a basis $\{c_j\}$ for $\mathcal{O}_\Gamma C$ such that each c_j spans a D-subcomodule isomorphic to $\gamma_j \in$, . Let $\Delta(c_j) = \sum_i c_i \otimes f_{ij}$. Then $\theta(f_{ij}) = \delta_{ij}\gamma_j$.

Let V is a C-comodule whose restriction to D is a direct sum of irreducible comodules indexed by elements in , . Then V is a subcomodule of a direct sum of copies of $\mathcal{O}_\Gamma C$. Thus, V is a comodule over $\mathrm{Cf}(\mathcal{O}_\Gamma C)$. Conversely, let V be a $\mathrm{Cf}(\mathcal{O}_\Gamma C)$-comodule with structure map τ. Let $v \in V$ span a D-subcomodule isomorphic to $\xi \in \Xi$, i. e., $(\mathrm{id} \otimes \theta)\tau(v) = v \otimes \xi$. Since V is a $\mathrm{Cf}(\mathcal{O}_\Gamma C)$-comodule, ξ must be a linear combination of γ_j's. It follows that $\xi = \gamma_j \in$, for some j, since Ξ is linear independent in D. Thus, the category comod-$\mathrm{Cf}(\mathcal{O}_\Gamma C)$ is the full subcategory of comod-C consisting of objects whose restrictions to D are direct sums of irreducible comodules indexed by elements in , .

Since $\mathrm{Cf}(\mathcal{O}_\Gamma C)$ is a $\mathrm{Cf}(\mathcal{O}_\Gamma C)$-comodule, the restriction of $\mathrm{Cf}(\mathcal{O}_\Gamma C)$ to D is a direct sum of irreducible comodules indexed by elements in , . Thus, $\mathrm{Cf}(\mathcal{O}_\Gamma C) \subset \mathcal{O}_\Gamma C$. The opposite inclusion is obvious, since we have $(\varepsilon \otimes \mathrm{id})\Delta = \mathrm{id}$. □

We also have the following very easy result on induction functors.

(7.2) Lemma. *Let $\theta: C \to D$ be a homomorphism of Hopf algebras. Let* θ_* :

comod-$D \to$ comod-C *be the induction functor, as defined in* [PW; (2.7)]. *Then* $\theta_* D \cong C$ *(with* Δ *as the structure map), and* $\theta\colon C \to D$ *is the evaluation homomorphism.*

Proof. When $D = k$, the trivial Hopf algebra over k, and $\theta\colon C \to k$ is the counit ε_C of C, this result is proved in [PW; (2.7)]. Thus, by the transitivity of induction, we have in the general situation that $\theta_* D \cong \theta_* \varepsilon_{D*} k \cong (\varepsilon_D \theta)_* k = \varepsilon_{C*} k \cong C$. It also follows that comultiplication $\Delta\colon C \to C \otimes C$ must define the structure map. Since $D \cong \varepsilon_{D*} k$, $\theta : C \to D$ is the unique morphism η of D-comodules satisfying $\varepsilon_D \eta = \varepsilon_C$. Hence, θ is the evaluation homomorphism, as required. $\square$

Now let $k[\mathfrak{U}]$ be the coordinate algebra of $\mathfrak{U}$, which, by definition, is the union of $(\mathfrak{U}/\operatorname{ann} V)^* \subset \mathfrak{U}^*$ for all finite dimensional integral $\mathfrak{U}$-modules of type 1. (See [L; §§ 1–2] and [APW; (1.33)].) Then $k[\mathfrak{U}]$ carries the natural structure of a Hopf algebra. The category of right $k[\mathfrak{U}]$-comodules is isomorphic to the category of left, integral $\mathfrak{U}$-modules of type 1. Similarly, we can form $k[\mathfrak{B}]$ (see [APW; (2.4)] for another definition of the algebra $k[\mathfrak{B}]$; two definitions are equivalent by [L]), and the embedding $\mathfrak{B} \to \mathfrak{U}$ induces a canonical surjective Hopf algebra homomorphisms $k[\mathfrak{U}] \twoheadrightarrow k[\mathfrak{B}]$ (see [APW; (2.7)]). For the maximal toral subalgebra $\mathfrak{H}$, the coordinate algebra $k[\mathfrak{H}]$ is nothing but the group algebra of the character group X. Clearly we also have a canonical surjection $k[\mathfrak{B}] \twoheadrightarrow k[\mathfrak{H}]$.

Now let , $_+$ be a finite ideal of X_+ and put , $= W,_+$. We apply (7.1) to the homomorphisms $k[\mathfrak{U}] \twoheadrightarrow k[\mathfrak{H}]$ and $k[\mathfrak{B}] \twoheadrightarrow k[\mathfrak{H}]$, yielding

$$\text{(7.3)} \qquad \mathfrak{U}\langle ,_+\rangle = \big(\mathcal{O}_{\Gamma_+} k[\mathfrak{U}]\big)^*, \quad \mathfrak{B}\langle , \rangle = \left(\mathcal{O}_{\Gamma} k[\mathfrak{B}]\right)^*.$$

Here we use $\mathcal{O}_{\Gamma_+}$ instead of $\mathcal{O}_{\Gamma}$ for $k[\mathfrak{U}]$-comodules.

The algebras $\mathfrak{U}\langle ,_+\rangle$ and $\mathfrak{B}\langle , \rangle$ are quotient algebras of $\mathfrak{U}$ and $\mathfrak{B}$, respectively. Moreover, the restriction functor from $\mathcal{C}_q \to \mathcal{B}_q$ gives a functor $\mathcal{C}_q[,_+] \to \mathcal{B}_q[,]$. It follows that the homomorphism $k[\mathfrak{U}] \twoheadrightarrow k[\mathfrak{B}]$ induces a coalgebra homomorphism $\Psi\colon \mathcal{O}_{\Gamma_+} k[\mathfrak{U}] \to \mathcal{O}_{\Gamma} k[\mathfrak{B}]$ (and hence an algebra homomorphism $\Psi^*\colon \mathfrak{B}\langle , \rangle \to \mathfrak{U}\langle ,_+\rangle$).

(7.4) Lemma. *The coalgebra homomorphism* $\Psi\colon \mathcal{O}_{\Gamma_+} k[\mathfrak{U}] \to \mathcal{O}_{\Gamma} k[\mathfrak{B}]$ *is surjective. Thus, the algebra homomorphism* $\varphi \equiv \Psi^*\colon \mathfrak{B}\langle , \rangle \to \mathfrak{U}\langle ,_+\rangle$ *is injective.*

Proof. By (7.2), $H^0(k[\mathfrak{B}]) = k[\mathfrak{U}]$, and the evaluation map is the canonical surjection $k[\mathfrak{U}] \twoheadrightarrow k[\mathfrak{B}]$. Thus, it suffices to show: (i) $H^0(\mathcal{O}_{\Gamma} k[\mathfrak{B}]) \subset \mathcal{O}_{\Gamma_+} k[\mathfrak{U}]$; and (ii) the evaluation map $H^0(\mathcal{O}_{\Gamma} k[\mathfrak{B}]) \to \mathcal{O}_{\Gamma} k[\mathfrak{B}]$ is surjective. By (7.3), $\mathcal{B}_q[,]$ is a highest weight category with poset $(, , \leq_e)$ and ∇-objects $N(\lambda)$, so the injective object $\mathcal{O}_{\Gamma} k[\mathfrak{B}]$ has an excellent filtration. By (6.1(7)), $H^0(\mathcal{O}_{\Gamma} k[\mathfrak{B}])$ has a good filtration with all sections having highest weights in $,_+$. Thus, (i) follows. Also, an easy induction starting from (6.1(7)) shows that the evaluation map $H^0(V) \to V$ is surjective for any $\mathfrak{B}$-module with excellent filtration. This implies (ii). $\square$

We now verify that $\mathfrak{U}\langle , ^+\rangle$ can be replaced by a Morita equivalent algebra having an excellent Borel subalgebra.

(7.5) Theorem. *With the above notation and assumptions, there exists a finite dimensional algebra Morita equivalent to $\mathfrak{U}\langle , ^{+}\rangle$ which has an excellent Borel subalgebra.*

Proof. This follows immediately from §4 and the previous results of this section. □

8. Classical q-Schur algebras and their Borel subalgebras

Continue to let q be a root of unity as before (7.4). Now we consider the classical q-Schur algebras and their Borel subalgebras. Thus, we assume that Φ has type A_{n-1}. Let $A_q(n)$ be coordinate algebra of the quantum $n \times n$-matrix semigroup with parameter q. That is, $A_q(n)$ is the algebra generated by n^2 generators X_{ij}, $i, j = 1, 2, \cdots, n$, subject to a set of well-known (homogeneous) relations (see, for example, [PW; (3.5)]). In fact, $A_q(n)$ is a bialgebra and is graded in the natural way, and the homogeneous component of degree r, denoted $A_q(n, r)$, is a subcoalgebra. The classical q-Schur algebra $S_q(n, r)$ is defined to be the dual algebra of $A_q(n, r)$. It is known that $S_q(n, r)$ can be realized as the centralizer $\mathrm{End}_{\mathcal{H}(\mathfrak{S}_r)}(V^{\otimes r})$ for an action of the Hecke algebra $\mathcal{H}(\mathfrak{S}_r)$ on $V^{\otimes r}$ for an n-dimensional vector space V; see [PW; (11.3.1)]. In fact, this was the approach taken in [DJ].

Let $B_q(n) = A_q(n)/I_q(n)$ (resp., $B_q^-(n) = A_q(n)/I_q^-(n)$, $T_q(n) = A_q(n)/J_q(n)$), where $I_q(n)$ (resp., $I_q^-(n)$, $J_q(n)$) is the ideal generated by all X_{ij} with $i > j$ (resp., $i < j$, $i \neq j$). These are quotient bialgebras. Also, $T_q(n)$ is spanned by its group-like elements; this set $\widetilde{X}(n)$ of group-like elements is the set of monomials $X_{11}^{r_1} X_{22}^{r_2} \cdots X_{nn}^{r_n}$ in the X_{ii}'s. We can also consider the homogeneous components $B_q(n, r)$, $B_q^-(n, r)$ and $T_q(n, r)$ of $B_q(n)$, $B_q^-(n)$, and $T_q(n)$, respectively, and form the dual algebras

$$S_q^+(n, r) = B_q(n, r)^*, \quad S_q^-(n, r) = B_q^-(n, r)^*, \quad S_q^0(n, r) = T_q(n, r)^*.$$

The set of group-like elements in $T_q(n, r)$ will be denoted by $\widetilde{X}(n, r)$. This set is stable under the action of the Weyl group $W = \mathfrak{S}_n$, and $X_{11}^{r_1} X_{22}^{r_2} \cdots X_{nn}^{r_n} \in \widetilde{X}(n, r)$ is *dominant* iff $r_1 \geq r_2 \geq \cdots \geq r_n$, i. e., $(r_1, r_2, \cdots, r_n)$ is a partition of r into at most n parts. The set of dominant elements in $\widetilde{X}(n, r)$ will be denoted by $\widetilde{X}(n, r)_+$.

The quotient algebra of $A_q(n)$ over its ideal generated by $\det_q - 1$, $\det_q$ being the quantum determinant (see, for example, [PW; (7.1.1)]), is the coordinate algebra $k[\mathfrak{U}]$ as defined in § 3 (see the Appendix of [APW], or [K2]). Similarly, $k[\mathfrak{B}]$ and $k[\mathfrak{H}]$ can be obtained from $B_q(n)$ and $T_q(n)$. There is a natural surjective map $\pi \colon \widetilde{X}(n) \to X$, sending $X_{11}^{r_1} X_{22}^{r_2} \cdots X_{nn}^{r_n}$ to $(r_1 - r_2)\omega_1 + (r_2 - r_3)\omega_2 + \cdots + (r_{n-1} - r_n)\omega_{n-1}$, where ω_i's are the fundamental dominant weights. Two elements in $\widetilde{X}(n)$ have the same image in X iff one of them is obtained from the other by multiplying by a power of $X_{11} X_{22} \cdots X_{nn}$. Thus, π is injective on the set $\widetilde{X}(n, r)$ for any r. In the sequel, when necessary, we will identify $\widetilde{X}(n, r)$ and its image in X.

(8.1) Theorem. *Let* $, _+ = \pi(\widetilde{X}(n, r)_+)$, $, \ = \pi(\widetilde{X}(n, r))$. *Then:*

(1) $,_{+}$ *is a finite ideal of* $(X_{+}, \leq)$, *and* $, = W,_{+}$ *is a finite ideal of* $(X, \leq_{\mathrm{e}})$;

(2) $S_q(n,r) \cong \mathfrak{U}\langle, _{+}\rangle$;

(3) $S_q^{+}(n,r) \cong \mathfrak{B}\langle, \rangle$.

Proof. (1) Since Weyl modules with highest weights in $\widetilde{X}(n,r)_{+}$ are comodules for $A_q(n,r)$, all statements in (1) are clear.

(2) The restriction of the canonical homomorphism $A_q(n) \to k[\mathfrak{U}]$ to $A_q(n,r)$ is injective, and has its image in $\mathcal{O}_{\Gamma_+}k[\mathfrak{U}]$. Thus, we need only to show they have the same dimension. Both the categories comod-$A_q(n,r)$ and comod-$\mathcal{O}_{\Gamma_+}k[\mathfrak{U}]$ are highest weight categories with weight poset $(,_{+}, \leq)$ (see [PW; (11.5.2)] for comod-$A_q(n,r)$). Via the canonical homomorphism $A_q(n,r) \to \mathcal{O}_{\Gamma_+}k[\mathfrak{U}]$, all irreducible objects, Δ-objects and ∇-objects in comod-$A_q(n,r)$ give rise to the corresponding objects in $\mathcal{O}_{\Gamma_+}k[\mathfrak{U}]$. Thus, by the Comparison Theorem [PW; (5.8)], the exact functor comod-$A(n,r) \to$ comod-$\mathcal{O}_{\Gamma_+}k[\mathfrak{U}]$ is an equivalence of highest weight categories, and so carries indecomposable injective objects to indecomposable injective objects. However, $A_q(n,r)$ is a direct sum of indecomposable injective objects, the multiplicity of an indecomposable injective object in the sum being the dimension of its socle. The coalgebra $\mathcal{O}_{\Gamma_+}k[\mathfrak{U}]$ has the same decomposition. Thus, they have the same dimension, as required.

(3) As in (2), we have an injective homomorphism $B_q(n,r) \hookrightarrow \mathcal{O}_{\Gamma}k[\mathfrak{B}]$. On the other hand, by (7.3) and (2), we have a surjective homomorphism $\Psi\colon A_q(n,r) \cong \mathcal{O}_{\Gamma_+}k[\mathfrak{U}] \twoheadrightarrow \mathcal{O}_{\Gamma}k[\mathfrak{B}]$. However, Ψ is induced by the canonical homomorphism $k[\mathfrak{U}] \to k[\mathfrak{B}]$ which sends X_{ij} with $i > j$ to 0. Thus, Ψ factors through $B_q(n,r)$, yielding a surjective homomorphism $B_q(n,r) \twoheadrightarrow \mathcal{O}_{\Gamma}k[\mathfrak{B}]$. This forces $B_q(n,r) \cong \mathcal{O}_{\Gamma}k[\mathfrak{B}]$. $\square$

In [PW; (11.5.6)], it is proved that, given $S_q(n,r)$-modules M, N, the groups $\mathrm{Ext}^{\bullet}_{S_q(n,r)}(M,N)$ agree with the corresponding $\mathrm{Ext}^{\bullet}$-group for the quantum general linear group. The analogous result below for the subalgebras $S_q^{+}(n,r)$ is new.

(8.2) Proposition. *Let* $\widetilde{\mathcal{B}}_q$ *be the category of rational modules for the "upper triangular" Borel subgroup* $\widetilde{B}_q$ *of the general quantum linear group* $\widetilde{G}_q = GL_q(n)$. *The full embedding*

$$S_q^{+}(n,r)\text{-mod} \longrightarrow \widetilde{\mathcal{B}}_q$$

induces a full embedding

$$D^{+}(S_q^{+}(n,r)\text{-mod}) \longrightarrow D^{+}(\widetilde{\mathcal{B}}_q)$$

of derived categories. If $M, N \in \mathrm{Ob}(S_q^{+}(n,r)\text{-mod})$, *then*

$$\mathrm{Ext}^{\bullet}_{S_q^{+}(n,r)}(M,N) \cong \mathrm{Ext}^{\bullet}_{\widetilde{\mathcal{B}}_q}(M,N).$$

Proof. It is easy to see that there is an epimorphism $B_q \times \mathbf{G}_m \to \widetilde{B}_q$, where B_q is the corresponding Borel subgroup in the quantum special linear group $G_q =$

$SL_q(n)$. (The corresponding statement for G_q is proved in [PW; (7.2.2)]. The same argument works for B_q, *mutatis mutandis.*) Let λ be a character (one-dimensional representation) for $\widetilde{B}_q$. The full subcategory $\widetilde{\mathcal{B}}_{q,\lambda}$ of $\widetilde{\mathcal{B}}_q$ consisting of all rational $\widetilde{B}_q$ modules with the property that all composition factors are isotypical of type λ as a $\mathbf{G}_m$-module via the homomorphism $\mathbf{G}_m \to \widetilde{B}_q$. Let $\mathcal{B}_{q,\lambda}$ be the image of $\widetilde{\mathcal{B}}_{q,\lambda}$ under the restriction functor $\widetilde{\mathcal{B}}_q \to \mathcal{B}_q$ given by the embedding $B_q \to \widetilde{B}_q$. One can see easily that $\widetilde{\mathcal{B}}_{q,\lambda}$ (resp., $\mathcal{B}_{q,\lambda}$) is a direct summand of $\widetilde{\mathcal{B}}_q$ (resp., $\mathcal{B}_q$), and the categories $\widetilde{\mathcal{B}}_{q,\lambda}$ and $\mathcal{B}_{q,\lambda}$ are equivalent under the restriction functor. As a result, we have $D^+(\mathcal{B}_{q,\lambda}) \cong D^+(\widetilde{\mathcal{B}}_{q,\lambda})$, and the latter is a full subcategory (even a direct summand) of $D^+(\widetilde{\mathcal{B}}_q)$. On the other hand, by (7.1(3)) and [CPS3; (3.9)], $D^+(S_q^+(n,r)\text{-mod})$ is a full subcategory of $D^+(\mathcal{B}_q)$. If λ is a character of $S_q^+(n,r)$, the image of the full embedding $D^+(S_q^+(n,r)\text{-mod}) \to D^+(\mathcal{B}_q)$ is contained in $D^+(\mathcal{B}_{q,\lambda})$, since any module for $S_q^+(n,r)$ is isotypical of type λ as a $\mathbf{G}_m$-module. This proves the proposition. □

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