STRATIFYING ENDOMORPHISM ALGEBRAS ASSOCIATED TO HECKE ALGEBRAS

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Abstract. Let G be a finite group of Lie type and let k be a field of characteristic *distinct* from the defining characteristic of G. In studying the *non-describing* representation theory of G, the endomorphism algebra $S(G,k) = \operatorname{End}_{kG}(\bigoplus_J \operatorname{ind}_{P_J}^G k)$ plays an increasingly important role. In type A, by work of Dipper and James, $S(G,k)$ identifies with a q-Schur algebra and so serves as a link between the representation theories of the finite general linear groups and certain quantum groups. This paper presents the first systematic study of the structure and homological algebra of these algebras for G of arbitrary type. Because $S(G,k)$ has a reinterpretation as a Hecke endomorphism algebra, it may be analyzed using the theory of Hecke algebras. Its structure turns out to involve new applications of Kazhdan-Lusztig cell theory. In the course of this work, we prove two stratification conjectures about Coxeter group representations made in [CPS4] and we formulate a new conjecture about the structure of $S(G,k)$. We verify this conjecture here in all rank 2 examples.

Let G be a finite group of Lie type. Thus, G is the subgroup of fixed points $\mathbf{G}^\sigma$ for a rational endomorphism σ of a reductive algebraic group $\mathbf{G}$ defined over an algebraically closed field F of positive characteristic. For a field k of characteristic *distinct* from that of F, consider the endomorphism algebra $S(G,k) = \operatorname{End}_{kG}(\bigoplus_{P\supseteq B} \operatorname{ind}_P^G k)$. Here $\operatorname{ind}_P^G k$ denotes the (right) permutation module over k defined by the set $\{Pg\}_{g\in G}$ of right cosets of the parabolic subgroup P in G. Work of Dipper and James for $\mathbf{G} = GL(n)$ (cf. below) suggests that the algebras $S(G,k)$ play an important role in the representation theory of kG for *all* types. This paper presents the first systematic study of these algebras. We establish new results concerning the structure and cohomology of $S(G,k)$ valid for all G. In the process, we also prove several conjectures made in [CPS4; §6] for finite Coxeter groups, and we indicate new directions for further study.

Let (W,S) be the Coxeter system defined by the BN-pair structure on G, let $\widetilde{H}_k$ be the corresponding Hecke algebra over k, and let $\widetilde{H}_{Jk}$, $J \subseteq S$, be the subalgebra of $\widetilde{H}_k$ associated to the Coxeter system (W_J, J). The index representation on $\widetilde{H}_{Jk}$

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is denoted IND_J. Dipper-James [DJ3; (2.24)] prove for $G = GL(n,q)$ that $S(G,k)$ can be reinterpreted as a *Hecke endomorphism algebra*:

$$\mathrm{End}_{kG}\left(\bigoplus_{P\supseteq B}\mathrm{ind}_P^G k\right) \cong \mathrm{End}_{\widetilde{H}_k}\left(\bigoplus_{J\subseteq S}\mathrm{ind}_{\widetilde{H}_{Jk}}^{\widetilde{H}_k}\mathrm{IND}_J\right). \tag{1}$$

In fact, their argument applies, almost without change, to establish (1) in the case of a general reductive group $\mathbf{G}$. For $G = GL(n,q)$, it follows from (1) that $S(G,k)$ is Morita equivalent to the q-Schur algebra $S_q(n,n)$ over k (as first defined by Dipper-James [DJ3]). Thus, the representation theory of $S(GL(n,q),k)$ is closely related to the representation theory of the quantum general linear group $GL_q(n,k)$. See [D2], [PW]. This connection between the representation theories of quantum groups (over fields of positive characteristic), q-Schur algebras, Hecke algebras, and general linear groups in non-describing characteristic has been developed in recent work of Dipper-James [DJ1,2,3], Dipper [D1,2], and others.

In addition, q-Schur algebras have been proved in [PW] to be quasi-hereditary in the sense of [CPS2]. Thus, they arise by means of a recursive construction closely associated, in spirit at least, to the geometric theory of perverse sheaves. (Conversely, by [PS], the category of perverse sheaves on a suitable topological space is equivalent to the module category for a quasi-hereditary algebra.) When $\mathbf{G} \neq GL(n)$, it is possible that $S(G,k)$ will again be quasi-hereditary, subject to restrictions on k and F.[1] However, in this paper, we are interested in proving results about $S(G,k)$ for *all* fields k and *all* types. Section 3 presents examples in which $S(G,k)$ is not quasi-hereditary. Fortunately, these examples suggest that the algebras do possess an interesting structure, not unrelated to quasi-heredity, which has been introduced in [CPS4] under the guise of *stratified algebras*. An algebra A has a stratification of length n if there is a sequence $0 = J_0 \subsetneq J_1 \subsetneq \cdots \subsetneq J_n = A$ of ideals such that, for $i \geq 1$, the inflation functor induces an isomorphism $\mathrm{Ext}^{\bullet}_{A/J_i}(M,N) \cong \mathrm{Ext}^{\bullet}_{A/J_{i-1}}(M,N)$ for all A/J_i-modules M, N. When each J_i/J_{i-1} is a projective (left) A/J_{i-1}-module, we call the stratification *standard*. An algebra A is quasi-hereditary if and only if it has a standard stratification of length equal to the number of irreducible A-modules.

In general, it is desirable to obtain stratifications of an algebra of as long a length as possible (that length never exceeding the number of irreducible modules). Block indecomposable symmetric algebras, such as the blocks of finite group algebras, do not even possess a non-trivial standard stratification of length ≥ 2. However, that situation changes when passing to endomorphism rings: $\mathrm{End}_{kG}(k \bigoplus kG)$, for G a non-trivial finite p-group and $p = \mathrm{char}\, k$, has a non-trivial standard stratifcation of length 2 [CPS4; (6.1)]. The generation of many substantial examples of stratifications of length ≥ 3 is a main consequence of the present paper. The stratifications

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[1] Suppose, for example, that $G = Sp(2n,q)$, $p \neq 2$, and q is not a primitive $2m$th root of 1 in k for some $m \leq n$. Then, by [DJ4; (4.17)], $\widetilde{H}_k$ is Morita equivalent to a finite direct product of Hecke algebras associated to direct products of (two) symmetric groups. Using this Morita equivalence, the quasi-heredity of q-Schur algebras, and the fact that a tensor product of quasi-hereditary algebras is quasi-hereditary [W; (1.3)], it follows that $S(G,k)$ is quasi-hereditary in this case. Geck and Hiss have pointed out to us that [GrH] contains a detailed argument.

we present are not, in general, standard. However, we conjecture that there are standard stratifications of related algebras of much longer length, closer to that obtained for the quasi-hereditary algebra. As discussed above, many of the stratified algebras we discuss appear to be important for the non-describing representation theory of finite groups of Lie type.

A main theme of [CPS4] concerned sufficient (and sometimes necessary) conditions, in terms of an abstract "Specht module" theory for an algebra R, guaranteeing that an endomorphism algebra $A = \mathrm{End}_R(T)$ have a stratification. In that direction, [CPS4] presented several conjectures involving finite Coxeter groups W. In brief, the first conjecture asserts that $\mathrm{End}_{kW}(\bigoplus_J \mathrm{ind}_{W_J}^W k)$ admits a stratification with at least three strata, while the second conjecture makes more precise assertions involving an appropriate Specht module theory. A third conjecture (which was joint work with the first author of this paper) was also briefly discussed—see §2.5 below. The present paper extends these conjectures to include Hecke algebras associated to finite Coxeter groups. We then prove the first two conjectures and lay the groundwork for an attack on the third conjecture. That conjecture (given in (2.5.2)) proposes a long stratification—of length equal to the number of two-sided Kazhdan-Lusztig cells in the Coxeter group—for a related algebra. In type A (now treated in [DPS]), the proposed construction can be chosen to yield a quasi-hereditary algebra, viz., the q-Schur algebra.

Our paper is organized as follows. Section 1 is largely foundational. In §1.1, we set up a general framework for a modular representation theory of Hecke algebras over regular rings of Krull dim. ≤ 2 based on a result of Auslander-Goldman [AG1]. In particular, we adapt in this more general setting some familiar results of Brauer. The remainder of §1 develops the theory of stratifying systems for algebras over general commutative, Noetherian rings, indicating somewhat stronger results when working over a regular ring of Krull dim. ≤ 2.

Section 2 takes up the theory of Hecke endomorphism algebras. Although our motivation comes from finite groups of Lie type, we consider more generally the generic Hecke algebras (over $\mathbb{Z}[t^2, t^{-2}]$) associated to an arbitrary finite Coxeter system and we allow specializations $t^2 \mapsto q$ in a field k in which q is not necessarily the image of a prime power. (Note that even in the finite group case, it is useful to allow q to be more general, e. g., an indeterminate, to allow for preliminary specializations to rings of Laurent polynomials over fields.) After some preliminary work in §§2.1, 2.2, we study various "cell filtrations" in §2.3 for induced representations. In this section, the Kazhdan-Lusztig cell theory for Coxeter groups [KL], [L2] plays an important role. The main result of this subsection, Theorem (2.3.9), establishes a remarkable homological property for cell filtrations of certain induced modules. These ideas combine with those of §1 to prove our main results in §2.4. Thus, by means of the isomorphism (1), Theorem (2.4.4) verifies, as a special case, that the algebras $S(G, k)$ above have a stratification of length ≥ 3; see (2.4.9). Another special case of this theorem proves the first two conjectures for Coxeter groups made in [CPS4]; see (2.4.5) and (2.4.6). A further interesting consequence is that the algebra $S(G, k)$ satisfies a kind of "non-describing characteristic" version of Kempf's vanishing theorem for the cohomology of homogeneous line bundles on flag varieties; see (2.4.7) and the discussion in (2.4.8). Finally, §2.5 considers the third conjecture [CPS4; fn. 26], proving one main formal consequence.

Section 3 treats in some detail some cases when $\mathbf{G}$ has type B_2 or G_2. In particular, in §3.5 we verify Conjecture (2.5.2) for all rank 2 groups of Lie type. (As mentioned above, [DPS] proves the conjecture in type A.) Finally, an appendix (§4) proves—largely for the convenience of the reader—a semisimplicity criterion for modular Hecke algebras. Although this criterion is known (see, e. g., Gyoja [Gy] and Fleischmann [F]), our proof (discovered independently) is more direct for our purposes.

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Notation. By a quasi-poset, we mean a set Λ having a reflexive and transitive relation $\leq$. A quasi-poset Λ always determines a surjective map $\pi : \Lambda \to \bar{\Lambda}$, $\lambda \mapsto \pi(\lambda) = \bar{\lambda}$, from Λ to a (unique, up to isomorphism) poset $\bar{\Lambda}$ in which, given $\lambda, \mu \in \Lambda$, $\lambda \leq \mu \iff \bar{\lambda} \leq \bar{\mu}$ in $\bar{\Lambda}$. Writing $\lambda \sim \mu$ if and only if $\bar{\lambda} = \bar{\mu}$ defines an equivalence relation $\sim$ on Λ—its equivalence classes are called the *cells* of Λ.

For an algebra $\widetilde{A}$ over a commutative ring $\mathcal{Z}$, let ${}_{\widetilde{A}}\mathcal{C}$ (resp., $\mathcal{C}_{\widetilde{A}}$) denote the category of finitely generated left (resp., right) $\widetilde{A}$-modules. If $\widetilde{A}$ is a finitely generated $\mathcal{Z}$-module, we say $\widetilde{A}$ is *finite* over $\mathcal{Z}$. If $\mathcal{Z} \to \mathcal{Z}'$ is a morphism of commutative rings, put $\widetilde{A}_{\mathcal{Z}'} = \widetilde{A} \otimes_{\mathcal{Z}} \mathcal{Z}'$, and, for $\widetilde{M} \in \mathrm{Ob}({}_{\widetilde{A}}\mathcal{C})$, put $\widetilde{M}_{\mathcal{Z}'} = \widetilde{M} \otimes_{\mathcal{Z}} \mathcal{Z}' \in \mathrm{Ob}({}_{\widetilde{A}_{\mathcal{Z}'}}\mathcal{C})$. Similar conventions hold for $\mathcal{C}_{\widetilde{A}}$. Given an abelian category $\mathcal{C}$, $\mathrm{proj}(\mathcal{C})$ denotes the full subcategory of projective objects. Unless otherwise explicitly stated, all commutative rings will assumed to be Noetherian. Thus, when $\widetilde{A}$ is finite over $\mathcal{Z}$, the categories ${}_{\widetilde{A}}\mathcal{C}$ and $\mathcal{C}_{{}_{\widetilde{A}}\mathcal{C}}$ are abelian categories.

1. INTEGRAL THEORY AND STRATIFICATIONS

We show in §1.1 how a "Brauer theory" can be developed for algebras defined over regular local rings of Krull dim. ≤ 2.[2] In §1.2, we discuss stratifying systems for ${}_{\widetilde{A}}\mathcal{C}$, where $\widetilde{A}$ is an algebra which is finite and projective over a commutative ring $\mathcal{Z}$. In §1.3, we specialize to assume that $\mathcal{Z}$ is a field.

1.1. Application of a result of Auslander-Goldman. A *local triple* is a triple $(\mathcal{O}, K, k)$ consisting of a commutative local domain $\mathcal{O}$ having fraction field K and residue field $k = \mathcal{O}/\mathfrak{m}$. A local triple $(\mathcal{O}, K, k)$ determines a second local triple $(\widehat{\mathcal{O}}, \widehat{K}, k)$ if the completion $\widehat{\mathcal{O}} = \varprojlim \mathcal{O}/\mathfrak{m}^i$ is a domain. Then $\mathcal{O} \subseteq \widehat{\mathcal{O}}$ and $\widehat{K}$ is an extension field of K. This occurs, for example, when $\mathcal{O}$ is a regular local ring. Then $\widehat{\mathcal{O}}$ is also a regular local ring and Krull dim. $\widehat{\mathcal{O}} =$ Krull dim. $\mathcal{O}$.

In the representation theory of finite groups, $\mathcal{O}$ is usually taken to be a discrete valuation ring. The general local triple will generally "give birth" to a number of further local triples involving discrete valuation rings, through localization or factoring out by ideals (see (1.1.4) below). It is not, however, convenient to restrict our general set-up to the discrete valuation ring case, because it is too restrictive for the modular representation theory of Hecke algebras. In that case, it is useful

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[2] A very general Brauer theory is developed in [GR]. For some of the discussion in this paper, e. g., (1.1.3), we could quote their results. However, when working over regular rings of Krull dimension ≤ 2 (the relevant case for Hecke algebras considered here), a simpler theory exists, which is in some ways stronger. For example, (1.1.1) shows the existence of lattices, just as in the classical theory over discrete valuation rings.

to allow $\mathcal{O}$ to be a regular local ring of Krull dim. ≤ 2. See §3 for some specific examples.

Our first result uses a theorem of Auslander-Goldman [AG1; Cor., p. 17]: *If $\mathcal{O}$ is a regular local ring of Krull dim. ≤ 2 and if $\widetilde{M}$ is a finitely generated $\mathcal{O}$-module, then the dual $\mathcal{O}$-module* $\mathrm{Hom}_{\mathcal{O}}(\widetilde{M}, \mathcal{O})$ *is projective (= free).* We adopt the convention that if $\widetilde{R}$ is a finite $\mathcal{O}$-algebra, then an $\widetilde{R}$*-lattice* $\widetilde{X}$ is an $\widetilde{R}$-module which is a finitely generated free $\mathcal{O}$-module.

(1.1.1) Proposition. *Let $\mathcal{O}$ be a regular local ring of Krull dim. ≤ 2 and form the local triple $(\mathcal{O}, K, k)$. Let $\widetilde{R}$ be an algebra which is finite and free over $\mathcal{O}$. Suppose that $\mathcal{X}$ is a finitely generated (right) $\widetilde{R}_K$-module. Then $\mathcal{X} = \widetilde{X}_K$ for some $\widetilde{R}$-lattice $\widetilde{X}$.*

Proof. By standard methods, there exists a finitely generated $\widetilde{R}$-submodule $\widetilde{X}_0$ of $\mathcal{X}$ such that $\mathcal{X} = (\widetilde{X}_0)_K$. As noted above, [AG1] implies that $\widetilde{X}_0^* = \mathrm{Hom}_{\mathcal{O}}(\widetilde{X}_0, \mathcal{O}) \in \mathrm{Ob}({}_{\widetilde{R}}\mathcal{C})$ is $\mathcal{O}$-free (of finite rank). Hence, $\widetilde{X} = \widetilde{X}_0^{**} \in \mathrm{Ob}(\mathcal{C}_{\widetilde{R}})$ is $\mathcal{O}$-free of finite rank. Since $\widetilde{X}_0$ is $\mathcal{O}$-torsion free, the natural evaluation map $\mathrm{Ev} : \widetilde{X}_0 \to \widetilde{X}_0^{**}$ defines an injection of $\widetilde{R}$-modules which becomes an isomorphism upon applying $- \otimes_{\mathcal{O}} K$. Identifying $\widetilde{X}$ with an $\widetilde{R}$-submodule of $\mathcal{X}$ gives $\widetilde{X}_K \cong \mathcal{X}$, as required. $\square$

Suppose that $\mathcal{X}_1, \cdots, \mathcal{X}_m$ are the distinct (up to isomorphism) irreducible $\widetilde{R}_K$-modules. By (1.1.1), $\mathcal{X}_i \cong \widetilde{X}_{iK}$ for $\widetilde{R}$-lattices $\widetilde{X}_i$, $1 \leq i \leq m$.

(1.1.2) Corollary. *Assume the hypotheses of (1.1.1). Let $\mathcal{X}_1 \cong \widetilde{X}_{1K}, \cdots, \mathcal{X}_m \cong \widetilde{X}_{mK}$ be as above. Then:–*

(a) Any irreducible $\widetilde{R}_k$-module L is a composition factor of some $\widetilde{X}_{ik}$, $1 \leq i \leq m$.

(b) If $\mathcal{X} \cong \widetilde{X}_K \cong \widetilde{X}'_K$ for $\widetilde{R}$-lattices $\widetilde{X}$ and $\widetilde{X}'$, then $\widetilde{X}_k$ and $\widetilde{X}'_k$ have the same $\widetilde{R}_k$-composition factors (with the same multiplicities).

Proof. (a) Every irreducible $\widetilde{R}_K$-module is (trivially!) a composition factor of some $\mathcal{X}_i$. This is the only property of $\mathcal{X}_1, \cdots, \mathcal{X}_m$ we use. Form the local triple $(\widehat{\mathcal{O}}, \widehat{K}, k)$. The system $\mathcal{X}_{1\widehat{K}}, \cdots, \mathcal{X}_{m\widehat{K}}$ of $\widetilde{R}_{\widehat{K}}$-modules has the property that any irreducible $\widetilde{R}_{\widehat{K}}$−module is a composition factor of some member. Thus, we can assume that $\mathcal{O}$ is complete.

Let $L \in \mathrm{Ob}(\mathcal{C}_{\widetilde{R}_k})$ be irreducible. Choose a projective cover $P \twoheadrightarrow L$ in $\mathcal{C}_{\widetilde{R}_k}$. Since $\mathcal{O}$ is complete, there exists $\widetilde{P} \in \mathrm{Ob}(\mathrm{proj}(\mathcal{C}_{\widetilde{R}}))$ with $\widetilde{P}_k \cong P$. For some i, $\mathrm{Hom}_{\widetilde{R}_K}(\widetilde{P}_K, \mathcal{X}_i) \neq 0$. Since $\widetilde{X}_i$ is $\mathcal{O}$-free and $\widetilde{P}$ is $\widetilde{R}$-projective, $\mathrm{Hom}_{\widetilde{R}}(\widetilde{P}, \widetilde{X}_i)$ is a projective $\mathcal{O}$-module of finite rank. Also, $\mathrm{Hom}_{\widetilde{R}_k}(P, \widetilde{X}_{ik}) \cong \mathrm{Hom}_{\widetilde{R}}(\widetilde{P}, \widetilde{X}_i)_k \neq 0$. Thus, L occurs with nonzero multiplicity in $\widetilde{X}_{ik}$, as required.

(b) As in (a), we can assume that $\mathcal{O}$ is complete. Let P be a projective indecomposable module in the category of $\widetilde{R}_k$-modules, and choose a projective $\widetilde{R}$-module

$\widetilde{P}$ such that $\widetilde{P}_k \cong P$. Then

$$\begin{aligned} \dim \operatorname{Hom}_{\widetilde{R}_k}(P, \widetilde{X}_k) &= \dim \operatorname{Hom}_{\widetilde{R}}(\widetilde{P}, \widetilde{X})_k = \dim \operatorname{Hom}_{\widetilde{R}}(\widetilde{P}, \widetilde{X})_K \\ &= \dim \operatorname{Hom}_{\widetilde{R}_K}(\widetilde{P}_K, \mathcal{X}) = \dim \operatorname{Hom}_{\widetilde{R}}(\widetilde{P}, \widetilde{X}')_k \\ &= \dim \operatorname{Hom}_{\widetilde{R}_k}(P, \widetilde{X}'_k). \end{aligned}$$

Thus, the head of P appears with the same multiplicity in $\widetilde{X}_k$ as it does in $\widetilde{X}'_k$. $\square$

Let $\widetilde{X}_1, \cdots, \widetilde{X}_m$ be $\widetilde{R}$-lattices for $\mathcal{X}_1, \cdots, \mathcal{X}_m$ as above. Let $L_1, \cdots, L_n$ be the distinct irreducible $\widetilde{R}_k$-modules. By (1.1.2(b)), the multiplicities $d_{ij} = [\widetilde{X}_{jk} : L_i]$ are independent of the choice of $\widetilde{X}_j$. Let $D = (d_{ij})$ be the corresponding $n \times m$ decomposition matrix. Also, let $C = (c_{ij})$ be the $n \times n$ Cartan matrix of $\widetilde{R}_k$, i. e., if $P(j)$ denotes the projective cover of L_j in the category of right $\widetilde{R}_k$-modules, then $c_{ij} = [P(j) : L_i]$. Now we can easily prove the following result:

(1.1.3) Theorem. *Let $(\mathcal{O}, K, k)$ be a local triple in which $\mathcal{O}$ is a regular local ring of Krull dim. ≤ 2. Let $\widetilde{R}$ be an algebra, finite and free over $\mathcal{O}$, such that both $\widetilde{R}_K$ and $\widetilde{R}_k/rad(\widetilde{R}_k)$ are split semisimple. Then $C = D \cdot D^T$.*

Proof. Since $\widetilde{R}_K$ is split semisimple, we can assume that $\mathcal{O}$ is complete. Thus, there exists $\widetilde{P}(i) \in \mathrm{Ob}(\mathrm{proj}(\mathcal{C}_{\widetilde{R}}))$ satisfying $\widetilde{P}(i)_k \cong P(i)$. Using $\widetilde{R}$-lattices $\widetilde{X}_1, \cdots, \widetilde{X}_t$ for $\mathcal{X}_1, \cdots, \mathcal{X}_m$, respectively, a standard calculation completes the proof:

$$\begin{aligned} c_{ij} &= \dim \operatorname{Hom}_{\widetilde{R}_k}(P(i), P(j)) = \dim \operatorname{Hom}_{\widetilde{R}_K}(\widetilde{P}(i)_K, \widetilde{P}(j)_K) \\ &= \sum_t [\widetilde{P}(i)_K : \mathcal{X}_t][\widetilde{P}(j)_K : \mathcal{X}_t] \\ &= \sum_t \dim \operatorname{Hom}_{\widetilde{R}_K}(\widetilde{P}(i)_K, \widetilde{X}_{tK}) \cdot \dim \operatorname{Hom}_{\widetilde{R}_K}(\widetilde{P}(j)_K, \widetilde{X}_{tK}) \\ &= \sum_t \dim \operatorname{Hom}_{\widetilde{R}_k}(P(i), \widetilde{X}_{tk}) \cdot \dim \operatorname{Hom}_{\widetilde{R}_k}(P(j), \widetilde{X}_{tk}) \\ &= \sum_t [\widetilde{X}_{tk} : L_i][\widetilde{X}_{tk} : L_j] = \sum_t d_{it} d_{jt}. \quad \square \end{aligned}$$

(1.1.4) Remarks. (a) The results (1.1.1), (1.1.2), and (1.1.3) generalize well-known work of Brauer. Use of [AG1] has allowed us to treat these results all in a concise and familiar way. As mentioned above, [GR] contains a more general Brauer theory, which has (1.1.3) as a special case, but contains no analogue of (1.1.1).

(b) Many discrete valuation rings *do* arise in the context of more general local algebras. For instance, suppose that $\mathfrak{p} \in \operatorname{Spec} \mathcal{O}$ has height 1. Then the localization $\mathcal{O}_{\mathfrak{p}}$ is always a discrete valuation ring, if $\mathcal{O}$ is integrally closed. Also, if $\mathcal{O}$ is regular of Krull dim. 2 and $\mathfrak{p} \not\subseteq \mathfrak{m}^2$, then $\mathcal{O}/\mathfrak{p}$ is a discrete valuation ring, since its maximal ideal $\mathfrak{m}/\mathfrak{p}$ is generated by one element.

1.2. Integral stratifying systems. Let $\mathcal{Z}$ be a fixed commutative ring. Let $\widetilde{A}$ be a $\mathcal{Z}$-algebra which is finite and projective over $\mathcal{Z}$.

(1.2.1) Lemma. *For $\widetilde{P} \in \mathrm{Ob}(proj({}_{\widetilde{A}}\mathcal{C}))$, the following statements are equivalent:*

(1) Every irreducible $\widetilde{A}$-module L appears in the head of $\widetilde{P}$ (i. e., L is a homomorphic image of $\widetilde{P}$).

(2) $\widetilde{P}$ is a progenerator of ${}_{\widetilde{A}}\mathcal{C}$; equivalently, $\widetilde{A}$ is a direct summand of $\widetilde{P}^{\oplus m}$ for some positive integer m.

(3) If $\mathcal{Z}'$ is any commutative $\mathcal{Z}$-algebra, then every irreducible $\widetilde{A}_{\mathcal{Z}'}$-module appears in the head of $\widetilde{P}_{\mathcal{Z}'}$.

Proof. First assume that (1) holds. To see (2), we show that $\widetilde{A}$ is a homomorphic image of a finite direct sum of copies of $\widetilde{P}$. Since $\widetilde{A}$ is Noetherian as a $\mathcal{Z}$-module, it has a left ideal $\widetilde{I}$ which is maximal with respect to being a homomorphic image of some $\widetilde{P}^{\oplus m}$ for positive integer m. If $\widetilde{I} \neq \widetilde{A}$, choose $a \in \widetilde{A} \backslash \widetilde{I}$ and let $\widetilde{J}$ be a maximal left ideal in $\widetilde{A}a + \widetilde{I}$ containing $\widetilde{I}$. By hypothesis, $(\widetilde{A}a + \widetilde{I})/\widetilde{J}$ is a homomorphic image of $\widetilde{P}$. Hence, there exists a morphism $\widetilde{P} \to \widetilde{A}a + \widetilde{I}$ whose image $\widetilde{K}$ is not contained in $\widetilde{I}$, so $\widetilde{K} + \widetilde{I}$ is a homomorphic image of a finite direct sum of copies of $\widetilde{P}$. This contradicts the maximality of $\widetilde{I}$, so $\widetilde{I} = \widetilde{A}$, proving (1) $\implies$ (2).

If (2) holds, then $\widetilde{A}_{\mathcal{Z}'}$ is a direct summand of $\widetilde{P}_{\mathcal{Z}'}^{\oplus m}$ for any commutative $\mathcal{Z}$-algebra $\mathcal{Z}'$. This implies that any irreducible $\widetilde{A}_{\mathcal{Z}'}$ module is a homomorphic image of $\widetilde{P}_{\mathcal{Z}'}$. Therefore, (2) $\implies$ (3). Finally, (3) $\implies$ (1) by taking $\mathcal{Z}' = \mathcal{Z}$. $\square$

(1.2.2) Lemma. *Let $\widetilde{P} \in \mathrm{Ob}(proj({}_{\widetilde{A}}\mathcal{C}))$. The following statements are equivalent for any $\widetilde{M} \in \mathrm{Ob}({}_{\widetilde{A}}\mathcal{C}))$:*

(1) $\mathrm{Hom}_{\widetilde{A}}(\widetilde{P}, \widetilde{M}) \neq 0$;

(2) $\mathrm{Hom}_{\widetilde{A}_k}(\widetilde{P}_k, \widetilde{M}_k) \neq 0$ for some field $k = \mathcal{Z}/\mathfrak{m}$;

(3) $\mathrm{Hom}_{\widetilde{A}_k}(\widetilde{P}_k, \widetilde{M}_k) \neq 0$ for some field k which is a $\mathcal{Z}$-algebra.

Proof. Assume (1). Replace $\mathcal{Z}$ by $\mathcal{Z}_{\mathfrak{m}}$ for a maximal ideal $\mathfrak{m} \in \mathrm{Supp}(\mathrm{Hom}_{\widetilde{A}}(\widetilde{P}, \widetilde{M}))$ to assume that $\mathcal{Z}$ is a local ring with residue field k. Suppose that $\mathrm{Hom}_{\widetilde{A}_k}(\widetilde{P}_k, \widetilde{M}_k) = 0$, so any $\widetilde{A}$-module morphism $\widetilde{P} \to \widetilde{M}$ has image contained in $\mathfrak{m}\widetilde{M}$. For some positive integer n (the number of generators of $\mathfrak{m}$ as an $\mathcal{O}$-module), there is a surjective morphism $\widetilde{M}^{\oplus n} \twoheadrightarrow \mathfrak{m}\widetilde{M}$. Since $\widetilde{P} \in \mathrm{Ob}(\mathrm{proj}({}_{\widetilde{A}}\mathcal{C}))$, it now follows that any morphism $\widetilde{P} \to \widetilde{M}$ has image contained in $\mathfrak{m}^2\widetilde{M}$. Continuing, we see that any morphism $\widetilde{P} \to \widetilde{M}$ has image contained in $\bigcap \mathfrak{m}^i\widetilde{M}$, which is 0 by Krull's theorem. This contradicts (1), so (1) $\implies$ (2). The implication (2) $\implies$ (3) is trivial.

For any field $k = \mathcal{Z}/\mathfrak{m}$, $\widetilde{M}_k$ is a $\widetilde{A}$-module homomorphic image of $\widetilde{M}$. Because $\widetilde{P}$ is projective, we obtain that (2) $\implies$ (1).

Assume that (3) holds for some field k which is a $\mathcal{Z}$-algebra. Then k is an extension field of $k' = \mathcal{Z}_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}$ for some $\mathfrak{p} \in \mathrm{Spec}\,\mathcal{Z}$. Since $\mathrm{Hom}_{\widetilde{A}_{k'}}(\widetilde{P}_{k'}, \widetilde{M}_{k'})_k \cong$

$\mathrm{Hom}_{\widetilde{A}_k}(\widetilde{P}_k, \widetilde{M}_k)$, so (3) $\implies$ (2) for $\widetilde{A}_{\mathfrak{p}}$ over $\mathcal{Z}_{\mathfrak{p}}$. Thus, (1) holds for $\widetilde{A}_{\mathfrak{p}}$, and so holds for $\widetilde{A}$ itself. Thus, (3) $\implies$ (1). $\square$

We will use below some notational conventions regarding filtrations of modules, closely adhering to that given in [CPS4; §§1.4,1.5] (where many further details on the discussion can be found). A decreasing filtration $\widetilde{G}$ on $\widetilde{X} \in \mathrm{Ob}({}_{\widetilde{A}}\mathcal{C})$ (of length m) is a sequence $\widetilde{G}: \ \widetilde{X} = \widetilde{G}_0 \supseteq \widetilde{G}_1 \supseteq \cdots \supseteq \widetilde{G}_m = 0$ of submodules of $\widetilde{X}$. By convention, put $\widetilde{G}_i = 0$ for $i > m$ and $\widetilde{G}_i = \widetilde{X}$ for $i < 0$. For any i, the ith section of $\widetilde{G}$ is $\mathrm{Gr}_i\widetilde{G} = \widetilde{G}_i/\widetilde{G}_{i+1}$.

Generally, we work with decreasing filtrations on ${}_{\widetilde{A}}\mathcal{C}$, and increasing filtrations on $\mathcal{C}_{\widetilde{A}}$. An increasing filtration $\widetilde{F}$ on $\widetilde{Y} \in \mathrm{Ob}(\mathcal{C}_{\widetilde{A}})$ consists of a sequence $\widetilde{F}: \ 0 = \widetilde{F}^0 \subseteq \widetilde{F}^1 \subseteq \cdots \subseteq \widetilde{F}^n = \widetilde{Y}$ of submodules of $\widetilde{Y}$. Define $\widetilde{F}^i = 0$ for $i < 0$ and $\widetilde{F}^i = \widetilde{Y}$ for $i > n$, and, for any i, put $\mathrm{Gr}^i\widetilde{F} = \widetilde{F}^{i+1}/\widetilde{F}^i$.

Now suppose that $\widetilde{T}$ is an $(\widetilde{A}, \widetilde{R})$-bimodule. Despite the ambiguity, it is convenient to let $(-)^\diamond$ denote *either* of the contravariant functors

$$(1.2.3) \qquad (-)^\diamond = \mathrm{Hom}_{\widetilde{A}}(-, \widetilde{T}) : {}_{\widetilde{A}}\mathcal{C} \to \mathcal{C}_{\widetilde{R}}, \qquad (-)^\diamond = \mathrm{Hom}_{\widetilde{R}}(-, \widetilde{T}) : \mathcal{C}_{\widetilde{R}} \to {}_{\widetilde{A}}\mathcal{C}.$$

These induce functors $(-)^\diamond : {}_{\widetilde{A}}\mathcal{C}_{\mathrm{filt}} \to \mathcal{C}^{\mathrm{filt}}_{\widetilde{R}}$ and $(-)^\diamond : \mathcal{C}^{\mathrm{filt}}_{\widetilde{R}} \to {}_{\widetilde{A}}\mathcal{C}_{\mathrm{filt}}$. Here ${}_{\widetilde{A}}\mathcal{C}_{\mathrm{filt}}$ (resp., $\mathcal{C}^{\mathrm{filt}}_{\widetilde{R}}$) denotes the category whose objects are pairs $(\widetilde{X}, \widetilde{G})$ (resp., $(\widetilde{Y}, \widetilde{F})$) consisting of $\widetilde{X} \in \mathrm{Ob}({}_{\widetilde{A}}\mathcal{C})$ (resp., $\widetilde{Y} \in \mathrm{Ob}(\mathcal{C}_{\widetilde{R}})$)), together with a decreasing (resp., increasing) filtration $\widetilde{G}$ on $\widetilde{X}$ (resp., $\widetilde{F}$ on $\widetilde{Y}$), and whose morphisms are defined in the natural way. For example, given $(\widetilde{Y}, \widetilde{F}) \in \mathrm{Ob}(\mathcal{C}^{\mathrm{filt}}_{\widetilde{R}})$, we have $(\widetilde{Y}, \widetilde{F})^\diamond = (\widetilde{Y}^\diamond, \widetilde{F}^\diamond) \in \mathrm{Ob}({}_{\widetilde{A}}\mathcal{C}_{\mathrm{filt}})$, where $(\widetilde{F}^\diamond)_i = (\widetilde{Y}/\widetilde{F}^i)^\diamond$. In particular, observe that $\mathrm{Gr}_i\widetilde{F}^\diamond \subseteq (\mathrm{Gr}^i\widetilde{F})^\diamond$, though equality need not hold. We will use several times below (see the proof of (1.2.10), for example) the following fact: *Suppose* $(\widetilde{Y}, \widetilde{F}) \in \mathrm{Ob}(\mathcal{C}^{\mathrm{filt}}_{\widetilde{R}})$ *and* $\mathrm{Ext}^1_{\widetilde{R}}(\widetilde{Y}/\widetilde{F}^{i+1}, \widetilde{T}) = 0$ *for some* i. *Then* $Gr_i\widetilde{F}^\diamond \cong (Gr^i\widetilde{F})^\diamond$. This follows easily from the long exact sequence of cohomology.

The definition below of a stratifying system for ${}_{\widetilde{A}}\mathcal{C}$ generalizes that given in [CPS4; §6.4] for fields.

(1.2.4) Definition. Let Λ be a finite quasi-poset. Suppose for $\lambda \in \Lambda$ there is given $\widetilde{\Delta}(\lambda) \in \mathrm{Ob}({}_{\widetilde{A}}\mathcal{C})$ which is projective over $\mathcal{Z}$. Then $\{\widetilde{\Delta}(\lambda)\}_\lambda$ is a *stratifying system* for ${}_{\widetilde{A}}\mathcal{C}$ provided that, for each $\lambda \in \Lambda$, there is given $\widetilde{P}(\lambda) \in \mathrm{Ob}(\mathrm{proj}({}_{\widetilde{A}}\mathcal{C}))$ and a surjective morphism $\widetilde{P}(\lambda) \twoheadrightarrow \widetilde{\Delta}(\lambda)$ satisfying the following three conditions:

(1) For $\lambda, \mu \in \Lambda$, if $\mathrm{Hom}_{\widetilde{A}}(\widetilde{P}(\lambda), \widetilde{\Delta}(\mu)) \neq 0$, then $\lambda \leq \mu$.

(2) For $\lambda \in \Lambda$, there is given a decreasing filtration $\widetilde{G}^\lambda$ of $\widetilde{P}(\lambda)$ (of length $t(\lambda)$) with $\mathrm{Gr}_i\widetilde{G}^\lambda \cong \widetilde{\Delta}(\nu_{\lambda,i})$ $(0 \leq i < t(\lambda))$ where $\lambda \leq \nu_{\lambda,i} \in \Lambda$ and $\nu_{\lambda,0} = \lambda$.

(3) Any irreducible $\widetilde{A}$-module is a homomorphic image of some $\widetilde{\Delta}(\lambda)$.

Suppose condition (2) is replaced by the stronger condition:

(2′) For $\lambda \in \Lambda$, there is given a decreasing filtration $\widetilde{G}^\lambda$ of $\widetilde{P}(\lambda)$ (of length $t(\lambda)$) satisfying $\mathrm{Gr}_i\widetilde{G}^\lambda \cong \widetilde{\Delta}(\nu_{\lambda,i})$ where $\bar{\lambda} < \bar{\nu}_{\lambda,i}$ for $0 < i < t(\lambda)$ and $\nu_{\lambda,0} = \lambda$.

In this case, we say the stratifying system is *strict*.

Suppose that we are given a stratifying system for ${}_{\widetilde{A}}\mathcal{C}$ with respect to a quasi-poset Λ. Define an equivalence relation on Λ by putting $\lambda \sim \mu$ if and only if $\widetilde{\Delta}(\lambda) \cong \widetilde{\Delta}(\mu)$. By (1.2.4(1)), the corresponding equivalence classes are contained in cells of Λ. Hence, there is a quasi-poset structure on the set $\Lambda_{\min}$ of equivalence classes so that the quotient map $\Lambda \to \Lambda_{\min}$, $\lambda \mapsto \tilde{\lambda}$, is a morphism of quasi-posets. If we put $\widetilde{\Delta}(\tilde{\lambda}) = \widetilde{\Delta}(\lambda)$ for all $\tilde{\lambda} \in \Lambda_{\min}$ and any $\lambda \in \tilde{\lambda}$, then $\{\widetilde{\Delta}(\tilde{\lambda})\}_{\tilde{\lambda}}$ is a stratifying system for ${}_{\widetilde{A}}\mathcal{C}$. We do not require Λ being minimal in (1.2.4) because, in practice, the larger quasi-posets Λ sometimes appear naturally; see (2.4.3) and (2.5.2) below. Also, in (1.2.4), we do not insist that $\widetilde{\Delta}(\tilde{\lambda})$ is indecomposable. When $\mathcal{Z}$ is a field and the stratification is strict, the indecomposable components of the $\widetilde{\Delta}(\tilde{\lambda})$ all have simple heads [CPS4; (6.4.6)], and can be uniquely defined in terms of their heads and the order relation, cf. [CPS4; (2.2.2(d)].

When $\{\widetilde{\Delta}(\lambda)\}_{\lambda}$ is a stratifying system for ${}_{\widetilde{A}}\mathcal{C}$, we call the $\widetilde{\Delta}(\lambda)$ the *standard objects* for ${}_{\widetilde{A}}\mathcal{C}$. We now give conditions for checking that a collection of modules is a stratifying system. (In particular, the notion behaves well under base change.)

(1.2.5) Lemma. *Let Λ be a finite quasi-poset, and for each $\lambda \in \Lambda$, let $\widetilde{\Delta}(\lambda) \in \mathrm{Ob}({}_{\widetilde{A}}\mathcal{C})$. For $\lambda \in \Lambda$, assume that $\widetilde{\Delta}(\lambda) \in \mathrm{Ob}(proj(\mathcal{C}_{\mathcal{Z}}))$ and that there is given $(\widetilde{P}(\lambda), \widetilde{G}^{\lambda}) \in \mathrm{Ob}({}_{\widetilde{A}}\mathcal{C}_{\mathrm{filt}})$ where $\widetilde{P}(\lambda) \in \mathrm{Ob}(proj({}_{\widetilde{A}}\mathcal{C}))$ and $\widetilde{G}^{\lambda}$ satisfies (1.2.4(2)). Then the following four statements are equivalent:*

(1) $\{\widetilde{\Delta}(\lambda)\}_{\lambda\in\Lambda}$ is a stratifying system for ${}_{\widetilde{A}}\mathcal{C}$.

(2) $\{\widetilde{\Delta}(\lambda)_k\}_{\lambda\in\Lambda}$ is a stratifying system for ${}_{\widetilde{A}_k}\mathcal{C}$, for any $k = \mathcal{Z}/\mathfrak{m}$, where $\mathfrak{m}$ is a maximal ideal of $\mathcal{Z}$.

(3) $\{\widetilde{\Delta}(\lambda)_k\}_{\lambda\in\Lambda}$ is a stratifying system for ${}_{\widetilde{A}_k}\mathcal{C}$, for each field k which is a $\mathcal{Z}$-algebra.

(4) $\{\widetilde{\Delta}(\lambda)_{\mathcal{Z}'}\}_{\lambda\in\Lambda}$ is a stratifying system for ${}_{\widetilde{A}_{\mathcal{Z}'}}\mathcal{C}$, for each commutative $\mathcal{Z}$ algebra $\mathcal{Z}'$.

Finally, assume that (1.2.4(2′)) holds and, in the statements of (1), (2), (3), (4) above, the phrase "stratifying system" is replaced by "strict stratifying system". Then the resulting statements are equivalent.

Proof. Assume that (1) holds, and let $\mathcal{Z}'$ be as in (4). Each $\widetilde{\Delta}(\lambda)_{\mathcal{Z}'}$ is a homomorphic image of $\widetilde{P}(\lambda)_{\mathcal{Z}'} \in \mathrm{Ob}(\mathrm{proj}({}_{\widetilde{A}_{\mathcal{Z}'}}\mathcal{C}))$. Also, each $\widetilde{\Delta}(\lambda)_{\mathcal{Z}'}$ is a projective $\mathcal{Z}'$-module. If $\mathrm{Hom}_{\widetilde{A}_{\mathcal{Z}'}}(\widetilde{P}(\lambda)_{\mathcal{Z}'}, \widetilde{\Delta}(\mu)_{\mathcal{Z}'}) \neq 0$, then, by (1.2.2), for some field k which is a $\mathcal{Z}'$-algebra, $\mathrm{Hom}_{\widetilde{A}_k}(\widetilde{P}(\lambda)_k, \widetilde{\Delta}(\mu)_k) \neq 0$; so that (1.2.2) implies that $\mathrm{Hom}_{\widetilde{A}}(\widetilde{P}(\lambda), \widetilde{\Delta}(\mu)) \neq 0$, whence $\lambda \leq \mu$ by hypothesis. Thus, (1.2.4(1)) holds over $\mathcal{Z}'$. Because each $\widetilde{\Delta}(\lambda) \in \mathrm{Ob}(\mathrm{proj}(\mathcal{Z}))$, $\widetilde{G}^{\lambda}_{\mathcal{Z}'}$ is a decreasing filtration of $\widetilde{P}(\lambda)_{\mathcal{Z}'}$ as required by (1.2.4(2)). Using (1.2.1) with $\widetilde{P} = \bigoplus_{\lambda} \widetilde{P}(\lambda)$ we see that any irreducible $\widetilde{A}_{\mathcal{Z}'}$-module is a homomorphic image of some $\widetilde{\Delta}(\lambda)_{\mathcal{Z}'}$; thus, (1.2.4(3)) holds for the $\widetilde{\Delta}(\lambda)_{\mathcal{Z}'}$. Hence, (1) $\implies$ (4). The other implications are clear. We leave the proof of the final assertion on strictness to the reader. $\square$

Later we will use the following result on Ext. We omit the standard proof (by dimension shifting). See [CPS4; (6.4.2)] for a similar argument.

(1.2.6) Lemma. *Suppose that* $\{\widetilde\Delta(\lambda)\}_{\lambda\in\Lambda}$ *is a stratifying system for* ${}_{\widetilde A}\mathcal{C}$. *Let* $\lambda,\mu\in\Lambda$ *be fixed elements satisfying* $\operatorname{Ext}^i_{\widetilde A}(\widetilde\Delta(\lambda),\widetilde\Delta(\mu))\neq 0$ *for some nonnegative integer* i. *Then* $\lambda\leq\mu$. *Suppose further, in the notation of (1.2.4(2)), that* $i>0$, *and* $\bar\lambda<\bar\nu_{\lambda,j}$ *for all* $j>0$.[3] *Then* $\bar\lambda<\bar\mu$.

An ideal $\widetilde J$ of $\widetilde A$ is called a *stratifying ideal* provided that $\widetilde J$ is a $\mathcal{Z}$-direct summand of $\widetilde A$ and, for any $\widetilde M,\widetilde N\in\operatorname{Ob}({}_{\widetilde A/\widetilde J}\mathcal{C})$, inflation defines an isomorphism $\operatorname{Ext}^\bullet_{\widetilde A/\widetilde J}(\widetilde M,\widetilde N)\cong\operatorname{Ext}^\bullet_{\widetilde A}(\widetilde M,\widetilde N)$. A sequence $0=\widetilde J_0\subsetneq\widetilde J_1\subsetneq\cdots\subsetneq\widetilde J_n=\widetilde A$ of stratifying ideals in $\widetilde A$ is called a *stratification of length* n. If each $\widetilde J_i/\widetilde J_{i-1}$ is a projective (left) $\widetilde A/\widetilde J_{i-1}$-module, the stratification is called *standard.*

If two algebras $\widetilde A$ and $\widetilde A'$ are Morita equivalent, it is obvious that a stratifying system for ${}_{\widetilde A}\mathcal{C}$ determines one for ${}_{\widetilde A'}\mathcal{C}$. The following result shows that if $\widetilde A$ has a stratification of length n, so does any Morita equivalent algebra. (See, for example, [CPS4; (2.1.6(a))] where a similar result is proved.)

(1.2.7) Lemma. *Suppose that* $\widetilde A$ *has a stratification* $0=\widetilde J_0\subsetneq\widetilde J_1\subsetneq\cdots\subsetneq\widetilde J_n=\widetilde A$ *of length* n. *Let* $\widetilde P\in\operatorname{Ob}(\mathit{proj}({}_{\widetilde A}\mathcal{C}))$ *be a progenerator of* ${}_{\widetilde A}\mathcal{C}$ *and consider the Morita equivalent algebra* $\widetilde A'=\operatorname{End}_{\widetilde A}(\widetilde P)$. *Then* $\widetilde A'$ *also has a stratification of length* n. *If the stratification for* $\widetilde A$ *is standard, that for* $\widetilde A'$ *can be taken to be standard also.*

Proof. It is well-known that $\widetilde J\mapsto\operatorname{Hom}_{\widetilde A}(\widetilde P,\widetilde J\widetilde P)$ defines an equivalence between the lattice of ideals of $\widetilde A$ and that of $\widetilde A'$. If $\widetilde J$ is a $\mathcal{Z}$-direct summand of $\widetilde A$ (i. e., the inclusion $\widetilde J\subseteq\widetilde A$ is split as $\mathcal{Z}$-modules), then it is easy to see that $\operatorname{Hom}_{\widetilde A}(\widetilde P,\widetilde J\widetilde P)$ is a $\mathcal{Z}$-direct summand of $\widetilde A'$. Also, since $\widetilde P$ is a progenerator, given an ideal $\widetilde J$ of $\widetilde A$, the full subcategory ${}_{\widetilde A/\widetilde J}\mathcal{C}$ of ${}_{\widetilde A}\mathcal{C}$ has objects which are precisely the homomorphic images of finite direct sums of copies of $\widetilde P/\widetilde J\widetilde P$. Therefore, the equivalence $\operatorname{Hom}_{\widetilde A}(\widetilde P,-):{}_{\widetilde A}\mathcal{C}\to{}_{\widetilde A'}\mathcal{C}$ restricts to define an equivalence ${}_{\widetilde A/\widetilde J}\mathcal{C}\xrightarrow{\sim}{}_{\widetilde A'/\widetilde J'}\mathcal{C}$, where $\widetilde J'=\operatorname{Hom}_{\widetilde A}(\widetilde P,\widetilde J\widetilde P)$. Thus, $\widetilde J$ is a stratifying ideal of $\widetilde A$ if and only if $\widetilde J'$ is a stratifying ideal of $\widetilde A'$. This establishes the first assertion of the lemma.

If $\widetilde J\in\operatorname{Ob}(\operatorname{proj}({}_{\widetilde A}\mathcal{C}))$, then $\widetilde J'=\operatorname{Hom}_{\widetilde A}(\widetilde P,\widetilde J\widetilde P)\in\operatorname{Ob}(\operatorname{proj}({}_{\widetilde A'}\mathcal{C}))$. Since $\widetilde A'/\widetilde J'\cong\operatorname{End}_{\widetilde A/\widetilde J}(\widetilde P/\widetilde J\widetilde P)$ and $\widetilde P/\widetilde J\widetilde P$ is a progenerator for ${}_{\widetilde A/\widetilde J}\mathcal{C}$, the second assertion follows. □

The argument for [CPS4; (6.4.5)] and (1.2.7) give the following result.

(1.2.8) Theorem. *Assume that* ${}_{\widetilde A}\mathcal{C}$ *has a stratifying system* $\{\widetilde\Delta(\lambda)\}_{\lambda\in\Lambda}$. *The algebra* $\widetilde A$ *has a stratification* $0=\widetilde J_0\subsetneq\widetilde J_1\subsetneq\cdots\subsetneq\widetilde J_n=\widetilde A$ *of length* $n=\#\bar\Lambda$. *If the stratifying system is strict, then the stratification can be chosen to be standard.*

The following hypothesis (generalizing [CPS4; (6.4.7)]) gives a criterion for strat-

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[3] Of course, this latter condition always holds if $\{\widetilde\Delta(\lambda)\}_\lambda$ is a strict stratifying system.

ifying ${}_{\widetilde{A}}\mathcal{C}$, when $\widetilde{A}$ is an endomorphism algebra.

(1.2.9) Hypothesis. *Let $\widetilde{R}$ be a finite and projective algebra over $\mathcal{Z}$. Let $\widetilde{T}$ be a finitely generated right $\widetilde{R}$-module, projective over $\mathcal{Z}$, and put $\widetilde{A} = \mathrm{End}_{\widetilde{R}}(\widetilde{T})$. Assume $\widetilde{T} = \bigoplus_{\lambda\in\Lambda} \widetilde{T}_\lambda^{\oplus m_\lambda}$ is a fixed direct sum decomposition, where Λ is a finite quasi-poset. (No assumption that the $\widetilde{T}_\lambda$ are indecomposable or even non-isomorphic is imposed.) For $\lambda \in \Lambda$, assume given an $\widetilde{R}$-submodule $\widetilde{S}_\lambda \hookrightarrow \widetilde{T}_\lambda$ and an increasing filtration $\widetilde{F}_\lambda$ of $\widetilde{T}_\lambda$ (of length $t(\lambda)$) such that the following three conditions hold:*

(1) For $\lambda \in \Lambda$, there is a fixed sequence $\nu_{\lambda,0}, \nu_{\lambda,1}, \cdots, \nu_{\lambda,t(\lambda)-1}$ in Λ such that $\nu_{\lambda,0} = \lambda$ and, for $i > 0$, $\nu_{\lambda,i} \geq \lambda$. For $0 \leq i < t(\lambda)$, there is given a fixed isomorphism $Gr^i\widetilde{F}_\lambda \cong \widetilde{S}_{\nu_{\lambda,i}}$.

(2) For $\lambda, \mu \in \Lambda$, $\mathrm{Hom}_{\widetilde{R}}(\widetilde{S}_\mu, \widetilde{T}_\lambda) \neq 0 \implies \lambda \leq \mu$.

(3) For all $\lambda \in \Lambda$, we have $\mathrm{Ext}\,^1_{\widetilde{R}}(\widetilde{T}_\lambda/\widetilde{F}^i_\lambda, \widetilde{T}) = 0$ for all i.

In establishing the following result, we use the notation (1.2.3).

(1.2.10) Theorem. *Assume that Hypothesis (1.2.9) holds. For $\lambda \in \Lambda$, put $\widetilde{\Delta}(\lambda) = \mathrm{Hom}_{\widetilde{R}}(\widetilde{S}_\lambda, \widetilde{T})$ and assume that each $\widetilde{\Delta}(\lambda)$ is $\mathcal{Z}$-projective. Then $\{\widetilde{\Delta}(\lambda)\}_{\lambda\in\Lambda}$ is a stratifying system for ${}_{\widetilde{A}}\mathcal{C}$ with respect to Λ. If the inequalities in (1.2.9(1)) can be replaced by strict inequalities $\bar{\nu}_{\lambda,i} > \bar{\lambda}$ for all $i > 0$, the system is strict.*

Proof. Put $(\widetilde{P}(\lambda), \widetilde{G}^\lambda) = (\widetilde{T}_\lambda, \widetilde{F}_\lambda)^\diamond$, i. e., $\widetilde{G}^\lambda_i = (\widetilde{T}_\lambda/\widetilde{F}^i_\lambda)^\diamond$, $\forall i$. The discussion immediately above (1.2.4) and (1.2.9(3)) imply that each $\mathrm{Gr}_i\widetilde{G}^\lambda \cong (\mathrm{Gr}^i\widetilde{F}_\lambda)^\diamond$, so (1.2.4(2)) holds. Since each $\widetilde{\Delta}(\lambda) \in \mathrm{Ob}(\mathrm{proj}(\mathcal{C}_\mathcal{Z}))$, each $\widetilde{P}(\lambda) \in \mathrm{Ob}(\mathrm{proj}(\mathcal{C}_\mathcal{Z}))$, too. Thus, $\widetilde{A} = \bigoplus_\lambda \widetilde{P}(\lambda)^{\oplus m_\lambda}$ is $\mathcal{Z}$-projective.

Observe that (1.2.9(3)) also implies that each $\widetilde{\Delta}(\lambda)$ is a homomorphic image of $\widetilde{P}(\lambda)$. Since $\mathrm{Hom}_{\widetilde{A}}(\widetilde{P}(\lambda), \widetilde{\Delta}(\mu)) \cong \mathrm{Hom}_{\widetilde{R}}(\widetilde{S}_\mu, \widetilde{T}_\lambda)$, (1.2.9(2)) $\implies$ (1.2.4(1)).

Finally, for any field k which is an $\mathcal{Z}$-algebra, $\widetilde{A}_k \cong \bigoplus_\lambda \widetilde{P}(\lambda)_k^{\oplus m_\lambda}$, so any irreducible $\widetilde{A}_k$-module is a homomorphic image of some $\widetilde{P}(\lambda)$. Since the sections of $\widetilde{G}^\lambda$ have the form $\widetilde{\Delta}(\mu)$, $\mu \in \Lambda$, any irreducible $\widetilde{A}$-module is a homomorphic image of some $\widetilde{\Delta}(\mu)$, so (1.2.4(3)) holds. The final assertion is clear. □

In (1.2.13) below we give a method for checking that (1.2.9(3)) holds when $\mathcal{Z}$ is a domain with fraction field K such that $\widetilde{R}_K$ is semisimple. When $\mathcal{Z} = k$ is a field (and $\widetilde{R}$ is not semisimple), it is generally not easy to verify that condition (1.2.9)(3)) holds—in fact, in many cases, it may fail, even though $\widetilde{A}$ has an interesting stratification; see §3.2 below. One way to obtain a stratification in the field case is to check (1.2.9) over a larger ring $\mathcal{Z}$ having k as a homomorphic image, using (1.2.10) over $\mathcal{Z}$ and (1.2.9). Of course, this requires at least that all data defining $\widetilde{A}$ over k exist over $\mathcal{Z}$ (something which often happens). For another, more complicated, way to work strictly over k itself, see §1.3 below.

In (1.2.10), it was necessary to assume that the $\widetilde{\Delta}(\lambda)$ are $\mathcal{Z}$-projective. In some important cases, this comes for free, using the following result. For $\widetilde{M} \in \mathrm{Ob}(\mathcal{C}_\mathcal{Z})$, put $\widetilde{M}^* = \mathrm{Hom}_\mathcal{Z}(\widetilde{M}, \mathcal{Z})$, and let $\mathrm{Ev} : \widetilde{M} \to \widetilde{M}^{**}$ be the natural evaluation map.

(1.2.11) Lemma. *Let $\mathcal{Z}$ be a regular ring of Krull dim. ≤ 2 having fraction field K. Let $\widetilde{Y} \in \mathrm{Ob}(\mathcal{C}_{\mathcal{Z}})$, and let $\mathcal{V}$ be a K-subspace of $\widetilde{Y}_K$. Assume that $\widetilde{Y} \cong \widetilde{X}^*$ for some $\widetilde{X} \in \mathrm{Ob}(\mathcal{C}_{\mathcal{Z}})$. (In particular, this holds if $\widetilde{Y} \cong \widetilde{Y}^{**}$.) Then $\widetilde{V} \overset{\mathrm{def}}{=} \widetilde{Y} \cap \mathcal{V}$ is $\mathcal{Z}$-projective.*

Proof. We show that $\mathrm{Ext}^1_{\mathcal{Z}}(\widetilde{V}, \widetilde{M}) = 0$, $\forall \widetilde{M} \in \mathrm{Ob}(\mathcal{C}_{\mathcal{Z}})$. If some $\mathrm{Ext}^1_{\mathcal{Z}}(\widetilde{V}, \widetilde{M}) \neq 0$, then $\mathrm{Ext}^1_{\mathcal{Z}}(\widetilde{V}, \widetilde{M})_{\mathfrak{p}} \cong \mathrm{Ext}^1_{\mathcal{Z}_{\mathfrak{p}}}(\widetilde{V}_{\mathfrak{p}}, \widetilde{M}_{\mathfrak{p}}) \neq 0$ for $\mathfrak{p} \in \mathrm{Supp}(\mathrm{Ext}^1_{\mathcal{Z}}(\widetilde{V}, \widetilde{M}))$. Thus, we can assume $\mathcal{Z}$ is local. By [AG1; Cor., p. 17], $\widetilde{Y}$ and $\widetilde{V}^*$ are $\mathcal{Z}$-free of finite rank. In particular, $\mathrm{Ev} : \widetilde{Y} \xrightarrow{\sim} \widetilde{Y}^{**}$ is an isomorphism. Form the commutative diagram

$$\begin{tikzcd}
\widetilde{V} \arrow[r, "\mathrm{Ev}"] \arrow[d, "\beta"] & \widetilde{V}^{**} \arrow[r, "\alpha"] \arrow[d, "\gamma"] & (\widetilde{V}^{**})_K \arrow[r, "\sim"] & \mathcal{V} \arrow[d, "\delta"] \\
\widetilde{Y} \arrow[r, "\sim", "\mathrm{Ev}"'] & \widetilde{Y}^{**} \arrow[r, "\zeta"'] & (\widetilde{Y}^{**})_K \arrow[r, "\sim"] & \widetilde{Y}_K
\end{tikzcd}$$

in which β and δ are the natural inclusions, $\gamma = \beta^{**}$, and α, ζ are the natural (injective) localization maps. Identifying $\widetilde{Y}$ and $\widetilde{Y}^{**}$, it follows that $\widetilde{V}^{**} \subseteq \widetilde{Y} \cap \mathcal{V}$, and so $\mathrm{Ev} : \widetilde{V} \to \widetilde{V}^{**}$ is an isomorphism. By [AG1; Cor., p. 17] again, $\widetilde{V}$ is $\mathcal{Z}$-projective, as required. $\square$

Thus, we can improve upon (1.2.10) when $\mathcal{Z}$ is a regular ring of Krull dim. ≤ 2.

(1.2.12) Corollary. *Assume that Hypothesis (1.2.9) holds and that $\mathcal{Z}$ is a regular ring of Krull dim. ≤ 2. For $\lambda \in \Lambda$, put $\widetilde{\Delta}(\lambda) = \mathrm{Hom}_{\widetilde{R}}(\widetilde{S}_\lambda, \widetilde{T})$. Then $\{\widetilde{\Delta}(\lambda)\}_\lambda$ is a stratifying system for ${}_{\widetilde{A}}\mathcal{C}$. If the inequalities $\nu_{\lambda,i} \geq \lambda$ in (1.2.9(1)) can be replaced by strict inequalities $\bar{\nu}_{\lambda,i} > \lambda$ for all $i > 0$, the system is strict.*

Proof. To apply (1.2.10), we show each $\widetilde{\Delta}(\lambda)$ is $\mathcal{Z}$-projective. We can assume that $\mathcal{Z}$ is local. Because $\widetilde{T}$ is $\mathcal{Z}$-free, $\mathrm{Hom}_{\mathcal{Z}}(\widetilde{S}_\lambda, \widetilde{T})$ is a direct sum of $\mathcal{Z}$-modules isomorphic to $\widetilde{S}^*_\lambda$, so is $\mathcal{Z}$-free by [AG1] again. Hence, $\mathrm{Hom}_{\mathcal{Z}}(\widetilde{S}_\lambda, \widetilde{T})^{**} \cong \mathrm{Hom}_{\mathcal{Z}}(\widetilde{S}_\lambda, \widetilde{T})$, so (1.2.11) applies with $\widetilde{Y} = \mathrm{Hom}_{\mathcal{Z}}(\widetilde{S}_\lambda, \widetilde{T})$ and $\mathcal{V} = \mathrm{Hom}_{\widetilde{R}_K}(\widetilde{S}_{\lambda K}, \widetilde{T}_K)$. $\square$

We conclude this subsection with a useful criterion for verifying that (1.2.9(3)) holds. In this result, we do not require that $\widetilde{A}$ be $\mathcal{Z}$-projective.

(1.2.13) Lemma. *Assume that $\mathcal{Z}$ is a commutative domain with fraction field K. Let $\widetilde{A}$ be a finite $\mathcal{Z}$-algebra such that $\widetilde{A}_K$ is semisimple. Let $\widetilde{M}, \widetilde{T}$ be finitely generated left (or right) $\widetilde{A}$-modules and assume that $\widetilde{T}$ is $\mathcal{Z}$-torsion free. The following two statements are equivalent:*

(1) For every $d \in \mathcal{Z}$, the natural map

$$\mathrm{Hom}_{\widetilde{A}}(\widetilde{M}, \widetilde{T}) \longrightarrow \mathrm{Hom}_{\widetilde{A}_{\mathcal{Z}/(d)}}(\widetilde{M}/d\widetilde{M}, \widetilde{T}/d\widetilde{T}) \tag{1.2.13.1}$$

is surjective.

(2) $\operatorname{Ext}^1_{\widetilde{A}}(\widetilde{M},\widetilde{T})=0$.

Proof. For $0\neq d\in\mathcal{Z}$, multiplication by d defines a short exact sequence $0\to\widetilde{T}\overset{d}{\to}\widetilde{T}\to\widetilde{T}/d\widetilde{T}\to 0$, and, therefore, an exact sequence

$$\operatorname{Hom}_{\widetilde{A}}(\widetilde{M},\widetilde{T})\overset{\alpha}{\to}\operatorname{Hom}_{\widetilde{A}}(\widetilde{M},\widetilde{T}/d\widetilde{T})\to\operatorname{Ext}^1_{\widetilde{A}}(\widetilde{M},\widetilde{T})\overset{d}{\to}\operatorname{Ext}^1_{\widetilde{A}}(\widetilde{M},\widetilde{T}). \tag{1.2.13.2}$$

Since $\operatorname{Hom}_{\widetilde{A}}(\widetilde{M},\widetilde{T}/d\widetilde{T})\cong\operatorname{Hom}_{\widetilde{A}_{\mathcal{Z}/(d)}}(\widetilde{M}/d\widetilde{M},\widetilde{T}/d\widetilde{T})$, α identifies with the map in (1.2.13.1).

Assume (1) holds. Since $\widetilde{A}_K$ is semisimple, $\operatorname{Ext}^1_{\widetilde{A}_K}(\widetilde{M}_K,\widetilde{T}_K)\cong\operatorname{Ext}^1_{\widetilde{A}}(\widetilde{M},\widetilde{T})_K=0$, so that $\operatorname{Ext}^1_{\widetilde{A}}(\widetilde{M},\widetilde{T})$ is a finitely generated torsion $\mathcal{Z}$-module. Choose $0\neq d\in\mathcal{Z}$ in the annihilator of $\operatorname{Ext}^1_{\widetilde{A}}(\widetilde{M},\widetilde{T})$. Since the map (1.2.13.1) is surjective, (1.2.13.2) shows that multiplication by d induces an injection $\operatorname{Ext}^1_{\widetilde{A}}(\widetilde{M},\widetilde{T})\hookrightarrow\operatorname{Ext}^1_{\widetilde{A}}(\widetilde{M},\widetilde{T})$. Hence, $\operatorname{Ext}^1_{\widetilde{A}}(\widetilde{M},\widetilde{T})=0$, and (2) holds.

The implication (2) $\implies$ (1) is also clear from (1.2.13.2) (and does not require that $\widetilde{A}_K$ be semisimple). $\square$

1.3. The stratification hypothesis over a field. Hypothesis (1.3.2) below provides a way to construct a standard stratification of an endomorphism algebra A over a field k. One way to do this has already been indicated in §1.2, using (1.2.10) over a larger ring $\mathcal{Z}$, together with (1.2.5) (and (1.2.8)). However, in the absence of an integral structure, the appropriate stratification hypothesis is quite complicated. Nevertheless, the more elaborate conditions in (1.3.2) can often be checked directly. We illustrate this explicitly in §3—the only place this subsection plays a role in this paper.

Let A be a finite dimensional algebra over k. We will say that ${}_A\mathcal{C}$ is *standardly stratified* if it has a strict stratifying system $\{\Delta(\lambda)\}_{\lambda\in\Lambda}$ (in the sense of (1.2.4), taking $\mathcal{Z}=k$, $\widetilde{A}=A$, etc.) such that: (i) each $\Delta(\lambda)$, $\lambda\in\Lambda$, has an irreducible head $L(\lambda)$; (ii) if $\lambda\neq\mu$, then $L(\lambda)\ncong L(\mu)$; and (iii) each $\widetilde{P}(\lambda)=P(\lambda)$, $\lambda\in\Lambda$, is indecomposable. By [CPS4; (2.2.3)], A has a standard stratification of length n if and only if ${}_A\mathcal{C}$ is standardly stratified with $\#\bar{\Lambda}=n$.[4]

Let A, R be finite dimensional algebras over k and let T be a finite dimensional (A,R)-bimodule. We assume that

$$\begin{cases}(1)\ \text{In } \mathcal{C}_R,\ T\cong\bigoplus_{\lambda\in\Lambda},\ Y_\lambda^{\oplus m_\lambda},\quad Y_\lambda\ \text{distinct indecomposable;}\\ (2)\ A=\operatorname{End}_R(T).\end{cases} \tag{1.3.1}$$

The $P(\lambda)=\operatorname{Hom}_R(Y_\lambda,T)$, $\lambda\in\Lambda$, are the distinct projective indecomposable modules for ${}_A\mathcal{C}$. The following hypothesis was first presented in [CPS4; (3.1.1)]:

(1.3.2) Hypothesis. Let R,T,A be as in (1.3.1). For each $\lambda\in\Lambda$, there is given a fixed R-submodule $S_\lambda\overset{\iota_\lambda}{\hookrightarrow}Y_\lambda$ and a fixed (increasing) filtration F_λ of Y_λ of length $t(\lambda)$. Let $\phi_\lambda=\operatorname{Hom}_R(\iota_\lambda,T):P(\lambda)\to\operatorname{Hom}_R(S_\lambda,T)$. Put $v_\lambda=\dim\operatorname{Im}\phi_\lambda$.

---

[4] One can similarly define a notion of standardly stratified category over a local triple, though a complete treatment requires the theory of semiperfect rings. We defer that project to a future paper. For some discussion in the discrete valuation ring case, see [CPS4; (4.7)].

These data are subject to the following four conditions:

(1) Λ has a fixed quasi-poset structure. For $\lambda \in \Lambda$, there is given a fixed sequence $\nu_{\lambda,0}, \nu_{\lambda,1}, \cdots, \nu_{\lambda,t(\lambda)-1}$ in Λ such that $\nu_{\lambda,0} = \lambda$ and, for $i > 0$, $\bar{\nu}_{\lambda,i} > \bar{\lambda}$. For $0 \leq i < t(\lambda)$, there is a fixed isomorphism $S_{\nu_{\lambda,i}} \cong \mathrm{Gr}^i F_\lambda$.

(2) For $\lambda, \mu \in \Lambda$, $0 \leq i < t(\lambda)$, let $Y_\lambda/F^i_\lambda \xrightarrow{f} Y_\mu$ be a morphism. Then there exists a morphism $Y_{\nu_{\lambda,i}} \xrightarrow{t} Y_\mu$ making the following diagram commutative:

(1.3.2.1)

$$\begin{tikzcd}
Y_\lambda/F^i_\lambda \arrow[r, "f"] & Y_\mu \\
S_{\nu_{\lambda,i}} \arrow[u, "\epsilon_{\nu_{\lambda,i}}"] \arrow[r, "\iota_{\nu_{\lambda,i}}"'] & Y_{\nu_{\lambda,i}} \arrow[u, "t"']
\end{tikzcd}\ .$$

Here $\epsilon_{\nu_{\lambda,i}}$ is the composite of $S_{\nu_{\lambda,i}} \xrightarrow{\sim} \mathrm{Gr}^i F_\lambda$ and the inclusion $\mathrm{Gr}^i F_\lambda \hookrightarrow Y_\lambda/F^i_\lambda$.

(3) For $\lambda, \mu \in \Lambda$, let $\phi_{\lambda,\mu} = \mathrm{Hom}_R(\iota_\lambda, Y_\mu) : \mathrm{Hom}_R(Y_\lambda, Y_\mu) \to \mathrm{Hom}_R(S_\lambda, Y_\mu)$. If $\phi_{\lambda,\mu} \neq 0$, then $\mu \leq \lambda$.

(4) For $\lambda, \mu \in \Lambda$, let $a_{\lambda,\mu} = \#\{i \mid \nu_{\lambda,i} = \mu\}$. Then $\sum_{\lambda,\mu} m_\lambda a_{\lambda,\mu} v_\mu = \dim A$.

The following result follows from [CPS4; (3.1.3)].

(1.3.3) Theorem. *Assume Hypothesis (1.3.2) holds. Then ${}_A\mathcal{C}$ is standardly stratified with respect to Λ, using the standard objects $\Delta(\lambda) = Im\,\phi_\lambda$, $\lambda \in \Lambda$.*

(1.3.4) Remark. [CPS4; (3.3.2)] provides a converse to (1.3.3). Thus, Hypothesis (1.3.2) is essentially *required* in order to standardly stratify ${}_A\mathcal{C}$, through an endomorphism ring structure on A.

2. HECKE ENDOMORPHISM ALGEBRAS

In §2.1, some generalities involving Hecke algebras are taken up. Quite general "cell filtrations" are briefly discussed in §2.2. These concepts are applied in §2.3 to establish important properties of the "dual left cell filtration" and the "dual right-set filtration" for certain induced representations. Using the results of §1, we prove the main stratification results in §2.4. In particular, (2.4.5) and (2.4.6) prove the first two conjectures in [CPS4; §6]. Finally, §2.5 discusses a stronger conjecture involving standard stratifications.

Except in §2.2, $\mathscr{Z}$ denotes the ring $\mathbb{Z}[t^2, t^{-2}]$ of Laurent polynomials.

2.1. Hecke algebras. To begin, let $\mathcal{G} = \{G(q)\}$ be a family of finite groups of Lie type having irreducible Coxeter system (W, S)—cf. [CR; (68.22)]. For each q, fix a Borel subgroup $B(q)$ and define index parameters $c_s \in \mathbb{Z}$, $s \in S$, by $[B(q) : {}^sB(q) \cap B(q)] = q^{c_s}$. The corresponding generic Hecke algebra $\widetilde{H}$ over $\mathscr{Z} = \mathbb{Z}[t^2, t^{-2}]$ has basis τ_w, $w \in W$, satisfying the relations:

$$
(2.1.1)\qquad \tau_s\tau_w = \begin{cases} \tau_{sw} & sw > w; \\ t^{2c_s}\tau_{sw} + (t^{2c_s}-1)\tau_w & sw < w. \end{cases}
$$

We call $\widetilde{H}$ a *Hecke algebra of Lie type* over $\mathcal{Z}$.

More generally, we will allow (W, S) to be *any* finite Coxeter system. Form the algebra $\widetilde{H}$ over $\mathcal{Z}$ with a basis τ_w, $w \in W$, satisfying (2.1.1) for some system $\{c_s\}_{s\in S}$ of integers satisfying $c_s = c_{s'}$ if s and s' are W-conjugate.[5]

Let $\mathcal{Z}'$ be a commutative $\mathcal{Z}$-algebra. Let q denote the image of t^2 in $\mathcal{Z}'$. (The element $q \in \mathcal{Z}'$ should not be confused with the parameter q above for $\mathcal{G}$.) The algebra $\widetilde{H}_{\mathcal{Z}'}$ has a basis, whose elements are still denoted τ_w, $w \in W$, satisfying the relations (2.1.1). In order to reign in the notation, it will be convenient to denote $\widetilde{H}_{\mathcal{Z}'}$ by $\widetilde{H}'$—this notation will alert the reader that we are considering Hecke algebras over a ring more general than $\mathcal{Z}$.

Two specific linear characters will often be used. These are the "trivial" representation IND and the "sign" representation SGN, defined on generators by:

$$
(2.1.2)\qquad \mathrm{IND} : \widetilde{H}' \to \mathcal{Z}',\ \tau_s \mapsto q^{c_s} \qquad \mathrm{SGN} : \widetilde{H}' \to \mathcal{Z}',\ \tau_s \mapsto -1 \quad (s \in S).
$$

For $w \in W$, put $q_w = \mathrm{IND}(\tau_w) \in \mathcal{Z}'$. (When it is clear from context, we also use this notation when $\mathcal{Z}' = \mathcal{Z}$, e. g., $q_s = t^{2c_s}$.)

Let Λ be the power set of S. For $\lambda \in \Lambda$, let $W_\lambda = \langle s \,|\, s \in \lambda\rangle$ and $\widetilde{H}'_\lambda = \langle \tau_s \,|\, s \in \lambda\rangle \subseteq \widetilde{H}'$. The set W^λ (resp., ${}^\lambda W$) of right (resp., left) distinguished coset representatives of W_λ in W indexes a basis $\{\tau_w\}_{w\in W^\lambda}$ (resp., $\{\tau_w\}_{w\in {}^\lambda W}$) for $\widetilde{H}'$ as a module for $\widetilde{H}'_\lambda$ for the left (resp., right) regular action.

The restriction functor $\mathrm{res}^{\widetilde{H}'}_{\widetilde{H}'_\lambda} : \mathcal{C}_{\widetilde{H}'} \to \mathcal{C}_{\widetilde{H}'_\lambda}$ has exact left adjoint

$$
(2.1.3)\qquad \mathrm{ind}^{\widetilde{H}'}_{\widetilde{H}'_\lambda} : \mathcal{C}_{\widetilde{H}'_\lambda} \to \mathcal{C}_{\widetilde{H}'} \qquad \widetilde{M} \mapsto \mathrm{ind}^{\widetilde{H}'}_{\widetilde{H}'_\lambda}(\widetilde{M}) = \widetilde{M} \otimes_{\widetilde{H}'_\lambda} \widetilde{H}'.
$$

Let IND_λ be the trivial representation for $\widetilde{H}'_\lambda$. Put $x_\lambda = \sum_{w\in W_\lambda} \tau_w$. Then

$$
(2.1.4)\qquad x_\lambda\tau_s = q_s x_\lambda, \qquad \tau_s x_\lambda = q_s x_\lambda,\ \forall s \in \lambda.
$$

Thus, the right ideal $\mathcal{Z}'x_\lambda = x_\lambda \widetilde{H}'_{\lambda\mathcal{Z}'}$ in $\widetilde{H}'_{\lambda\mathcal{Z}'}$ realizes IND_λ. Because $\{\tau_w\}_{w\in W^\lambda}$ is a basis for $\widetilde{H}' \in \mathrm{Ob}({}_{\widetilde{H}'_\lambda}\mathcal{C})$, multiplication defines an isomorphism

$$
(2.1.5)\qquad \mathrm{ind}^{\widetilde{H}'}_{\widetilde{H}'_\lambda}\mathrm{IND}_\lambda \xrightarrow{\sim} x_\lambda \widetilde{H}'.
$$

Of course, similar results hold for categories of left modules. We record the following characterization of the ideals $x_\lambda\widetilde{H}'$ and $\widetilde{H}'x_\lambda$.

---

[5] We observe that these algebras are a special case of the Hecke algebras H_ϕ considered in [L2] with $\phi(s) = t^{2c_s}$, so the results there apply to the present situation. The algebra H_ϕ exists for functions ϕ defined on S with values in an abelian group , —in the present case, consisting of integer powers of t^2—which are constant on intersections of S with W-conjugacy classes.

(2.1.6) Lemma. *For* $\lambda \in \Lambda$, *we have* $\widetilde{H}'x_\lambda = \{\xi \in \widetilde{H}' \mid \xi\tau_s = q_s\xi, \forall s \in \lambda\}$ *and* $x_\lambda\widetilde{H}' = \{\xi \in \widetilde{H}' \mid \tau_s\xi = q_s\xi, \forall s \in \lambda\}$.

Proof. If $\xi \in x_\lambda\widetilde{H}'$, (2.1.4) implies that $\tau_s\xi = q_s\xi$ for all $s \in \lambda$. Conversely, suppose that $\xi \in \widetilde{H}'$ satisfies $\tau_s\xi = q_s\xi$ for all $s \in \lambda$. Write $\xi = \sum_w b_w\tau_w$, where $b_w \in \mathcal{Z}'$. For $w \in W$ and $s \in \lambda$ satisfying $w < sw$, the equality $\tau_s\xi = q_s\xi$ implies that

$$\begin{aligned} q_s b_w\tau_w + q_s b_{sw}\tau_{sw} &= \tau_s(b_w\tau_w + b_{sw}\tau_{sw}) \\ &= q_s b_{sw}\tau_w + (b_w + (q_s - 1)b_{sw})\tau_{sw}. \end{aligned}$$

Thus, $b_{sw} = b_w$. It follows for $v \in W^\lambda$ and $u \in W_\lambda$ that $b_{uv} = b_v$. Hence, $\xi \in x_\lambda\widetilde{H}'$, proving the assertion for $x_\lambda\widetilde{H}'$. The other assertion follows similarly. □

The algebra $\widetilde{H}'$ admits a symmetric, associative pairing $\langle\, , \rangle : \widetilde{H}' \times \widetilde{H}' \to \mathcal{Z}'$ defined on basis elements by

$$\langle\tau_w, \tau_v\rangle = \begin{cases} q_w & w = v^{-1} \\ 0 & \text{otherwise.} \end{cases} \tag{2.1.7}$$

(See [DJ1; (2.3)] in type A—the argument is the same in general.) Consider the duality functors

$$\begin{cases} (-)^* = (-)^*_{\mathcal{Z}'} : \mathcal{C}_{\widetilde{H}'} \to {}_{\widetilde{H}'}\mathcal{C} \\ (-)^* = (-)^*_{\mathcal{Z}'} : {}_{\widetilde{H}'}\mathcal{C} \to \mathcal{C}_{\widetilde{H}'} \end{cases} \text{given by } \widetilde{M} \mapsto \widetilde{M}^* \overset{\text{def}}{=} \mathrm{Hom}_{\mathcal{Z}'}(\widetilde{M}, \mathcal{Z}'). \tag{2.1.8}$$

(2.1.9) Lemma. *For* $\lambda \in \Lambda$, *we have* $(x_\lambda\widetilde{H}'_\lambda)^* \cong \widetilde{H}'_\lambda x_\lambda$ *and* $(\widetilde{H}'_\lambda x_\lambda)^* \cong x_\lambda\widetilde{H}'_\lambda$.

Proof. If $\widetilde{M} \in \mathrm{Ob}(\mathcal{C}_{\widetilde{H}})$ is $\mathcal{Z}$-projective, then $(M^*)_{\mathcal{Z}'} \cong (M_{\mathcal{Z}'})^*$. Therefore, it suffices to prove the result when $\mathcal{Z}' = \mathcal{Z}$.

If $d_\lambda = \mathrm{IND}(x_\lambda) \in \mathcal{Z}$, then $x_\lambda^2 = d_\lambda x_\lambda \neq 0$. For $u \in W^\lambda$ and $v \in {}^\lambda W$, (2.1.7) and the equality $(W^\lambda)^{-1} = {}^\lambda W$ imply that $\langle x_\lambda\tau_u, \tau_v x_\lambda\rangle = d_\lambda q_u\delta_{u,v^{-1}}$. Hence, $(\, ,) = \frac{1}{d_\lambda}\langle\, , \rangle : x_\lambda\widetilde{H} \times \widetilde{H}x_\lambda \to \mathcal{Z}$ defines a perfect pairing, with the basis $\{x_\lambda\tau_u\}_{u \in W^\lambda}$ of $x_\lambda\widetilde{H}$ dual to the basis $\{q_v^{-1}\tau_v x_\lambda\}_{v^{-1} \in W^\lambda}$ of $\widetilde{H}x_\lambda$. Thus, $x_\lambda\widetilde{H} \to (\widetilde{H}x_\lambda)^*$, $\xi \mapsto (\xi, -)$, gives the required isomorphism of $\widetilde{H}$-modules. This proves one claim in (2.1.9); the second is similar. □

2.2. Cell filtrations. In this subsection, $\widetilde{R}$ is an algebra over $\mathcal{Z}$. Here $\mathcal{Z}$ can be an arbitrary commutative, Noetherian ring (and not just the ring $\mathbb{Z}[t^2, t^{-2}]$ of Laurent polynomials as in the previous subsection). Let $\{\tau_s\}_{s \in S}$ be a fixed generating set for $\widetilde{R}$. Let $\widetilde{Y} \in \mathrm{Ob}(\mathcal{C}_{\widetilde{R}})$ be $\mathcal{Z}$-free with basis $\{\theta_\lambda\}_{\lambda \in \Lambda}$ indexed by a finite quasi-poset Λ. In some situations, the cell structure on Λ leads to *cell filtrations* on $\widetilde{Y}$. For example, assume that

$$\theta_\lambda\tau_s = \sum_{\mu \leq \lambda} a^{(s)}_{\lambda\mu}\theta_\mu, \ (a^{(s)}_{\lambda\mu} \in \mathcal{Z}) \quad \forall s \in S, \forall \lambda \in \Lambda. \tag{2.2.1}$$

Fix a listing

$$\mathfrak{C}_1, \cdots, \mathfrak{C}_m \tag{2.2.2}$$

of the cells in Λ such that, given $\lambda \in \mathfrak{C}_i$ and $\mu \in \mathfrak{C}_j$, if $\bar{\lambda} < \bar{\mu}$, then $i < j$—i. e., the "larger" cells are listed to the *right* of the "smaller" cells. The (increasing) *cell filtration* $\widetilde{F} = \widetilde{F}_{\widetilde{Y}}$ on $\widetilde{Y}$ (of length m) is defined by putting

$$\widetilde{F}^i = \operatorname{span}\left(\theta_\lambda \mid \lambda \in \mathfrak{C}_j, j \leq i\right), \ 0 \leq i \leq m \ \ (\text{Let } \mathfrak{C}_0 = \emptyset). \tag{2.2.3}$$

For $0 < i < m$, $\operatorname{Gr}^i\widetilde{F}$ is a free $\mathcal{Z}$-module with basis $\{\theta_\lambda + \widetilde{F}^i\}_{\lambda \in \mathfrak{C}_{i+1}}$.

We can also consider the category ${}_{\widetilde{R}}\mathcal{C}$ of left modules for $\widetilde{R}$. Suppose $\widetilde{X} \in \operatorname{Ob}({}_{\widetilde{R}}\mathcal{C})$ is $\mathcal{Z}$-free with basis $\{\rho_\lambda\}_{\lambda\in\Lambda}$ indexed by the finite quasi-poset Λ and satisfying

$$\tau_s\rho_\lambda = \sum_{\mu \leq \lambda} b^{(s)}_{\mu\lambda}\rho_\mu, \ (b^{(s)}_{\mu\lambda} \in \mathcal{Z}) \quad \forall s \in S, \forall \lambda \in \Lambda. \tag{2.2.4}$$

The (decreasing) *cell filtration*[6] $\widetilde{E} = \widetilde{E}^{\widetilde{X}}$ on $\widetilde{X}$ (of length m) is defined by putting

$$\widetilde{E}_i = \operatorname{span}\left(\rho_\lambda \mid \lambda \in \mathfrak{C}_j, j \leq m - i\right). \tag{2.2.5}$$

Suppose that $\widetilde{X} \in \operatorname{Ob}({}_{\widetilde{R}}\mathcal{C})$ is $\mathcal{Z}$-free with basis $\{\rho_\lambda\}_{\lambda\in\Lambda}$ satisfying (2.2.4) for a finite quasi-poset Λ. The dual $\widetilde{Y} = (\widetilde{X})^* = \operatorname{Hom}_{\mathcal{Z}}(\widetilde{X}, \mathcal{Z}) \in \operatorname{Ob}(\mathcal{C}_{\widetilde{R}})$ is a free $\mathcal{Z}$-module with basis $\{\theta_\lambda\}_{\lambda\in\Lambda}$ defined by $\theta_\lambda(\rho_\mu) = \delta_{\lambda,\mu}$, $\mu \in \Lambda$, which satisfies (2.2.1) for the *opposite* quasi-poset Λ^{op}, putting $a^{(s)}_{\lambda\mu} = b^{(s)}_{\lambda\mu}$. Letting $\mathfrak{C}^{\text{op}}_i = \mathfrak{C}_{m-i+1}$, we obtain $(\widetilde{X}^*, \widetilde{F}) \in \operatorname{Ob}(\mathcal{C}^{\text{filt}}_{\widetilde{R}})$ as described in (2.2.3) for the listing $\mathfrak{C}^{\text{op}}_1, \cdots, \mathfrak{C}^{\text{op}}_m$. On the other hand, the decreasing filtration $\widetilde{E}$ of $\widetilde{X}$ defined in (2.2.5) determines, after taking duals, an increasing filtration $\widetilde{E}^*$ on $\widetilde{X}^*$ by setting $(\widetilde{E}^*)^i = (\widetilde{X}/\widetilde{E}_i)^*$. We call $\widetilde{E}^*$ the *dual left cell filtration* of the *right* $\widetilde{R}$-module $\widetilde{X}^*$.

(2.2.6) Proposition. *In the notation of the previous paragraph,* $\widetilde{F} = (\widetilde{E})^*$.

Proof. The left $\widetilde{R}$-module $\widetilde{X}/\widetilde{E}_i$ has basis consisting of the cosets $\rho_\lambda + \widetilde{E}_i$, $\lambda \in \mathfrak{C}_j$ with $j > m - i$. Hence, $(\widetilde{X}/\widetilde{E}_i)^* \subseteq \widetilde{X}^*$ has basis consisting of the θ_λ, $\lambda \in \mathfrak{C}_j$ with $j > m - i$. On the other hand, $\widetilde{F}^i$ is spanned by the θ_λ, $\lambda \in \mathfrak{C}^{\text{op}}_j$ with $j \leq i$. Since $\mathfrak{C}^{\text{op}}_j = \mathfrak{C}_{m-j+1}$, it follows that $(\widetilde{E}^*)^i = \widetilde{F}^i$, as required. $\square$

2.3. Cell filtrations for Hecke algebras. Return to the setting of §2.1 for a generic algebra $\widetilde{H}$ associated to a general finite Coxeter system (W, S) and system $\{c_s \in \mathbb{Z}\}_{s\in S}$ satisfying $c_s = c_t$ if s and t are W-conjugate. Recall that Λ denotes the power set of S.

In order to apply §2.2, we will need to modify the Kazhdan-Lusztig basis for the generic Hecke algebra

$$\widetilde{H}_0 = \widetilde{H} \otimes \mathcal{Z}_0, \qquad \text{where } \mathcal{Z}_0 = \mathbb{Z}[t, t^{-1}]. \tag{2.3.1}$$

---

[6] We denote the filtration by $\widetilde{E}$, rather than our customary $\widetilde{G}$, because the latter is reserved for the filtrations in (1.2.4(2)).

By [KL] (in the "untwisted" case $c_s = 1$, all s) and [L2] (in the "twisted" case), there is a $\mathcal{Z}_0$-basis $\{C'_w\}_{w\in W}$ for $\widetilde{H}_0$ satisfying

$$\tau_s C'_w = \begin{cases} t^{2c_s} C'_w, & \text{if } sw < w \\ -C'_w + t^{c_s} C'_{sw} + t^{c_s} \sum_z M^s_{z,w} C'_z, & \text{otherwise} \end{cases} \tag{2.3.1a}$$

for $s \in S, w \in W$. The sum in (2.3.1a) is over $z < w$ such that $sz < z$, and the $M^s_{z,w} \in \mathcal{Z}$ are certain recursively defined elements (which are symmetric and may vanish). Also,

$$C'_w \tau_s = \begin{cases} t^{2c_s} C'_w, & \text{if } ws < w \\ -C'_w + t^{c_s} C'_{ws} + t^{c_s} \sum_z M^s_{z^{-1},w^{-1}} C'_z, & \text{otherwise} \end{cases} \tag{2.3.1b}$$

for $s \in S, w \in W$. Here the sum is over $z < w$ satisfying $zs < z$. (Apply the involution j [L2; p. 104] to formulas (6.3), (6.4) in *op. cit.*) When $\widetilde{H}_0$ is untwisted, the formulas simplify somewhat; cf. [KL; §2].

By the definition of the basis elements C'_y (see [KL] and [L2; p. 101]), we clearly obtain a basis $\{C^+_w\}_{w\in W}$ for $\widetilde{H}$, where

$$C^+_y = q_y^{1/2} C'_y, \qquad y \in W. \tag{2.3.2}$$

It will be convenient to also call this basis the Kazhdan-Lusztig basis for $\widetilde{H}$. Writing $q = t^{2c_s}$, we can rewrite (2.3.1a) as an equation in $\widetilde{H}$ as follows:–

$$\tau_s C^+_w = \begin{cases} q_s C^+_w, & \text{if } sw < w \\ -C^+_w + C^+_{sw} + \sum_z \widetilde{M}^s_{z,w} C^+_z, & \text{otherwise} \end{cases} \tag{2.3.2a}$$

for $s \in S, w \in W$. Here $\widetilde{M}^s_{z,w} \in \mathcal{Z}$. There is an analogous version of (2.3.1b):–

$$C^+_w \tau_s = \begin{cases} q_s C^+_w, & \text{if } ws < w \\ -C^+_w + C^+_{ws} + \sum_z \widetilde{M}^s_{z^{-1},w^{-1}} C^+_z, & \text{otherwise.} \end{cases} \tag{2.3.2b}$$

The formulas (2.3.1a,b) lead to three quasi-poset structures $\leq_L$, $\leq_R$ and $\leq_{LR}$ on W. See [KL; p. 167], [L2; p. 105] for precise definitions. The corresponding cells are the *left*, *right* and *two-sided* Kazhdan-Lusztig cells, respectively, in W (relative to the system $\{c_s\}_{s\in S}$—if W is not simply laced, cells corresponding to different systems may be different). The left (resp., right) quasi-poset $(W, \leq_L)$ (resp., $(W, \leq_R)$) satisfies the condition (2.2.4) (resp., (2.2.1)) for the left (resp., right) regular action of the algebra $\widetilde{R} = \widetilde{H}$ on $\widetilde{Y} = \widetilde{H}$, using the generators $\{\tau_s\}_{s\in S}$ for $\widetilde{R}$ and the distinguished basis $\{C^+_w\}_{w\in W}$ for $\widetilde{Y}$. Also, (2.2.4) is valid using the quasi-poset $(W, \leq_{LR})$ and the algebra $\widetilde{R} = \widetilde{H} \otimes_{\mathcal{Z}} \widetilde{H}^{\text{op}}$ with generators $\{\tau_s \otimes 1, 1 \otimes \tau_s\}_{s\in S}$ and the left module $\widetilde{Y} = \widetilde{H}$ with basis $\{C^+_w\}_{w\in W}$.

In §2.4, we will use yet another quasi-poset structure, the *right-set order* $\leq^{\mathcal{R}}$, on W. Given $w \in W$, its right-set (resp., left-set) is defined by $\mathcal{R}(w) = \{s \in S \mid ws <$

$w\}$ (resp., $\mathcal{L}(w) = \{s \in S \,|\, sw < w\}$). Then $u \leq^{\mathcal{R}} v \iff \mathcal{R}(u) \supseteq \mathcal{R}(v)$ defines a quasi-poset structure on W. It is well-known ([KL; (2.4)], [X; (1.20)]) that

$$u \leq_L v \implies u \leq^{\mathcal{R}} v, \tag{2.3.3}$$

so we can define the right-set of a left cell ω by putting $\mathcal{R}(\omega) = \mathcal{R}(y)$ for any $y \in \omega$.

Let $\mathcal{Z}'$ be a commutative $\mathcal{Z}$-algebra. As in §2.1, denote $\widetilde{H}_{\mathcal{Z}'}$ by simply $\widetilde{H}'$. The basis $\{C_w^+\}_w$ determines a basis, whose elements are still denoted C_w^+, for $\widetilde{H}'$. The formulas (2.3.2a,b) remain valid.

(2.3.4) Lemma. *Let $s \in S$, $x \in \widetilde{H}'$. If $x\tau_s = q_s x$ (resp., $\tau_s x = q_s x$), then $x = \sum_y a_y C_y^+$ where $a_y \neq 0 \implies s \in \mathcal{R}(y)$ (resp., $a_y \neq 0 \implies s \in \mathcal{L}(y)$).*

Proof. Suppose $x\tau_s = q_s x$, but that $x \notin \operatorname{span}\left(C_y^+ \,|\, s \in \mathcal{R}(y)\right)$. By (2.3.2b), we can assume that $a_y \neq 0 \implies s \notin \mathcal{R}(y)$. Among those y with $a_y \neq 0$, choose y_0 of maximal length. Since $y_0 s > y_0$, (2.3.2b) implies that $C_{y_0 s}^+$ appears with non-zero coefficient in $x\tau_s = q_s\xi$, a contradiction. A similar argument applies when $\tau_s x = q_s x$. $\square$

Using this fact, we establish the following simple, but important, result.

(2.3.5) Lemma. *For any $\lambda \in \Lambda$, $\{C_y^+ \,|\, \lambda \subseteq \mathcal{R}(y)\}$ (resp., $\{C_y^+ \,|\, \lambda \subseteq \mathcal{L}(y)\}$) is a $\mathcal{Z}'$-basis for $\widetilde{H}'x_\lambda$ (resp., $x_\lambda\widetilde{H}'$).*

Proof. If $\lambda \subseteq \mathcal{R}(y)$, then $C_y^+\tau_s = q_s C_y^+$ by (2.3.2b). Thus, $C_y^+ \in \widetilde{H}'x_\lambda$ by (2.1.6). Now apply (2.3.4) (and (2.1.6) again). $\square$

By (2.3.5) and (2.3.3), condition (2.2.4) applies to $\widetilde{H}'x_\lambda \in \mathrm{Ob}({}_{\widetilde{H}'}\mathcal{C})$, using the basis $\{C_y^+ \,|\, \mathcal{R}(y) \supseteq \lambda\}$, the quasi-poset structure $\leq^{\mathcal{R}}$ induced on its indexing set, and the generators $\{\tau_s\}_{s\in S}$ for $\widetilde{H}'$. The cells of $\leq^{\mathcal{R}}$ correspond bijectively to the elements of Λ, so, following (2.2.2), fix an ordering $S = \lambda_1, \cdots, \lambda_{2^{\#S}} = \emptyset$ on Λ such that $\lambda_i \subseteq \lambda_j \implies i \geq j$. Then $\lambda \in \Lambda$ determines a subsequence $\lambda_{i_1}, \cdots, \lambda_{i_{m_\lambda}} = \lambda$ consisting of those $\mu \in \Lambda$ satisfying $\lambda \subseteq \mu$. Thus, the (decreasing) *right-set filtration* $\widetilde{E}^{\lambda,\mathcal{R},\mathcal{Z}'}$ on $\widetilde{H}'x_\lambda$ and the (increasing) *dual right-set filtration* $\widetilde{F}_{\lambda,\mathcal{R},\mathcal{Z}'}$ on $x_\lambda\widetilde{H}' \cong (\widetilde{H}'x_\lambda)^*$ are defined as per (2.2.5)—explicitly,

$$\begin{cases} \widetilde{E}_i^{\lambda,\mathcal{R},\mathcal{Z}'} = \operatorname{span}\left(C_y^+ \,|\, \mathcal{R}(y) = \lambda_{i_j}, j \leq m_\lambda - i\right), \ 0 \leq i < m_\lambda, \\ \quad \text{and } \widetilde{E}_{m_\lambda}^{\lambda,\mathcal{R}} = 0; \\ \widetilde{F}_{\lambda,\mathcal{R},\mathcal{Z}'}^i = (\widetilde{H}'x_\lambda / \widetilde{E}_i^{\lambda,\mathcal{R},\mathcal{Z}'})^*, \ \ 0 \leq i \leq m_\lambda. \end{cases} \tag{2.3.6}$$

To simplify notation, we will usually denote $\widetilde{E}^{\lambda,\mathcal{R},\mathcal{Z}'}$ and $\widetilde{F}_{\lambda,\mathcal{R},\mathcal{Z}'}$ by $\widetilde{E}^{\lambda,\mathcal{R}}$ and $\widetilde{F}_{\lambda,\mathcal{R}}$, respectively. It will be clear from context that we are working over $\mathcal{Z}'$.

In §2.5, we use a finer filtration on $x_\lambda\widetilde{H}'$. Let Ω be the set of left (Kazhdan-Lusztig) cells of W (for the system $\{c_s \,|\, s \in S\}$). For $\lambda \in \Lambda$, put $\Omega_\lambda = \{\omega \in \Omega \,|\, \lambda \subseteq \mathcal{R}(\omega)\}$. By (2.3.3) and (2.3.5), the $\mathcal{Z}'$-basis $\{C_y^+ \,|\, \lambda \subseteq \mathcal{R}(y)\}$ of $\widetilde{H}'x_\lambda$ partitions into subsets $\{C_y^+\}_{y\in\omega}$, $\omega \in \Omega_\lambda$. Fix a listing $\omega_1, \cdots, \omega_m$ of Ω as in (2.2.2). Thus, if $x \in \omega_i, y \in \omega_j$ and $x \leq_L y$, then $i \leq j$. This listing induces a similar listing

$\omega_{\lambda,1},\cdots,\omega_{\lambda,n_\lambda}$ of Ω_λ. (Observe that $\omega_{\lambda,n_\lambda}$ is the left cell ω containing the longest word in W_λ.) Then the *left cell filtration* $\widetilde{E}^{\lambda,\mathcal{Z}'}=\widetilde{E}^{\lambda}$ of $\widetilde{H}'x_\lambda$ and the *dual left cell filtration* $\widetilde{F}_{\lambda,\mathcal{Z}'}=\widetilde{F}_\lambda$ of $x_\lambda\widetilde{H}'$ are defined by putting:

$$\text{(2.3.7)}\qquad \begin{cases} \widetilde{E}^{\lambda}_i=\operatorname{span}\left(C^+_y\,|\,y\in\omega_{\lambda,j},\,j\leq n_\lambda-i\right),\ \ 0\leq i<n_\lambda,\\ \quad\text{and }\ \widetilde{E}^{\lambda}_{n_\lambda}=0;\\ \widetilde{F}^{i}_{\lambda}=(\widetilde{H}'x_\lambda/\widetilde{E}^{\lambda}_i)^*\ \ 0\leq i\leq n_\lambda. \end{cases}$$

The following lemma establishes a very strong property of these filtrations.

(2.3.8) Lemma. *For $\lambda\in\Lambda$, consider filtrations $\widetilde{F}_{\lambda,\mathcal{R}}$ and $\widetilde{F}_\lambda$ defined in (2.3.6) and (2.3.7), respectively, over $\mathcal{Z}$. For $\mu\in\Lambda$, and any commutative $\mathcal{Z}$-algebra $\mathcal{Z}'$, base change defines isomorphisms*

$$\text{(2.3.8.1)}\qquad \begin{cases} (1)\ \operatorname{Hom}_{\widetilde{H}}(x_\lambda\widetilde{H}/\widetilde{F}^{i}_{\lambda,\mathcal{R}},x_\mu\widetilde{H})_{\mathcal{Z}'}\xrightarrow{\sim}\operatorname{Hom}_{\widetilde{H}_{\mathcal{Z}'}}((x_\lambda\widetilde{H}/\widetilde{F}^{i}_{\lambda,\mathcal{R}})_{\mathcal{Z}'},x_\mu\widetilde{H}_{\mathcal{Z}'})\\ (2)\ \operatorname{Hom}_{\widetilde{H}}(x_\lambda\widetilde{H}/\widetilde{F}^{i}_{\lambda},x_\mu\widetilde{H})_{\mathcal{Z}'}\xrightarrow{\sim}\operatorname{Hom}_{\widetilde{H}_{\mathcal{Z}'}}((x_\lambda\widetilde{H}/\widetilde{F}^{i}_{\lambda})_{\mathcal{Z}'},x_\mu\widetilde{H}_{\mathcal{Z}'}) \end{cases}$$

for all i. Also, the $\mathcal{Z}'$-modules in (2.3.8.1(1)) (resp., (2.3.8.1(2))) are free of rank $r^{\mathcal{R}}_{\lambda,\mu,i}\overset{\text{def}}{=}\#\{y\in W\,|\,\mathcal{R}(y)=\lambda_{i_t},t\leq m_\lambda-i\text{ and }\mu\subseteq\mathcal{L}(y)\}$ (resp., $r_{\lambda,\mu,i}\overset{\text{def}}{=}\#\{y\in W\,|\,y\in\omega_{\lambda,j},j\leq n_\lambda-i\text{ and }\mu\subseteq\mathcal{L}(y)\}$). Finally, for $i<j$, the natural maps

$$\text{(2.3.8.2)}\qquad \begin{cases} \operatorname{Hom}_{\widetilde{H}}(x_\lambda\widetilde{H}/\widetilde{F}^{j}_{\lambda,\mathcal{R}},x_\mu\widetilde{H})_{\mathcal{Z}'}\to\operatorname{Hom}_{\widetilde{H}}(x_\lambda\widetilde{H}/\widetilde{F}^{i}_{\lambda,\mathcal{R}},x_\mu\widetilde{H})_{\mathcal{Z}'}\\ \operatorname{Hom}_{\widetilde{H}}(x_\lambda\widetilde{H}/\widetilde{F}^{j}_{\lambda},x_\mu\widetilde{H})_{\mathcal{Z}'}\to\operatorname{Hom}_{\widetilde{H}}(x_\lambda\widetilde{H}/\widetilde{F}^{i}_{\lambda},x_\mu\widetilde{H})_{\mathcal{Z}'} \end{cases}$$

have $\mathcal{Z}'$-free cokernels, of ranks $r^{\mathcal{R}}_{\lambda,\mu,i}-r^{\mathcal{R}}_{\lambda,\mu,j}$ and $r_{\lambda,\mu,i}-r_{\lambda,\mu,j}$, respectively.

Proof. We give the argument for $\widetilde{F}_{\lambda,\mathcal{R}}$—that for $\widetilde{F}_\lambda$ is the same.

If $\widetilde{M},\widetilde{N}\in\operatorname{Ob}(\mathcal{C}_{\widetilde{H}})$ are $\mathcal{Z}$-projective, then $\operatorname{Hom}_{\widetilde{H}}(\widetilde{M},\widetilde{N})\cong\operatorname{Hom}_{\widetilde{H}}(\widetilde{N}^*,\widetilde{M}^*)$. A similar statement holds for $\widetilde{H}'$-modules. Since $(x_\lambda\widetilde{H}/\widetilde{F}^{i}_{\lambda,\mathcal{R}})^*\cong\widetilde{E}^{\lambda,\mathcal{R}}_i$ and $(x_\mu\widetilde{H})^*\cong\widetilde{H}x_\mu$, (2.3.8.1) is equivalent to showing that base change

$$\text{(2.3.8.3)}\qquad \operatorname{Hom}_{\widetilde{H}}(\widetilde{H}x_\mu,\widetilde{E}^{\lambda,\mathcal{R}}_i)_{\mathcal{Z}'}\xrightarrow{\sim}\operatorname{Hom}_{\widetilde{H}'}(\widetilde{H}'x_\mu,\widetilde{E}^{\lambda,\mathcal{R}}_{i\mathcal{Z}'}),\ \forall i,$$

defines an isomorphism. By Frobenius reciprocity, a morphism $f:\widetilde{H}'x_\mu\to\widetilde{E}^{\lambda,\mathcal{R}}_{i\mathcal{Z}'}$ is completely determined by specifying $x=f(x_\mu)\in\widetilde{E}^{\lambda,\mathcal{R}}_{i\mathcal{Z}'}$ satisfying $\tau_sx=q_sx$ for all $s\in\mu$. By (2.3.4) and (2.3.6), the possible x are merely all $\mathcal{Z}'$-linear combinations of those C^+_y such that $\mathcal{R}(y)=\lambda_{i_t}$ for some $t\leq m_\lambda-i$ and $\mu\subseteq\mathcal{L}(x)$. Thus, both sides of (2.3.8.3) are $\mathcal{Z}'$-free of the same rank, and base change carries a basis of the left hand side to one of the right hand side.

Finally, the discussion in the previous paragraph shows that the maps in (2.3.8.2) carry a basis to a subset of a basis of the image spaces. $\square$

With an eye to using (1.2.9), we now establish the following important homological result for the filtrations $\widetilde{F}_{\lambda,\mathcal{R}}$ and $\widetilde{F}_\lambda$ defined in (2.3.6) and (2.3.7), respectively. Recall that in defining these filtrations, we have fixed a linear ordering compatible with inclusion on the power set Λ of S. Also, observe that $\widetilde{H}_{\mathbb{Q}(t^2)}$ is a separable algebra. This fact follows immediately from [AG2; (4.7)], since the finite group algebra $\mathbb{Q}W$ is separable. Hence, if $\mathcal{Z} \to \mathcal{Z}'$ is an inclusion of integral domains, $\widetilde{H}'_{K'}$ is semisimple over the quotient field K' of $\mathcal{Z}$. This fact justifies the parenthetic assertion in the statement of the theorem.

(2.3.9) Theorem. *Let $\mathcal{Z} \to \mathcal{Z}'$ be a homomorphism into an integral domain $\mathcal{Z}'$ with fraction field K'. Put $\widetilde{H}' = \widetilde{H}_{\mathcal{Z}'}$ and assume that $\widetilde{H}'_{K'}$ is semisimple. (In particular, $\widetilde{H}'_{K'}$ is semisimple if $\mathcal{Z} \to \mathcal{Z}'$ is an inclusion.) Then:–*

(a) For $\lambda, \mu \in \Lambda$, we have

$$\operatorname{Ext}^1_{\widetilde{H}'}(x_\lambda \widetilde{H}'/\widetilde{F}^i_{\lambda,\mathcal{R}}, x_\mu \widetilde{H}') = 0 \ \forall i. \tag{2.3.9.1}$$

Similarly,

$$\operatorname{Ext}^1_{\widetilde{H}'}(x_\lambda \widetilde{H}'/\widetilde{F}^i_{\lambda}, x_\mu \widetilde{H}') = 0 \ \forall i. \tag{2.3.9.2}$$

(b) For $\lambda, \mu \in \Lambda$, both $\operatorname{Hom}_{\widetilde{H}'}(\widetilde{F}^j_{\lambda,\mathcal{R}}, x_\mu \widetilde{H}')$ and $\operatorname{Hom}_{\widetilde{H}'}(\widetilde{F}^j_{\lambda}, x_\mu \widetilde{H}')$ are $\mathcal{Z}'$-free for all j. They have ranks $r_{\lambda,\mu} - r^{\mathcal{R}}_{\lambda,\mu,j}$ and $r_{\lambda,\mu} - r_{\lambda,\mu,j}$, respectively, where $r_{\lambda,\mu} = r^{\mathcal{R}}_{\lambda,\mu,0} = r_{\lambda,\mu,0}$ is the number $\#W_\mu\backslash W/W_\lambda$ of W_μ-W_λ double cosets in W.

Proof. Assertion (a) follows from (1.2.13) and (2.3.8).

Next, observe that for any j, (2.3.9.1) implies that $\operatorname{Hom}_{\widetilde{H}'}(\widetilde{F}^j_{\lambda,\mathcal{R}}, x_\mu \widetilde{H}')$ identifies with the cokernel of the natural map

$$\operatorname{Hom}_{\widetilde{H}'}(x_\lambda \widetilde{H}'/\widetilde{F}^j_{\lambda,\mathcal{R}}, x_\mu \widetilde{H}') \to \operatorname{Hom}_{\widetilde{H}'}(x_\lambda \widetilde{H}', x_\mu \widetilde{H}').$$

Thus, (b) follows from the final assertion of (2.3.8), together with elementary properties of distinguished double coset representatives. ☐

We remark that the semisimplicity criterion (4.2.2) below gives necessary and sufficient conditions for the algebra $\widetilde{H}'_{K'}$ in the theorem above to be semisimple.

2.4. Stratifying Hecke endomorphism algebras. As before, let Λ be the power set of S; we will make Λ into a quasi-poset in (2.4.3) below. For $\lambda \in \Lambda$, let $\widetilde{T}_\lambda = x_\lambda \widetilde{H}$, so that $\widetilde{T}^*_\lambda \cong \widetilde{H} x_\lambda$ by (2.1.9). Put

$$\widetilde{T} = \bigoplus_{\lambda\in\Lambda} \widetilde{T}_\lambda \in \operatorname{Ob}(\mathcal{C}_{\widetilde{H}}). \tag{2.4.1}$$

For $\lambda \in \Lambda$, let $\widetilde{F}_{\lambda,\mathcal{R}}$ be as in (2.3.6), and define

$$\widetilde{S}^{\mathcal{R}}_\lambda = \widetilde{F}^1_{\lambda,\mathcal{R}} \cong \operatorname{Gr}^0 \widetilde{F}_{\lambda,\mathcal{R}}. \tag{2.4.2}$$

From the definitions, $\widetilde{S}^{\mathcal{R}}_\lambda$ is independent of ordering of Λ. Also, any nonzero section $\operatorname{Gr}^i \widetilde{F}_{\lambda,\mathcal{R}}$ has the form $\widetilde{S}^{\mathcal{R}}_\mu$ for some $\mu \in \Lambda$ satisfying $\lambda \subseteq \mu$. Each $\widetilde{S}^{\mathcal{R}}_\lambda$ is $\mathcal{Z}$-free.

(2.4.3) Lemma. *Assume that $\#S > 1$, and let $\xi = S \in \Lambda$ and $\zeta = \emptyset \in \Lambda$. Let $\leq$ be the smallest quasi-poset structure on Λ containing the set $\{(\lambda,\mu) \in \Lambda \times \Lambda \,|\, \lambda \subseteq \mu \text{ or } \mathrm{Hom}_{\widetilde{H}}(\widetilde{S}^{\mathcal{R}}_\lambda, \widetilde{S}^{\mathcal{R}}_\mu) \neq 0\}$. Then $\#\bar{\Lambda} \geq 3$. Also, ξ (resp., ζ) is the unique maximal (resp., minimal) element in the quasi-poset Λ. Further, we have*

$$\widetilde{S}^{\mathcal{R}}_\xi \cong \mathrm{IND}, \qquad \widetilde{S}^{\mathcal{R}}_\zeta \cong \mathrm{SGN}. \tag{2.4.3.1}$$

Proof. Clearly, $\mathrm{Gr}_0\widetilde{E}^{\xi,\mathcal{R}} = \widetilde{E}^{\xi,\mathcal{R}}_0$ is spanned by $C^+_{w_0}$, where $w_0 \in W$ is the longest word. By (2.3.2a), $\mathrm{Gr}_0\widetilde{E}^{\xi,\mathcal{R}} \cong \mathrm{IND}$, so $\widetilde{S}^{\mathcal{R}}_\xi \cong \mathrm{IND}^* \cong \mathrm{IND}$. Also, $\mathrm{Gr}_0\widetilde{E}^{\zeta,\mathcal{R}}$ is spanned by the coset $C^+_1 + \widetilde{E}^{\zeta,\mathcal{R}}_1$, so $\mathrm{Gr}_0\widetilde{E}^{\zeta,\mathcal{R}} \cong \mathrm{SGN}$ by (2.3.2a) again. Thus, $\widetilde{S}^{\mathcal{R}}_\zeta \cong \mathrm{SGN}$. This proves (2.4.3.1).

For $\lambda \neq \xi$, we claim that $\mathrm{Hom}_{\widetilde{H}}(\widetilde{S}^{\mathcal{R}}_\xi, \widetilde{S}^{\mathcal{R}}_\lambda) = 0$. In fact, any nonzero morphism $\widetilde{S}^{\mathcal{R}}_\xi \to \widetilde{S}^{\mathcal{R}}_\lambda$, upon taking duals and composing with the quotient morphism $\widetilde{H}x_\lambda \to \widetilde{S}^{\mathcal{R}*}_\lambda$, yields a nonzero morphism $f : \widetilde{H}x_\lambda \to \widetilde{S}^{\mathcal{R}*}_\xi \cong \mathrm{IND}$ vanishing on $\mathrm{Gr}_1\widetilde{E}^{\lambda,\mathcal{R}}$. By Frobenius reciprocity, f is determined by $f(x_\lambda) = cx_\xi$ $(0 \neq c \in \mathcal{Z})$. Thus, $f(x_\xi) = f(\sum_{w \in {}^\lambda W} \tau_w \cdot x_\lambda) = c\mathrm{IND}(\sum_{w\in {}^\lambda W} \tau_w)x_\xi \neq 0$. Since $x_\xi \in \mathrm{Gr}_1\widetilde{E}^{\lambda,\mathcal{R}}$, this contradiction proves our claim. It follows ξ is the unique maximal element in Λ.

Since $\widetilde{H}_{\mathbb{Q}(t)}$ is semisimple,

$$\mathrm{Hom}_{\widetilde{H}}(\widetilde{S}^{\mathcal{R}}_\lambda, \widetilde{S}^{\mathcal{R}}_\mu) \neq 0 \iff \mathrm{Hom}_{\widetilde{H}}(\widetilde{S}^{\mathcal{R}}_\mu, \widetilde{S}^{\mathcal{R}}_\lambda) \neq 0 \qquad \forall \lambda, \mu \in \Lambda.$$

If $\lambda \in \Lambda$ satisfies $\mathrm{Hom}_{\widetilde{H}}(\widetilde{S}^{\mathcal{R}}_\lambda, \widetilde{S}^{\mathcal{R}}_\zeta) \neq 0$, then $\mathrm{Hom}_{\widetilde{H}}(\mathrm{SGN}, \widetilde{S}^{\mathcal{R}}_\lambda) \neq 0$. As above, this implies there is a nonzero morphism $\widetilde{H}x_\lambda \to \mathrm{SGN}$. By Frobenius reciprocity, $\lambda = \zeta$. Hence, $\zeta = \emptyset$ is the unique minimal element in Λ.

Finally, $\#S > 1$, so $\#\Lambda > 3$, and $\#\bar{\Lambda} \geq 3$. $\square$

We are now ready to establish that the algebra $\widetilde{A} = \mathrm{End}_{\widetilde{H}}(\widetilde{T})$ has a stratification of length ≥ 3. By (1.2.8), it suffices to show that the module category ${}_{\widetilde{A}}\mathcal{C}$ has a stratifying system $\{\widetilde{\Delta}(\gamma)^{\mathcal{R}}\}_{\gamma\in\Gamma}$ for a quasi-poset Γ satisfying $\#\bar{\Gamma} \geq 3$. In fact, we will take $\Gamma = \Lambda$, the power set of S with the quasi-poset structure defined by (2.4.3).

(2.4.4) Theorem. *Assume that $\#S > 1$, let $\mathcal{Z}'$ be a commutative $\mathcal{Z}$-algebra, and put $\widetilde{H}' = \widetilde{H}_{\mathcal{Z}'}$. Let $\widetilde{A}' = \mathrm{End}_{\widetilde{H}'}(\widetilde{T}_{\mathcal{Z}'})$. Then:–*

(a) The data consisting of $\widetilde{T} = \bigoplus_{\lambda\in\Lambda} \widetilde{T}_\lambda$, $\widetilde{S}^{\mathcal{R}}_\lambda$ and $\widetilde{F}_{\lambda,\mathcal{R}}$ satisfy Hypothesis (1.2.9).

(b) For $\lambda \in \Lambda$, put $\widetilde{\Delta}(\lambda)^{\mathcal{R}} = \mathrm{Hom}_{\widetilde{H}}(\widetilde{S}^{\mathcal{R}}_\lambda, \widetilde{T})$. Then each $\widetilde{\Delta}(\lambda)^{\mathcal{R}}_{\mathcal{Z}'}$ is $\mathcal{Z}'$-free, and $\{\widetilde{\Delta}(\lambda)^{\mathcal{R}}_{\mathcal{Z}'}\}_\lambda$ is a stratifying system for ${}_{\widetilde{A}'}\mathcal{C}$ with respect to the quasi-poset structure defined on the power set Λ of S in (2.4.3). Also, there is an isomorphism

$$\mathrm{End}_{\widetilde{H}}(\widetilde{T})_{\mathcal{Z}'} \cong \mathrm{End}_{\widetilde{H}'}(\widetilde{T}_{\mathcal{Z}'}). \tag{2.4.4.1}$$

Finally, if $\xi = S$ and $\zeta = \emptyset$, then $\widetilde{\Delta}(\xi)^{\mathcal{R}}_{\mathcal{Z}'}$ is $\widetilde{A}'$-projective and has $\mathcal{Z}'$-rank $2^{\#S}$, while $1 = \mathrm{rank}\,\widetilde{\Delta}(\zeta)^{\mathcal{R}}_{\mathcal{Z}'} < \mathrm{rank}\,\widetilde{\Delta}(\mu)^{\mathcal{R}}_{\mathcal{Z}'}$, $\mu \neq \zeta$.

Proof. Fix an ordering—as per (2.2.2)—on Λ compatible with the quasi-poset structure defined in (2.4.3). The isomorphism (2.4.4.1) follows from the $i = 0$ case of (2.3.8). Next, consider the case $\mathcal{Z}' = \mathcal{Z}$, writing $\widetilde{A} = \mathrm{End}_{\widetilde{H}}(\widetilde{T})$. For $\lambda \in \Lambda$, we have defined $\widetilde{S}^{\mathcal{R}}_{\lambda} \hookrightarrow \widetilde{T}_{\lambda}$ and the increasing filtration $\widetilde{F}_{\lambda,\mathcal{R}}$ on $\widetilde{T}_{\lambda}$ which satisfies (1.2.9(1)) (relative to some sequence $\nu_{\lambda,0} = \lambda, \nu_{\lambda,1}, \cdots$) for the quasi-poset structure $\leq$ on Λ defined in (2.4.3). Also, (2.3.9.1) implies that (1.2.9(3)) holds. Next, suppose that $\mathrm{Hom}_{\widetilde{H}}(\widetilde{S}^{\mathcal{R}}_{\lambda}, \widetilde{T}_{\mu}) \neq 0$ for some $\lambda, \mu \in \Lambda$. Then for some $\nu_{\mu,i} \in \Lambda$ (corresponding to a section $\mathrm{Gr}^{i}\widetilde{F}_{\mu,\mathcal{R}}$ of $\widetilde{T}_{\mu}$), $\mathrm{Hom}_{\widetilde{H}}(\widetilde{S}^{\mathcal{R}}_{\lambda}, \widetilde{S}^{\mathcal{R}}_{\nu_{\mu,i}}) \neq 0$. Thus, by the semisimplicity of $\widetilde{H}_{\mathbb{Q}(t)}$, $\mathrm{Hom}_{\widetilde{H}}(\widetilde{S}^{\mathcal{R}}_{\nu_{\mu,i}}, \widetilde{S}^{\mathcal{R}}_{\lambda}) \neq 0$, so that $\mu \leq \nu_{\mu,i} \leq \lambda$ by (2.4.3). Thus, (1.2.9(2)) holds, completing the proof of (a).

By (2.3.9(b)), each $\widetilde{\Delta}(\lambda)^{\mathcal{R}}$ is $\mathcal{Z}$-free. Thus, by (1.2.10), $\{\widetilde{\Delta}(\lambda)^{\mathcal{R}}\}_{\lambda}$ is a stratifying system for ${}_{\widetilde{A}}\mathcal{C}$. Now apply (1.2.5) to complete the proof that $\{\widetilde{\Delta}(\lambda)^{\mathcal{R}}_{\mathcal{Z}'}\}_{\lambda}$ is a stratifying system for ${}_{\widetilde{A}'}\mathcal{C}$. The final assertion of (b) follows from the definitions. □

In general, $\widetilde{\Delta}(\lambda)^{\mathcal{R}}_{\mathcal{Z}'} = \mathrm{Hom}_{\widetilde{H}}(\widetilde{S}^{\mathcal{R}}_{\lambda}, \widetilde{T})_{\mathcal{Z}'}$ does *not* coincide with $\mathrm{Hom}_{\widetilde{H}'}(\widetilde{S}^{\mathcal{R}}_{\lambda\mathcal{Z}'}, \widetilde{T}_{\mathcal{Z}'})$. (In fact, it identifies with the image of $\mathrm{Hom}_{\widetilde{H}'}(x_{\lambda}\widetilde{H}', \widetilde{T}_{\mathcal{Z}'})$ in $\mathrm{Hom}_{\widetilde{H}'}(\widetilde{S}^{\mathcal{R}}_{\lambda\mathcal{Z}'}, \widetilde{T}_{\mathcal{Z}'})$.) This is a main reason for working in an integral context, even if one is primarily interested in the endomorphism algebra $\widetilde{A}'$ over a field.

We indicate some important consequences of this result.

(2.4.5) Corollary. *Let Z be a discrete valuation ring of characteristic zero. Then Conjecture* [CPS4; (6.4.10)] *holds for the group algebra ZW of the finite Coxeter group W.*

Proof. Indeed, [CPS4; (6.4.10)] is essentially the same as the conclusion of (2.4.4) for the group algebra ZW, specializing $t \mapsto 1$ and using $\mathcal{Z}' = Z$.

(2.4.6) Corollary. *Assume $\#S > 1$ and let $\mathcal{Z}'$ be as in (2.4.4). Then the algebra $\widetilde{A}' = \mathrm{End}_{\widetilde{H}'}(\widetilde{T}_{\mathcal{Z}'})$ has a stratification of length ≥ 3. In particular,* [CPS4; Conjecture (6.3.1)] *holds.*

Proof. The first assertion is immediate from (2.4.4) and (1.2.8), and, again the conjecture mentioned is the case $\mathcal{Z} = k$, $t^2 \mapsto 1$. □

(2.4.7) Corollary. *With ζ as in (2.4.3) and $\widetilde{A}'$ as in (2.4.4),*

$$\mathrm{Ext}^{\,i}_{\widetilde{A}'}(\widetilde{\Delta}(\zeta)^{\mathcal{R}}_{\mathcal{Z}'}, \widetilde{\Delta}(\zeta)^{\mathcal{R}}_{\mathcal{Z}'}) = 0 \qquad \forall i > 0.$$

Proof. ζ is the unique minimal element Λ and $\widetilde{S}^{\mathcal{R}}_{\zeta}$ occurs with multiplicity 1 in $\widetilde{F}_{\zeta,\mathcal{R}}$, so apply (1.2.6) and (2.4.4). □

(2.4.8) Remark. Suppose (W, S) has type A_{r-1} and $c_s = 0$, $\forall s \in S$. Let k be a field of characteristic $p > 0$. Then $\mathrm{End}_{\widetilde{H}'}(\widetilde{T}_k)$ is Morita equivalent to the Schur algebra $S(r, r)$ over k. Also, $\widetilde{\Delta}(\zeta)^{\mathcal{R}}_{k}$ corresponds to the determinant representation det for $S(r, r)$. If $\mathcal{C}$ denotes the category of rational $SL_r(k)$-modules, there is a full embedding $D^b({}_{S(r,r)}\mathcal{C}) \hookrightarrow D^b(\mathcal{C})$ of derived categories (see, e. g., [D; §2], [CPS1; (4.5), (4.6)]), so (2.4.7) is closely related to the vanishing

$$\text{(2.4.8.1)} \qquad \operatorname{Ext}^i_{SL_r(k)}(\det, \det) \cong H^i(SL_r(k), k) = 0 \quad \forall i > 0.$$

See [CPSK; (3.4)], where the proof of (2.4.8.1) requires the deep Kempf vanishing theorem for homogeneous line bundles. Thus, taking note of the isomorphism (1), we might view (2.4.7) as a kind of "aftershock" of Kempf's theorem in the non-describing representation theory of finite groups of Lie type! (Similar remarks hold for the q-Schur algebras.)

As a special case of (2.4.4), we have the following result concerning the algebras $S(G, k)$ defined in the introduction.

(2.4.9) Corollary. *Consider a family $\mathcal{G} = \{G(q)\}$ of finite groups of Lie type, and let k be a field of characteristic p relatively prime to q. Assume that $G(q)$ has rank > 1. Let $B(q)$ be a Borel subgroup of $G(q)$. Then the algebras*

$$\text{(2.4.9.1)} \qquad S(G(q), k) = \operatorname{End}_{kG(q)}\left(\bigoplus_{P(q)\supseteq B(q)} \operatorname{ind}_{P(q)}^{G(q)} k\right)$$

have a stratification of length ≥ 3.

Proof. Using the isomorphism (1) in the introduction, we see that $S(G(q), k) \cong \operatorname{End}_{\widetilde{H}_k}(\widetilde{T}_k)$, where $\widetilde{H}$ is the generic Hecke algebra associated to $\mathcal{G}$ and k is viewed as a $\mathcal{Z}$-module by the mapping $t^2 \mapsto q$. Now apply (2.4.4). $\square$

We next sketch a further interesting application of (2.4.4); a second argument is indicated after the proof.

(2.4.10) Corollary. *Let $\mathcal{Z}'$ be a commutative $\mathcal{Z}$-algebra. For $\lambda \in \Lambda$, $\widetilde{S}^{\mathcal{R}}_{\lambda\mathcal{Z}'}$ has a (cohomological) resolution by $\widetilde{H}'$-modules each of which is a direct sum of copies of modules $\widetilde{T}_{\mu\mathcal{Z}'}$ with $\mu \geq \lambda$.*

Proof. We can assume that $\mathcal{Z}' = \mathcal{Z}$. A standard argument (see [CPS4; (2.3.1)]) based on (1.2.4(2)) shows there is a projective resolution $\widetilde{P}^\bullet \twoheadrightarrow \widetilde{\Delta}(\lambda)^{\mathcal{R}}$, in which each $\widetilde{P}^i$ is a direct sum of modules $\widetilde{P}(\mu)$ with $\mu \geq \lambda$. In this resolution, all kernels and cokernels are filtered by $\widetilde{\Delta}(\mu)^{\mathcal{R}}$'s. The filtration $\widetilde{G}^{\lambda,\mathcal{R}} = \widetilde{F}^{\diamond}_{\lambda,\mathcal{R}}$ of $\widetilde{P}(\lambda)$ obtained by applying $(-)^\diamond = \operatorname{Hom}_{\widetilde{H}}(-, \widetilde{T})$ to the filtration $\widetilde{F}_{\lambda,\mathcal{R}}$ of $\widetilde{T}_\lambda$ has sections obtained by applying $(-)^\diamond$ to the sections of $\widetilde{F}_{\lambda,\mathcal{R}}$. The argument for [CPS4; (6.4.8)] shows this process reverses and that, consequently, $\operatorname{Ext}^1_{\widetilde{A}}(\widetilde{\Delta}(\mu)^{\mathcal{R}}, \widetilde{T}) = 0$. Thus, the desired resolution is obtained by applying the functor $\operatorname{Hom}_{\widetilde{A}}(-, \widetilde{T})$ to $\widetilde{P}^\bullet \twoheadrightarrow \widetilde{\Delta}(\lambda)^{\mathcal{R}}$. $\square$

One can prove (2.4.10) more directly, using (2.3.8), and even show the resolutions obtained can be taken to be finite—details are left to the reader. These resolutions appear to be dual to the generalizations of the Coxeter complex obtained in [M].

We conclude this subsection with the following rank 2 result. We will consider some explicit examples when $\widetilde{H}$ is a Hecke algebra of Lie type in §3.

(2.4.11) Corollary. *Assume that $\widetilde{H}$ is a generic Hecke algebra over $\mathcal{Z}$ associated to a finite Coxeter system (W, S) of rank 2 together with a system $\{c_s\}_{s\in S}$ of integral parameters. For any commutative $\mathcal{Z}$-algebra $\mathcal{Z}'$, let $\widetilde{A}' = \mathrm{End}_{\widetilde{H}'}(\widetilde{T}_{\mathcal{Z}'})$. Then the stratifying system $\{\widetilde{\Delta}(\lambda)^{\mathcal{R}}_{\mathcal{Z}'}\}_\lambda$ (given in (2.4.4)) is strict. In particular, $\widetilde{A}'$ has a standard stratification of length ≥ 3.*

Proof. By (2.4.4(a)), Hypothesis (1.2.9) holds for the data $\widetilde{T}_\lambda$,$\widetilde{S}^{\mathcal{R}}_\lambda$, $\widetilde{F}_{\lambda,\mathcal{R}}$, etc. By (1.2.8) and the last assertion of (1.2.10), it suffices to show in (1.2.9(1)) that

$$\bar{\nu}_{\lambda,i} > \bar{\lambda}, \qquad \forall\, \lambda \in \Lambda,\ i > 0. \tag{2.4.11.1}$$

If $S = \{u, v\}$, then Λ consists of the four elements $\zeta = \emptyset, \mu = \{u\}, \nu = \{v\}$, and $\xi = \{u, v\}$. Also, ζ is the unique minimal element and ξ is the unique maximal element. Thus, (2.4.11.1) holds trivially if $\lambda = \xi$ or ζ. If $\lambda = \mu$ or ν, (2.3.6) implies that $\mathrm{Gr}^0\widetilde{F}_{\lambda,\mathcal{R}} = \widetilde{S}^{\mathcal{R}}_\lambda$ and $\mathrm{Gr}^1\widetilde{F}_{\lambda,\mathcal{R}} = \widetilde{S}^{\mathcal{R}}_\xi$ are the two nonzero sections of $\widetilde{F}_{\lambda,\mathcal{R}}$. Hence, (2.4.11.1) holds in this case, too. □

Consideration of type B_3 shows that this result does not extend to higher rank. Instead, a role for standard stratifications, as well as dual left cell modules, is suggested by Conjecture (2.5.3) in the next subsection.

2.5. More on cell filtrations.

We discuss possible stratifications arising from a refinement of the "Specht modules" $\widetilde{S}^{\mathcal{R}}_\lambda$ (2.4.2). Even in type A, the module $\widetilde{S}^{\mathcal{R}}_\lambda$ is not always a classical Specht module (e. g., as defined in [DJ1; §4]), though this is true for the modules $\widetilde{S}_\lambda$ defined below.

As in §2.3, let Λ be the power-set of S and let Ω be the set of left (Kazhdan-Lusztig) cells in W. The quasi-poset structure $\leq_L$ on W induces a poset structure, still denoted $\leq_L$, on Ω. (In fact, $(\Omega, \leq_L)$ is the poset associated to the quasi-poset $(W, \leq_L)$.)

Consider the dual left cell filtration $\widetilde{F} = \widetilde{F}_S$ on $\widetilde{H} = x_\emptyset\widetilde{H}$ defined in (2.3.7). For $\omega \in \Omega$, let $\widetilde{S}_\omega$ be the corresponding *dual left cell* module—i. e., if $\omega = \omega_i$, then $\widetilde{S}_\omega = \mathrm{Gr}^{m-i}\widetilde{F}$. For $\lambda \in \Lambda$, $\widetilde{S}_{\omega_{\lambda,i}} \cong \mathrm{Gr}^{n_\lambda - i}\widetilde{F}_\lambda$, where the $\omega_{\lambda,i}$ and the (increasing) dual left cell filtration $\widetilde{F}_\lambda$ of $\widetilde{T}_\lambda = x_\lambda\widetilde{H}$ is defined in (2.3.7).

For $\omega \in \Omega$, $\widetilde{\Delta}(\omega) = \mathrm{Hom}_{\widetilde{H}'}(\widetilde{S}_\omega, \widetilde{T}) \in \mathrm{Ob}({}_{\widetilde{A}}\mathcal{C})$ is the corresponding *left cell module* for $\widetilde{A} = \mathrm{End}_{\widetilde{H}}(\widetilde{T})$.

(2.5.1) Theorem. *Let $\mathcal{Z}'$ be a commutative $\mathcal{Z}$-algebra, and put $\widetilde{A}' = \mathrm{End}_{\widetilde{H}'}(\widetilde{T}_{\mathcal{Z}'})$. Then $\widetilde{A}' = \bigoplus_{\lambda\in\Lambda}\widetilde{P}(\lambda)_{\mathcal{Z}'}$, where $\widetilde{P}(\lambda) = \mathrm{Hom}_{\widetilde{H}}(\widetilde{T}_\lambda, \widetilde{T})$. Each $\widetilde{P}(\lambda)_{\mathcal{Z}'}$ has a (decreasing) filtration $\widetilde{G}^{\lambda,\mathcal{Z}'}$ satisfying $Gr_{n_\lambda - i}\widetilde{G}^{\lambda,\mathcal{Z}'} \cong \widetilde{\Delta}(\omega_{\lambda,i})_{\mathcal{Z}'}$. Finally, for $\omega \in \Omega$, $\widetilde{\Delta}(\omega)_{\mathcal{Z}'}$ is $\mathcal{Z}'$-free.*

Proof. First, suppose $\mathcal{Z}' = \mathcal{Z}$. The filtration $\widetilde{G}^\lambda = (\widetilde{F}_\lambda)^\diamond$ (in the notation of (1.2.3)) has the required property, using (2.3.9.2). By (2.3.9(b)), for $\omega \in \Omega$, $\widetilde{\Delta}(\omega)$ is $\mathcal{Z}$-free.

This proves the theorem when $\mathcal{Z}' = \mathcal{Z}$. The general case follows by base change and (2.4.4.1). $\square$

Except in type A, one does not know that (1.2.4(2)) holds for appropriate summands $\widetilde{P}(\omega)$ of $\widetilde{P}(\lambda)$. We expect that this filtration condition does hold if we replace $\widetilde{A}'$ by a larger endomorphism algebra:

(2.5.2) Conjecture. *Suppose that $\widetilde{H}$ is a Hecke algebra of Lie type over $\mathcal{Z}$ (cf. §2.1). There exists $\widetilde{X} \in \mathrm{Ob}(\mathcal{C}_{\widetilde{H}})$ such that:*

(1) $\widetilde{X}$ has a (increasing) filtration with sections of the form $\widetilde{S}_\omega$, $\omega \in \Omega$.

(2) Let $\widetilde{T}^+ = \widetilde{T} \bigoplus \widetilde{X}$, and let $\mathcal{Z}'$ be any commutative $\mathcal{Z}$-algebra. Put $\widetilde{A}^+_{\mathcal{Z}'} = \mathrm{End}_{\widetilde{H}'}(\widetilde{T}^+_{\mathcal{Z}'})$ and, for $\omega \in \Omega$, $\widetilde{\Delta}^+(\omega)_{\mathcal{Z}'} = \mathrm{Hom}_{\widetilde{H}'}(\widetilde{S}_\omega, \widetilde{T}^+)_{\mathcal{Z}'}$. Then $\{\widetilde{\Delta}^+(\omega)_{\mathcal{Z}'}\}_\omega$ is a strict stratifying system relative to the quasi-poset $(\Omega, \leq^{\mathrm{op}}_{LR})$ for the module category ${}_{\widetilde{A}^+_{\mathcal{Z}'}}\mathcal{C}$.

(2.5.3) Remarks. (a) Of course, $\widetilde{A}' = e\widetilde{A}^+_{\mathcal{Z}'}e$ for the idempotent projection $e : \widetilde{T}^+_{\mathcal{Z}'} \to \widetilde{T}_{\mathcal{Z}'}$. Thus, $\widetilde{A}'$ is in some sense approximated by $\widetilde{A}^+_{\mathcal{Z}'}$. The approximation is good if $\widetilde{X}$ is small or naturally defined. Corollary (2.4.11) and its proof show that the Conjecture is true (with $\widetilde{X} = 0$) in the rank 2 cases when $\widetilde{H}$ is untwisted. (Observe that $\leq_L = \leq^{\mathcal{R}}$ for dihedral groups in the untwisted cases.)

(b) Observe that the existence of filtrations by $\widetilde{\Delta}(\omega)_{\mathcal{Z}'}$'s in (2.5.1) is a non-trivial consequence of (2.5.2), since $\widetilde{\Delta}(\omega) \cong e\widetilde{\Delta}^+(\omega)$.

(c) We will prove that Conjecture (2.5.2) holds for all rank 2 Hecke algebras of Lie type in §3 below. In [DPS], we give a proof of the conjecture in type A. Of course, in that case, it is known that $\widetilde{A}$ is $\mathcal{Z}$-quasi-hereditary.

3. RANK 2 EXAMPLES

Section 3.1 lays the foundation—and, at the same time, nicely illustrates the Brauer theory from §1.1—for checking (1.3.2) when W has type B_2 or G_2. We then consider three examples §§3.2—3.4 where Hypothesis (1.3.2) holds, and hence the algebra A has an interesting *standard* stratification. In fact, this is true for *all* rank 2 examples by (2.4.11), though our verification through (1.3.2) has the advantage of yielding more detailed information. In particular, we can split the modules $\widetilde{T}_{\lambda k}$ and $\widetilde{S}^{\mathcal{R}}_{\lambda k}$ of §2.4 into indecomposable components. This works, on a case-by-case basis, for all Lie type rank 2 examples, though we are content here to illustrate the method in the three cases presented. Finally, in §3.5 we verify that Conjecture (2.5.2) holds in all rank 2 cases.

3.1. Hecke algebras of rank 2. Consider a Coxeter system (W, S) with $S = \{u, v\}$, where $u^2 = v^2 = 1$, and uv has order m. In (2.1.1), we write $q_s = t^{2c_s} \in \mathcal{Z} = \mathbb{Z}[t^2, t^{-2}]$, $s \in S$, where the index parameters c_s are associated to a family $\mathcal{G}$ of finite groups of Lie type. We consider the two cases $m = 4, 6$.

Form the local triple $(\mathcal{O}, K, k)$ in which $\mathcal{O} = \mathbb{Z}[t^2]_{(p,\Phi)}$, where p is a prime integer and $\Phi \in \mathbb{Z}[t^2]$ is a cyclotomic polynomial. Thus, $\mathcal{O}$ is a regular local ring of Krull dim. 2. Also, q, q_u, q_v will denote the images of t^2, q_u, q_v in either K or k. As noted in §4, $\widetilde{H}_K$ is a split semisimple algebra over K. Write $H = \widetilde{H}_k$ and let $\mathrm{Irr}(H)$ denote the set of distinct irreducible H-modules.

There are four linear characters for $\widetilde{H}_K$, described as follows:

$$\begin{gathered} \phi_1 = \mathrm{IND} : \begin{cases} \tau_u \mapsto q_u \\ \tau_v \mapsto q_v \end{cases}, \qquad \phi_{-1} = \mathrm{SGN} : \begin{cases} \tau_u \mapsto -1 \\ \tau_v \mapsto -1 \end{cases} \\ \phi_{1'} : \begin{cases} \tau_u \mapsto q_u \\ \tau_v \mapsto -1 \end{cases}, \qquad \phi_{-1'} : \begin{cases} \tau_u \mapsto -1 \\ \tau_v \mapsto q_v. \end{cases} \end{gathered} \tag{3.1.1}$$

Each of $\phi_{\pm 1}, \phi_{\pm 1'}$ defines a linear character for H, denoted by the same symbol. They need not be distinct, but all linear characters are included among them.

When $m = 4$, there is one further irreducible representation for $\widetilde{H}_K$. It has degree 2 and is defined by:

$$\phi_2 : \tau_u \mapsto \begin{pmatrix} -1 & 1 \\ 0 & q_u \end{pmatrix}, \qquad \tau_v \mapsto \begin{pmatrix} q_v & 0 \\ q_u+q_v & -1 \end{pmatrix}. \tag{3.1.2}$$

When $m = 6$, there are two further irreducible representations for $\widetilde{H}_K$:

$$\phi_2 : \begin{cases} \tau_u \mapsto \begin{pmatrix} -1 & 1 \\ 0 & q_u \end{pmatrix} \\ \tau_v \mapsto \begin{pmatrix} q_v & 0 \\ b & -1 \end{pmatrix} \end{cases}, \qquad \phi_{2'} : \begin{cases} \tau_u \mapsto \begin{pmatrix} -1 & 1 \\ 0 & q_u \end{pmatrix} \\ \tau_v \mapsto \begin{pmatrix} q_v & 0 \\ d & -1 \end{pmatrix} \end{cases}, \tag{3.1.3}$$

where $b = q_u + q_v + \sqrt{q_u q_v}$ and $d = q_u + q_v - \sqrt{q_u q_v}$. See [CR; (67.14)].

Formulas (3.1.1)–(3.1.3) already give natural $\widetilde{H}_{\mathcal{O}}$-lattices for the representations involved, so they define representations for H, denoted by the same symbols. By (1.1.2(a)), any irreducible H-module is a composition factor of some ϕ_i. Since $\dim \phi_i \leq 2$ and the 1-dimensional modules for $\widetilde{H}_{k'}$, k' an extension field of k, are already defined over k, *if a given ϕ_i is an irreducible H-module, then it is absolutely irreducible.* Hence, $H/\mathrm{rad}(H)$ is split semisimple, so (1.1.3) applies. In practice, the decomposition matrix D can be easily determined, then the Cartan matrix $C = D \cdot D^T$ computed. Given ϕ_i, let $P(i)$ denote the projective cover of ϕ_i. Since H is a symmetric algebra (cf. (2.1.7)), $\mathrm{soc}(P(i)) \cong \mathrm{head}(P(i))$. When informally describing modules by their Loewy series, we abbreviate ϕ_i to i.

By (4.2.2), if p is good for (W, S) and $d_W(q) \neq 0$ in k, then H is semisimple. For the possible $\mathcal{G}$ of type B_2 and G_2, the d_W are:

$$d_W = \begin{cases} (1)\ (1+q)^2(1+q^2) & m = 4, q_u = q_v = q \\ (2)\ (1+q)(1+q^2)(1+q^3) & m = 4, q_u = q, q_v = q^2 \\ (3)\ (1+q^2)(1+q^3)(1+q^5) & m = 4, q_u = q^2, q_v = q^3 \\ (4)\ (1+q)^2(1+q^2+q^4) & m = 6, q_u = q_v = q \\ (5)\ (1+q)(1+q^3)(1+q^4+q^8) & m = 6, q_u = q, q_v = q^3. \end{cases} \tag{3.1.4}$$

For $s = u, v$, let $x_s = \tau_s + 1$. Put $T = \bigoplus_{\lambda \subseteq S} \mathrm{ind}_{H_\lambda}^H \phi_1 = H \oplus x_u H \oplus x_v H \oplus \phi_1$. We consider possible standard stratifications for the algebra $A = \mathrm{End}_H(T)$ by verifying Hypothesis (1.3.2) and applying (1.3.3). The quasi-poset Λ will always satisfy $\bar{\Lambda} = \{\overline{-1}, \bar{0}, \bar{1}\}$ with $\overline{-1} < \bar{0} < \bar{1}$. Denote elements of the fiber of $\Lambda \to \bar{\Lambda}$ over $\bar{i}$ by i, i', i'', etc.

In each of the examples below, we provide enough detail so that the reader can easily verify that the four conditions in (1.3.2) hold. In each case, the Y_λ are sufficiently small that the required filtration F_λ is clear once the "Specht modules" S_μ have all been indicated.

3.2. Example: $m = 4$, $q = q_u = q_v = -1$, $p > 2$. This Hecke algebra H over k is associated to the finite groups $Sp_4(q)$, where here q is a prime power satisfying $q \equiv -1 \pmod{p}$. Here $\mathrm{Irr}(H) = \{\phi_1, \phi_2\}$, and $C = \left(\begin{smallmatrix} 4 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ (indexing the rows and columns by ϕ_1, ϕ_2). By direct calculation, $x_u H \cong \phi_2 \oplus \xi$ and $x_v H \cong \phi_2 \oplus \zeta$, where ξ, ζ are 2 dimensional modules which define a basis for $\mathrm{Ext}^1_H(\phi_1, \phi_1) \cong k^2$. (Note that $\dim \mathrm{Hom}_H(x_u H, x_v H) = 2$.) Thus, $T \cong P(1) \oplus \phi_2^{\oplus 4} \oplus \xi \oplus \zeta \oplus \phi_1$ and $P(1) = \begin{array}{|c|c|c|}\hline & 1 & \\ \hline 1 & & 1 \\ \hline & 1 & \\ \hline \end{array}$. (In this diagrammatic representation, successive rows represent Loewy layers in the module; empty boxes are present largely for visual clarity.) Put $Y_{-1} = P(1)$, $Y_0 = \xi$, $Y_{0'} = \zeta$, $Y_{0''} = \phi_2$, and $Y_1 = \phi_1$. If $S_\lambda = \mathrm{soc}(Y_\lambda)$, $\lambda \in \Lambda$, it is easy to check that (1.3.2) holds. (However, condition (1.2.9(3)) obviously fails over k, since $\mathrm{Ext}^1_H(\phi_1, \phi_1) \neq 0$.)

It is also easily seen from the above that the modules $\widetilde{S}^{\mathcal{R}}_{\lambda k}$ obtained from (2.4.2) are $S_{-1}, S_{0''} \oplus S_{0'}$, $S_{0''} \oplus S_0$, and S_1 for $\lambda = \emptyset$, $\{u\}$, $\{v\}$, and $\{u, v\}$, respectively. The modules $\widetilde{S}^{\mathcal{R}}_{\lambda k}$ are also dual left cell modules $\widetilde{S}_{\omega k}$ as defined in §2.5. The algebra $A = \mathrm{End}_H(T)$ is quasi-hereditary, as can be verified using [CPS4; (3.1.5)]. The latter provides necessary and sufficient conditions for an endomorphism algebra as in (1.3.2) to be quasi-hereditary.

3.3. Example: $m = 4$, $q_u = q, q_v = q^2 = -1, p > 2$. The Hecke algebra H is that associated to the finite groups $SU_4(q)$, where $q^2 \equiv -1 \pmod{p}$. Here $\mathrm{Irr}(H) = \{\phi_{\pm 1}, \phi_2\}$, $C = \mathrm{diag}(2, 2, 1)$ and $T \cong P(1)^{\oplus 2} \oplus \phi_{-1} \oplus P(-1) \oplus \phi_2^{\oplus 4} \oplus \phi_1^{\oplus 2}$. Here $P(i)$, $i = \pm 1$, is a self-extension of ϕ_i by itself. Observe that $x_u H \cong P(1) \oplus \phi_2$ and $x_v H \cong \phi_1 \oplus \phi_{-1} \oplus \phi_2$. Let $Y_{-1} = P(-1), Y_0 = \phi_{-1}, Y_{0'} = P(1), Y_{0''} = \phi_2$, and $Y_1 = \phi_1$. Then (1.3.2) holds if $S_\lambda = \mathrm{soc}(Y_\lambda)$ for all λ.

Here $\widetilde{S}^{\mathcal{R}}_{\lambda k}$ identifies with S_{-1}, $S_{0''} \oplus S_{0'}$, $S_{0''} \oplus S_0$ and S_1 for $\lambda = \emptyset, \{u\}, \{v\}$, and $\{u, v\}$, respectively. However, the dual left cell modules $\widetilde{S}_{\omega k}$ identify with the S_i. The algebra $A = \mathrm{End}_R(T)$ is quasi-hereditary in this case.

3.4. Example: $m = 6$, $q_u = q \neq -1$, $q_v = q^3 = -1$, $q^2 - q + 1 = 0$, $p > 2$. Here H is associated to the finite Steinberg groups ${}^3D_4(q, q^3)$, where $q \not\equiv -1$ and $q^3 \equiv -1 \pmod{p}$. Then $\mathrm{Irr}(H) = \{\phi_1, \phi_{-1}, \phi_2\}$ and $C = \left(\begin{smallmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$. Thus, $P(1) = \begin{array}{|c|c|c|}\hline & 1 & \\ \hline -1 & & 1 \\ \hline & 1 & \\ \hline \end{array}$ and $P(-1) = \begin{array}{|c|c|c|}\hline & -1 & \\ \hline 1 & & -1 \\ \hline & -1 & \\ \hline \end{array}$, respectively. Since $e = \frac{1}{2} x_v$ is an idempotent, $x_v H \cong P(1) \oplus \phi_2$. Also, $x_u H \cong Y \oplus \phi_2$ with $Y = \begin{array}{|c|c|}\hline 1 & -1 \\ \hline -1 & 1 \\ \hline \end{array}$. (Use that $\dim \mathrm{End}_H(x_u H) = 4$.) So we have $T = P(-1) \oplus P(1)^{\oplus 2} \oplus \phi_2^{\oplus 4} \oplus Y \oplus \phi_1$. Put

$Y_{-1} = P(-1)$ with $S_{-1} = \mathrm{soc}(Y_{-1})$, $Y_0 = P(1)$ with $S_0 = \mathrm{rad}(Y_0)$, $Y_{0'} = Y$ with $S_{0'} = \begin{array}{|c|c|c|}\hline & -1 & \\ \hline -1 & & 1 \\ \hline\end{array}$, and $Y_{0''} = \phi_2 = S_{0''}$. Finally, let $Y_1 = S_1 = \phi_1$. Then (1.3.2) holds.

Here the modules $\widetilde{S}^{\mathcal{R}}_{\lambda k}$ identify with S_{-1}, $S_{0'} \oplus S_{0''}$, $S_0 \oplus S_{0''}$, and S_1 for $\lambda = \emptyset$, $\{u\}$, $\{v\}$, and $\{u, v\}$, respectively. Each module $S_0 \oplus S_{0''}$ and $S_{0'} \oplus S_{0''}$ has a length two filtration with two dual left cell modules $\widetilde{S}_{\omega k}$ as sections, corresponding to the left cells $\omega = B, C_1$, and C_2 in the notation of (3.5) below). The filtrations are of the form $\boxed{\begin{matrix} B \\ C_1 \end{matrix}}$ and $\boxed{\begin{matrix} B \\ C_2 \end{matrix}}$, and the order cannot be changed (i. e., B cannot appear below C_i.) The modules S_0 and $S_{0'}$ are not direct summands of any dual left cell modules. The algebra $A = \mathrm{End}_H(T)$ is not quasi-hereditary.

In the same spirit, the other Hecke algebras of Lie type can be analyzed for the various families $\mathcal{G} = \{G(q)\}$ of rank 2 groups of Lie type and the various possible congruence relations on $q \pmod p$. There are a total of 26 non-semisimple examples, as follows by applying (4.2.2) below.

3.5 The conjecture in rank 2 case. We now verify Conjecture (2.5.2) in the rank 2 case. The untwisted rank 2 case follows from (2.4.11), since the right set cells agree with the Kazhdan-Lusztig left cells. Thus, the conjecture holds in this case with $\widetilde{X} = 0$. There are total 4 twisted cases, three of which are described in (3.1.4(2,3,5)). The following is a complete list for all cases, each of which is followed by the system (c_u, c_v).

$$(3.5.1) \qquad \begin{cases} (1)\ {}^2A_3, & (1, 2) \\ (2)\ {}^2A_4, & (2, 3) \\ (3)\ {}^3D_4, & (1, 3) \\ (4)\ {}^2F_4, & (2, 4). \end{cases}$$

By [L2] and [X; p.17][7], the left cells for all cases are of the form

$$(3.5.2) \qquad \begin{gathered} A = \{w_0\}, B = \{uw_0\}, C_1 = \{w \in W \mid \mathcal{R}(w) = u\} \backslash \{u\} \\ C_2 = C_1 w_0, D = Bw_0, E = Aw_0 \end{gathered}$$

and the two-sided cells are $A, B, C = C_1 \cup C_2, D, E$.

(3.5.3) Theorem. *Conjecture (2.5.2) holds for the Hecke algebras of types given in (3.5.1).*

Proof. Using (3.5.2), we explicitly write down the dual left cell filtrations on the various $\widetilde{T}_\lambda = x_\lambda \widetilde{H}$:

$$\widetilde{H} = \boxed{\begin{matrix} A \\ B \\ C_1 \oplus C_2 \\ D \\ E \end{matrix}} \qquad x_{\{u\}}\widetilde{H} = \boxed{\begin{matrix} A \\ C_1 \\ D \end{matrix}} \qquad x_{\{v\}}\widetilde{H} = \boxed{\begin{matrix} A \\ B \\ C_2 \end{matrix}} \qquad x_S \widetilde{H} = \boxed{A}.$$

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[7] The cell structure for ${}^2F_4$ has been worked out in [DR].

Here dual left cell modules are denoted by the same symbol as the left cell itself. Also, the modules C_1 and C_2 are isomorphic. This follows since the module C_1 is isomorphic to the submodule M_{vu} of $x_{\{v\}}\widetilde{H}$ spanned by $\{C^+_{vu}C^-_w\}_{w\in C_2}$, while the module C_2 is spanned by $\{x_{\{v\}}C^-_y\}_{y\in C_1}$. Here $C^-_w = q_w^{-1/2}C_w$ is the modified Kazhdan-Lusztig C_w-basis for H (cf. (2.3.2)). Now $M_{vu} = C_2$ gives the isomorphism between C_1 and C_2. Since B apparently does not appear as a *submodule* in any of the filtrations above, there is no obvious way to standardly stratify ${}_{\widetilde{A}}\mathcal{C}$ using the $\widetilde{S}_\lambda$. (See also §3.4 above.) However, let $\widetilde{T}^+ = \widetilde{T}\oplus\widetilde{X}$ where $\widetilde{X} = \boxed{\begin{matrix} A \\ B \end{matrix}}$. We claim that (1.2.9) applies and therefore, the conclusion of Conjecture (2.5.3) is satisfied.

Indeed, all the conditions of (1.2.9) clearly hold, except possibly the homological condition (1.2.9(3)). By (2.3.9.2), it suffices to prove that $\operatorname{Ext}^1_{\widetilde{H}}(\widetilde{T}_\lambda/F^i_\lambda, \widetilde{X}) = 0$ for all i. Using (1.2.13), this vanishing property will follow provided we prove that base change defines a surjection

$$\operatorname{Hom}_{\widetilde{H}}(\widetilde{T}_\lambda/F^i_\lambda, \widetilde{X})_{\mathcal{Z}'} \to \operatorname{Hom}_{\mathcal{H}'}(\widetilde{T}_{\lambda\mathcal{Z}'}/F^i_{\lambda\mathcal{Z}'}, \widetilde{X}_{\mathcal{Z}'}).$$

Taking duals, it is equivalent to show that

$$\operatorname{Hom}_{\widetilde{H}}(\widetilde{E}^1_{\{v\}}, \widetilde{E}^i_\lambda)_{\mathcal{Z}'} \to \operatorname{Hom}_{\mathcal{H}'}(\widetilde{E}^1_{\{v\}\mathcal{Z}'}, \widetilde{E}^i_{\lambda\mathcal{Z}'}) \tag{3.5.3.1}$$

is surjective. Since $\widetilde{E}^1_{\{v\}\mathcal{Z}'}$ is a submodule of $\widetilde{H}x_{\{v\}}$ generated by $C^+_{w_0}, C^+_{uw_0}$, each map f belonging to the right hand side of (3.5.3.1) is defined by the images $x = f(C^+_{uw_0})$ and $y = f(C^+_{w_0})$. Clearly, $y = \beta C^+_{w_0}$ for some $\beta\in\mathcal{Z}'$ and $\tau_v x = q_v x$. So we may write

$$x = \sum_{\mathcal{L}(w)=\{v\},\mathcal{R}(w)=\lambda} \alpha_w C^+_w.$$

Since $\tau_u C^+_{uw_0} = -C^+_{uw_0} + C^+_{w_0}$ by (2.3.2a), we have

$$\tau_u x = f(\tau_u C^+_{uw_0}) = -f(C^+_{uw_0}) + \beta C^+_{w_0}.$$

On the other hand, using (2.3.2a) again, we obtain

$$\begin{aligned} \tau_u x &= \sum_{w\neq w_0} \alpha_w(-C^+_w + C^+_{uw} + h_w) + q_u\alpha_{w_0}C^+_{w_0} \\ &= -f(C^+_{uw_0}) + \sum_{w\neq w_0}\alpha_w(C^+_{uw} + h_w) + q_u\alpha_{w_0}C^+_{w_0}. \end{aligned}$$

Thus, $\sum_{w\neq w_0}\alpha_w(C^+_{uw}+h_w)+q_u\alpha_{w_0}C^+w_0 = -\alpha_{w_0}C^+_{w_0}+\beta C^+_{w_0}$. So, if $w_0 = uw$ for some w with $\mathcal{L}(w) = \{v\}$ and $\mathcal{R}(w) = \lambda(= \{v\})$, then $h_w = 0$ and $\alpha_w + q_u\alpha_{w_0} = -\alpha_{w_0} + \beta$, and all other $\alpha_w = 0$ by induction on the length of w. Thus, in this case, $x = (\beta - (q_u+1)\alpha)C^+_{uw_0} + \alpha C^+_{w_0}$ for some $\alpha\in\mathcal{Z}'$. Hence, each f is defined by a pair (α,β) and therefore, the elements defined by $(1,0)$ and $(0,1)$ form a basis which is independent of $\mathcal{Z}'$.

If $w_0 \neq uw$ for all such $w\neq w_0$, then $\alpha_w = 0$ for all $w\neq w_0$ and $(q_u+1)\alpha_{w_0} = \beta$. So $\alpha = \alpha_{w_0} = 1$ and $\beta = q_u+1$ define a map which forms a basis over any $\mathcal{Z}'$. This proves (3.5.3.1) □

4. Appendix: Modular Hecke algebras

The semisimplicity criterion (4.2.2) below was obtained by Gyoja [Gy; (3.9)], who discussed more general symmetric algebras over commutative rings and multiparameter Hecke algebras. (See also [G; (1.3.8)] and [GR; (4.3), (5.3)]. The most essential cases were also discussed in [F].) Our proof, obtained independently, is similar in spirit to [Gy], but differs in our use of matrix units, the Brauer theory (especially the existence of lattices) of §1.1, as well as our focus on the Hecke algebra case. The resulting argument is shorter and more direct for our context; in particular, no results on W-graphs are required.

4.1 Generic degrees. For a system $\mathcal{G}$ of finite groups of Lie type, consider the generic algebra $\widetilde{H}_0$[8] over $\mathcal{Z}_0=\mathbb{Z}[t,t^{-1}]$ as given in (2.3.1). Assume that $\widetilde{H}_{\mathbb{Q}(t)}\cong\widetilde{H}_0\otimes_{\mathcal{Z}_0}\mathbb{Q}(t)$ is a split semisimple algebra over $\mathbb{Q}(t)$. By [BC] and [L1], this holds for all Lie types except ${}^2F_4$. (For a discussion of the exceptional case, see (4.2.3) below.) Each $\chi\in\mathrm{Irr}(\widetilde{H}_{\mathbb{Q}(t)})$ (= set of irreducible characters of $\widetilde{H}_{\mathbb{Q}(t)}$) determines a central primitive idempotent $e_\chi\in\widetilde{H}_{\mathbb{Q}(t)}$ given by

$$e_\chi=\frac{d_\chi}{d_W}\sum_w\alpha_w^{-1}\chi(\tau_{w^{-1}})\tau_w,\ \alpha_w=\mathrm{IND}(\tau_w)\in\mathcal{Z}\tag{4.1.1}$$

where $d_\chi\in\mathbb{Z}[t^2]$ is the generic degree of χ and $d_W=\sum_w\alpha_w$. (In this section, we reserve the notation q_w for the image of α_w in a field k.) It is known that

$$d_\chi=\frac{1}{m}t^a\Phi_1\cdots\Phi_b,\tag{4.1.2}$$

where the Φ_i are cyclotomic polynomials in t^2 and m is a product of bad primes for the ambient algebraic group associated to $\mathcal{G}$; see [C; Ch. 13].[9]

Summing (4.1.1) over $\mathrm{Irr}(\widetilde{H}_{\mathbb{Q}})$ gives $\sum_\chi\frac{d_\chi}{d_W}\alpha_w^{-1}\chi(\tau_w)=\delta_{w,1}$, after equating τ_w-coefficients (since $\alpha_w=\alpha_{w^{-1}}$ and $\chi(\tau_w)=\chi(\tau_{w^{-1}})$). The coefficient of τ_1 in $\tau_w\tau_{y^{-1}}$ is $\alpha_w\delta_{w,y}$, so $\sum_\chi\frac{d_\chi}{d_W}\alpha_w^{-1}\chi(\tau_w\tau_{y^{-1}})=\delta_{w,y}$. Hence,

$$\sum_\chi\sum_w\frac{d_\chi}{d_W}\alpha_w^{-1}\chi(h\tau_{w^{-1}})\tau_w=h,\ \forall h\in\widetilde{H}_{\mathbb{Q}(t)}.\tag{4.1.3}$$

For $\chi\in\mathrm{Irr}(\widetilde{H}_{\mathbb{Q}(t)})$, let $\rho^\chi:\widetilde{H}_{\mathbb{Q}(t)}\to M_{n_\chi}(\mathbb{Q}(t))$, $h\mapsto(\rho^\chi_{ij}(h))$, be a matrix realization. Let $f_{ij}\in\widetilde{H}_{\mathbb{Q}(t)}e_\chi$ satisfy $\rho^\chi(f_{ij})=e_{ij}$, the (i,j)-th matrix unit in $M_{n_\chi}(\mathbb{Q}(t))$. For $\psi\in\mathrm{Irr}(\widetilde{H}_{\mathbb{Q}(t)})$, $\psi(f_{ij}\tau_{w^{-1}})=\delta_{\psi,\chi}\rho^\chi_{ji}(\tau_{w^{-1}})$, so (4.1.3) implies that

$$f_{ij}=\frac{d_\chi}{d_W}\sum_w\alpha_w^{-1}\rho^\chi_{ji}(\tau_{w^{-1}})\tau_w,\quad 1\le i,j\le n_\chi.\tag{4.1.4}$$

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[8] One reason for choosing $\widetilde{H}_0$ over $\widetilde{H}$ here is that, when $\widetilde{H}_0$ is untwisted (i. e., when all $c_s=1$), Lusztig [L1] has defined an isomorphism $\widetilde{H}_{0\mathbb{Q}(t)}\xrightarrow{\sim}\mathbb{Q}(t)W$; hence $\widetilde{H}_{0\mathbb{Q}(t)}$ is a split semisimple algebra. Except in types E_7,E_8, the same result holds for the algebras $\widetilde{H}_{\mathbb{Q}(t^2)}$. This fact was proved earlier by Benson-Curtis [BC], who also consider the twisted $\widetilde{H}$.

[9] A prime which is not bad is called good. In types $A_n,{}^2A_n$ all primes are good. In types $B_n,C_n,D_n,{}^2D_n,{}^2B_n,{}^3D_4$ all primes are good, except $p=2$. For F_4,E_6, ${}^2F_4$, ${}^2E_6$, E_7, G_2, ${}^2G_2$ all primes are good, except $p=2,3$. Finally, in type E_8, all primes are good, except $p=2,3,5$.

4.2. Modular Hecke algebras. Fix a field k of characteristic $p > 0$, choose $0 \neq q^{\frac{1}{2}} \in k$, and let $\pi : \mathcal{Z} \to k$ be the ring homomorphism satisfying $\pi(t) = q^{\frac{1}{2}}$. We distinguish two cases:

CASE 1: $q^{\frac{1}{2}}$ *is transcendental over* $\mathbb{F}_p$. Form the local triple $(\mathcal{O}, \mathbb{Q}(t), \mathbb{F}_p(t))$, for the discrete valuation ring $\mathcal{O} = \mathcal{Z}_{(p)}$.

CASE 2: $q^{\frac{1}{2}}$ *is algebraic over* $\mathbb{F}_p$. Form the local triple $(\mathcal{O}, \mathbb{Q}(t), k_0)$, where $\mathcal{O} = \mathcal{Z}_{(p,\Phi)}$ is a regular local ring of Krull dim. 2 (and Φ is a suitable cyclotomic polynomial). Also, we need the local triple $(\mathcal{O}', K', k_0)$, where $\mathcal{O}' = \mathcal{O}/(p)$ is a discrete valuation ring and $K' = \mathbb{F}_p(t)$.

List $\mathrm{Irr}(\widetilde{H}_{\mathbb{Q}(t)})$ as $\chi_1, \cdots, \chi_n$, and, using (1.1.1), let $\widetilde{X}_1, \cdots, \widetilde{X}_n$ be $\widetilde{H}_{\mathcal{O}}$-lattices, respectively, for the corresponding irreducible $\widetilde{H}_{\mathbb{Q}(t)}$-modules $\mathcal{X}_1, \cdots, \mathcal{X}_n$. For each t, put $\bar{X}_t = \widetilde{X}_{tk_0} \in \mathrm{Ob}(\mathcal{C}_{\widetilde{H}_{k_0}})$. Now we prove the following irreducibility result.

(4.2.1) Theorem. *Suppose for some* t*,* $1 \leq t \leq n$*, we have that* $d_{\chi_t}/d_W \in \mathcal{O}$*. Then the* $\widetilde{H}_{k_0}$*-module* $\bar{X}_t$ *is an absolutely irreducible projective module.*

Proof. Write $\chi = \chi_t$, $\mathcal{X} = \mathcal{X}_t$, etc. In §4.1, the matrix realization ρ^χ can taken relative to an $\mathcal{O}$-basis of $\widetilde{X}$. By (4.1.4), each $f_{ij} \in \widetilde{H}_{\mathcal{O}}$ because $d_\chi/d_W \in \mathcal{O}$. Thus, $\rho^\chi(e_\chi \widetilde{H}_{\mathcal{O}}) = M_{n_\chi}(\mathcal{O})$, so $\bar{X}$ is absolutely irreducible. Finally, $e_\chi \in \widetilde{H}_{\mathcal{O}}$ defines a central idempotent $\bar{e}_\chi \in \widetilde{H}_{k_0}$. If $s \neq t$, $\bar{X}_s \bar{e}_\chi = 0$ since $\rho^{\chi_s}(e_\chi) = 0$, while $\bar{e}_\chi$ acts as the identity on $\bar{X}$. So (1.1.2) implies that $\bar{e}_\chi$ is a central primitive idempotent in $\widetilde{H}_{k_0}$. Since $\dim \widetilde{H}_{k_0}\bar{e}_\chi = (\dim \bar{X})^2$, $\bar{X}$ is projective. $\square$

We now prove a general semisimplicity criterion. Compare [F] and [Gy]. We emphasize that in the following theorem q denotes an arbitrary nonzero element in a field k, and *not* necessarily the image of a prime power (associated to the family $\mathcal{G}$) in k. Even if one is ultimately interested in the prime power case, the semisimplicity of the other specializations can be quite relevant (e. g., one might want to apply (2.3.9) to the Hecke algebra over the ring of Laurent polynomials over a finite field, where q would again be a prime power).

(4.2.2) Theorem. *Let* k *be a field of characteristic* $p > 0$ *and let* $0 \neq q^{\frac{1}{2}} \in k$*. Then* $\widetilde{H}_k$ *is semisimple if and only if* p *is a good prime for* (W, S) *and*

$$\sum_{w \in W} q_w = \pi(d_W) \neq 0. \tag{4.2.2.1}$$

When $\widetilde{H}_k$ *is semisimple, it is split semisimple.*

Proof. Assume that p is good and $d_W(q) \neq 0$. Then (4.1.2) implies each $d_{\chi_t}/d_W \in \mathcal{O}$. By (4.2.1), $\bar{X}_t \in \mathrm{Ob}(\mathrm{proj}(\widetilde{H}_{k_0}))$ is absolutely irreducible. By (1.1.2), every irreducible $\widetilde{H}_{k_0}$-module must be isomorphic to some $\bar{X}_t$. Therefore, $\widetilde{H}_{k_0}$ and hence $\widetilde{H}_k = \widetilde{H}_{k_0} \otimes_{k_0} k$ are split semisimple.

Conversely, assume that $\widetilde{H}_k$ is semisimple. Then $\widetilde{H}_{k_0}$ must also be semisimple, so can assume that $k = k_0$. We treat the two CASES above separately:

CASE 1: Since $\widetilde{H}_{\mathbb{Q}(t)}$ is split semisimple, [CR; 1, Ex. 16, p. 142] implies that any idempotent $\bar{e} \in \widetilde{H}_k$ lifts to an idempotent $e \in \widetilde{H}_{\mathcal{O}}$. Suppose $\bar{e}$ is primitive. Then e is primitive in $\widetilde{H}_{\mathbb{Q}(t)}$ (by (1.1.2(b))). Thus, $e\widetilde{H}_{\mathbb{Q}(t)}e \cong \mathbb{Q}(t)$, whence $e\widetilde{H}_{\mathcal{O}}e \cong \mathcal{O}$ and $\bar{e}\widetilde{H}_k\bar{e} \cong k$. Thus, $\widetilde{H}_k$ is split. Also, central primitive idempotents in $\widetilde{H}_k$ lift to central primitive idempotents in $\widetilde{H}_{\mathcal{O}}$ (*op. cit.*), and every e_χ is the lift of an idempotent (so lies in $\widetilde{H}_{\mathcal{O}}$).

Suppose that p is not good. By [C; Ch. 13], we can choose $\chi_i \in \operatorname{Irr}(\widetilde{H}_{\mathbb{Q}(t)})$ so that in (4.1.2) $p|m$. Necessarily, $\bar{X}_i$ is irreducible; see, e. g., [CPS4; (1.5.2(d,e)]. Since $e_{\chi_i} \in \widetilde{H}_{\mathcal{O}}$, (4.1.1) implies $\chi_i(\tau_w) \in \mathfrak{m}$, $\forall w \in W$, Thus, any $h \in \widetilde{H}_k$ has zero trace on $\bar{X}_i$, which is absurd. Hence, p is good, while $d_W(q) \neq 0$, trivially.

CASE 2: Since $\widetilde{H}_k$ is semisimple, $\widetilde{H}_{K'}$ is also semisimple. By CASE 1, p is good. Finally, let $\bar{e} \in \widetilde{H}_{k_0}$ be the primitive idempotent corresponding to IND. Then $\bar{e} = c\sum_w \tau_w$, $0 \neq c \in k_0$ so $\bar{e}^2 = cd_W(q)\bar{e}$ and $d_W(q) \neq 0$, as required. $\square$

(4.2.3) Remarks. (a) When W has type A, (4.2.2) has been proved by Dipper-James [DJ2]. In type B, see Dipper-James [DJ4] and Dipper [D1].

(b) In type ${}^2F_4$, $\widetilde{H}_{\mathbb{Q}(t)}$ splits after $\sqrt{2}$ is adjoined. One can easily modify the arguments above to conclude that (4.2.2) holds in case $\sqrt{2} \in k$.

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