QUANTUM WEYL RECIPROCITY AND TILTING MODULES

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Abstract. Quantum Weyl reciprocity relates the representation theory of Hecke algebras of type A with that of q-Schur algebras. This paper establishes that Weyl reciprocity holds integrally (i. e., over the ring $\mathbb{Z}[q,q^{-1}]$ of Laurent polynomials) and that it behaves well under base-change. A key ingredient in our approach involves the theory of tilting modules for q-Schur algebras. New results obtained in that direction include an explicit determination of the Ringel dual algebra of a q-Schur algebra in all cases. In particular, in the most interesting situation, the Ringel dual identifies with a natural quotient algebra of the Hecke algebra.

Weyl reciprocity refers to the connection between the representation theories of the general linear group $GL_n(k)$ and the symmetric group $\mathfrak{S}_r$. Let V be a vector space (over a field k) of dimension n and form the tensor space $V^{\otimes r}$. The natural (left) action of $GL_n(k)$ on $V^{\otimes r}$ commutes with the (right) permutation action of $\mathfrak{S}_r$. Let A (resp., R) be the algebra generated by the image of $GL_n(k)$ (resp., $\mathfrak{S}_r$) in the algebra $\mathrm{End}(V^{\otimes r})$ of linear operators on $V^{\otimes r}$. Classically [We], when $k=\mathbb{C}$, these algebras satisfy the double centralizer property

$$(1) \qquad a) \quad A=\mathrm{End}_R(V^{\otimes r}) \quad \text{and} \qquad b) \quad R=\mathrm{End}_A(V^{\otimes r}).$$

Further, the set $\Lambda^+(n,r)$ of partitions of r into at most n nonzero parts indexes both the irreducible A-modules $L(\lambda)$ and the irreducible R-modules S_λ. The $L(\lambda)$ are the irreducible polynomial representations of $GL_n(\mathbb{C})$ of homogeneous degree r, while the S_λ are Specht modules for $\mathfrak{S}_r$. Weyl reciprocity also entails the decomposition

$$(2) \qquad V^{\otimes r}=\bigoplus_{\lambda\in\Lambda^+(n,r)} L(\lambda)\otimes S_\lambda$$

of the tensor space into irreducible (A,R^{op})-bimodules.

When k has positive characteristic p, property (1) remains true, but it is more difficult to establish; see [CL; (3.1)] for the equality (1a) and [dCP; (4.1)] or [D2; §2 Cor.] for (1b). (The latter *is* easy when $n\geq r$.) The set $\Lambda^+(n,r)$ still indexes

1991 *Mathematics Subject Classification.* Primary 17B37,20G05; Secondary 16S80.
Research supported by the National Science Foundation and the Australian Research Council.

$\mathrm{Irr}(A)$, while $\mathrm{Irr}(R)$ is indexed by the subset $\Lambda^+(n,r)_{p-\mathrm{reg}}$ of p-regular partitions. The decomposition (2) no longer holds, in general.

The Hecke algebra H of type A_{r-1} arises as a q-deformation of the group algebra $k\mathfrak{S}_r$. Motivated by physics, Jimbo [Ji] gave a corresponding action of H on $V^{\otimes r}$ deforming the permutation action of $\mathfrak{S}_r$. In the generic case ($k = \mathbb{C}$ and $q \in \mathbb{C}$ is "general"—i. e., not a root of 1), he gave a quantum version of Weyl reciprocity, similar to the classical situation. In his theory, the quantum enveloping algebra $U_q(\mathfrak{gl}_n)$ played the role of $GL_n(k)$ above (while A is the q-Schur algebra $S_q(n,r)$). Later, entirely different considerations in finite group representation theory led Dipper and James [DJ1&2] independently to make similar constructions; the name "q-Schur algebra" is due to them.[1]

This paper considers quantum Weyl reciprocity when k and q are arbitrary, and the related theory of tilting modules for q-Schur algebras $S_q(n,r)$. There is a surjective homomorphism $U_{q^{1/2}} = U_{q^{1/2}}(\mathfrak{gl}_n) \to S_q(n,r)$. A self-dual $S_q(n,r)$-module X is a *tilting module* if, when regarded as a $U_{q^{1/2}}$-module, it has a filtration with sections isomorphic to q-Weyl modules. In the more general context of quasi-hereditary algebras, Ringel [R] established the existence of a rich supply of tilting modules. Tilting modules have remarkable homological properties, giving rise, for example, to interesting equivalences of derived categories $D^b(A) \xrightarrow{\sim} D^b(B)$ if B is the endomorphism algebra of a *full* tilting module for A. In this case, B is called the Ringel dual of A (though it is only determined up to Morita equivalence); it is also a quasi-hereditary algebra.

In [DPS1,2], we used Kazhdan-Lusztig cell theory methods to study Hecke endomorphism algebras of importance in the representation theory of finite groups G of Lie type over fields of positive characteristic distinct from the defining characteristic of G. Those methods remain effective in the present paper. In particular, the "strong homological property" of cell filtrations, discovered in [DPS1] and reviewed in (3.5) below, plays an essential role in several places, e. g., in our generalization of (2) above in Theorem 6.8. Also, new and simple proofs of a number of results (from [dCP], [D1], [E1]) involving the representation theory of GL_n and $\mathfrak{S}_r$ result by setting $q = 1$. Because we largely work in an integral, or characteristic-free, setting, many of our results are new even in the $q = 1$ case.

We now outline the contents of this paper. Section 1 discusses general representation theoretic facts for Hecke algebras associated to a finite Coxeter system. We apply these in §2 to study certain Hecke endomorphism algebras A. In particular, Theorem 2.5 presents a candidate for a full tilting module for A. Section 3 treats various "cell modules" which play an important role in the theory of q-Schur algebras in §5. Section 4 collects information concerning quasi-hereditary algebras and their tilting modules. In (5.5), we give a direct proof that $V^{\otimes r}$ is a tilting module for $S_q(n,r)$. A consequence, Theorem 5.7, establishes an important base change property for q-tensor space which plays an essential role in §6.

Section 6 takes up quantum Weyl reciprocity. Making use of an interesting new basis for the q-Specht modules given in (6.1), Theorem 6.2 describes R^{op} very explicitly as a quotient algebra of the Hecke algebra H. The proof is almost self-

[1] If k has positive characteristic p, the q-Schur algebras play a central role in the representation theory of the finite general linear groups $GL_n(\mathbb{F}_q)$ when p and q are relatively prime; see [DJ2].

contained, using only (5.7) mentioned above. The double centralizer property (1) then follows easily in Theorem 6.3. In the special case $q = 1$, we obtain a new proof of the double centralizer result of [dCP]. Although the decomposition (2) fails in general, Theorem 6.8 establishes that $V^{\otimes r}$ does have an $(A, R^{\rm op})$-bimodule *filtration* analogous to (2). The existence of such a filtration plays a crucial role in our determination in Theorem 7.9 of §7 of the Ringel duals of q-Schur algebras. In the classical $q = 1$ case, Donkin [D1; (3.6)] observed a connection between tilting modules and "twisted" permutation modules. [CPS2; (5.2)] used a similar idea to give a combinatorial[2] approach to Schur algebra tilting modules, realizing the latter in characteristic $\neq 2$ as modules of intertwining operators between "twisted" permutation modules and permutation modules. While we find that this description generally remains valid here, a second (and perhaps equally interesting) realization of tilting modules given in Theorem 2.5 proves essential in treating all specializations of q and all characteristics; see (7.4), (7.7). Also, (7.10) describes connections with more recent work of Donkin.

Section 8 first recasts, in a q-setting, recent work of Erdmann [E1] concerning decomposition numbers for symmetric groups and Weyl modules. As in the classical case, the decomposition numbers of the Hecke algebra H are determined in terms of filtration multiplicities for tilting modules; see (8.3). Finally, suppose that q is a primitive lth root of unity satisfying $n < l$. Then Theorem 8.6 establishes that $H(n,r) \stackrel{\rm def}{=} {\rm End}_{S_q(n,r)}(V^{\otimes r})^{\rm op}$ is quasi-hereditary; in fact, $H(n,r)$ identifies with the Ringel dual of $S_q(n,r)$.

Some notation. Unless otherwise stated, $\mathcal{Z} = \mathbb{Z}[q, q^{-1}]$, the ring of integral Laurent polynomials in a variable q, and $\mathbb{Q}(q)$ is its field of fractions. If $\mathcal{Z}'$ is a commutative $\mathcal{Z}$-algebra and $\widetilde{M}$ is a $\mathcal{Z}$-module, put $\widetilde{M}_{\mathcal{Z}'} = \widetilde{M} \otimes_{\mathcal{Z}} \mathcal{Z}'$. When clear from context, denote $\widetilde{M}_{\mathcal{Z}'}$ by $\widetilde{M}'$.

The reader should take careful note of the "tilde" notation used thoughout this paper: Usually, the notation $\widetilde{M}$ denotes a $\mathcal{Z}'$-module, where $\mathcal{Z}'$ is some fixed commutative $\mathcal{Z}$-algebra (e. g., $\mathcal{Z}' = \mathcal{Z}$). However, if a field k is a $\mathcal{Z}'$-algebra, we often write M for $\widetilde{M}_k$.

Given a ring R, ${}_R\mathcal{C}$ (resp., $\mathcal{C}_R$) is the category of finitely generated left (resp., right) R-modules. If an R-module M has a composition series, then $[M : L]$ is the multiplicity of the irreducible module L as a composition factor of M. Let $R^{\rm op}$ be the opposite ring of R. If M is a left (resp., right) R-module, let $M^{\rm op}$ be the right (resp., left) $R^{\rm op}$-module obtained from the action of R on M.

For an R-module M, we consider (finite) filtrations $F_\bullet : 0 = F_0 \subset F_1 \subset \cdots \subset F_t = M$ of M by submodules F_i. The F_i/F_{i-1} are called the *sections* of $F_\bullet$, with $F_1 = F_1/F_0$ the *bottom* section and F_t/F_{t-1} the *top* section. If Δ is a fixed family of R-modules and if each section $F_i/F_{i-1} \in \Delta$, then $F_\bullet$ is called a Δ-filtration. Let ${}_R\mathcal{C}(\Delta)$ denotes the subclass of ${\rm Ob}({}_R\mathcal{C})$ consisting of objects which have a Δ-filtration.

If $e \in R$ is an idempotent and $M \in {\rm Ob}({}_R\mathcal{C})$, ${\rm Hom}_R(Re, M)$ identifies with the eRe-module eM by means of $f \mapsto f(e)$, $f \in {\rm Hom}_R(Re, M)$. It will be convenient

[2]The method of [D1] involved algebraic groups and Hopf algebra structures, whereas [CPS2] and the present paper are essentially combinatorial in nature.

to have a general characterization of this property in the following way: Given $a \in R$ and $M \in \mathrm{Ob}({}_R\mathcal{C})$, let $\mathrm{Tr}_R(a, M) = \{f(a) \mid f \in \mathrm{Hom}_R(Ra, M)\}$. Since any module map $f : Ra \to M$ is determined by the image $f(a)$, $f \mapsto f(a)$ defines an additive group isomorphism $\mathrm{Hom}_R(Ra, M) \xrightarrow{\sim} \mathrm{Tr}_R(a, M)$. For $m \in M$, the map $Ra \to M$, $ra \mapsto ram$, is an R-module map (the restriction to Ra of an obvious map $R \to M$). Hence, aM is a subgroup of $\mathrm{Tr}_R(a, M)$. By definition, $a \in R$ is *almost-idempotent w.r.t.* M if $\mathrm{Tr}_R(a, M) = aM$, or, equivalently, if $f(a) \in aM$ for every $f \in \mathrm{Hom}_R(Ra, M)$. The usefulness of such a property was perhaps first suggested, though not formalized, by [DJ2; (2.7)].

(0.1) Lemma. *Let R be a ring, $e \in R$ an idempotent, and $M \in \mathrm{Ob}({}_R\mathcal{C})$. Let $a \in eRe$ be almost-idempotent w.r.t. the eRe-module eM. Then a is almost-idempotent w.r.t. the R-module M, and restriction $\mathrm{Hom}_R(Ra, N) \stackrel{\mathrm{res}}{\to} \mathrm{Hom}_{eRe}(eRa, eN)$ is a natural isomorphism of abelian groups for every R-submodule $N \subseteq M$. In particular, for $M = Re$ and $a, b \in eRe$ with a almost-idempotent w.r.t. eM, we have an isomorphism*

$$\mathrm{Hom}_R(Ra, Rb) \cong \mathrm{Hom}_{eRe}(eRa, eRb). \tag{0.1.1}$$

We leave the proof to the reader as an exercise, using the definition above. Observe in the second assertion that $\mathrm{Hom}_{eRe}(eRa, eN)$ identifies with $eN \cap aeM = N \cap aM$.

Although the above has been cast for ${}_R\mathcal{C}$, an evident version holds for $\mathcal{C}_R$.

(0.2) Remark. In the context of [DJ2; (2.7)], or any situation in which M is a faithful R-module, the property that a is almost-idempotent is equivalent to the *double annihilator* property for aM in M:

$$aM = \{m \in M \mid rm = 0 \text{ for every } r \in R \text{ with } raM = 0)\}. \tag{0.2.1}$$

In general, even if M is not faithful, the double annihilator property for aM in M implies that a is almost-idempotent with respect to M.

Yet a third property, one which implies the double annihilator property, is the *annihilator property* which says that for some subset $S \subseteq R$, we have

$$aM = \{m \in M \mid sm = 0 \text{ for all } s \in S\}. \tag{0.2.2}$$

This property is often the most natural one to check, cf. the proof of (1.4e) below, but sometimes (0.1) is more natural, in the presence of a natural idempotent e. (See the proof of (2.3b).) With regard to (0.1), observe that if (0.2.2) applies for the ring eRe for some $S \subseteq eRe$ and $a \in eRe$, then to show that a is almost-idempotent in R w.r.t. M, we can again use (0.2.2) with the (somewhat unnatural) subset $S + (1 - e)$.

1. Hecke algebras. Let (W, S) be a finite Coxeter system. Let $\leq$ be the Bruhat-Chevalley poset structure on W and let $\ell : W \to \mathbb{Z}$ be the usual length function.

Let $\mathcal{P}(S)$ denote the power set of S. Consider a finite poset Λ and a fixed (arbitrary) function $J : \Lambda \to \mathcal{P}(S)$. The poset Λ will serve as a "weight" poset in our theory and the role of the function J is to adjust the multiplicities of direct summands in a "tensor" space. We will assume for simplicity that J is *surjective.* For $\lambda \in \Lambda$, $W_\lambda = \langle s \,|\, s \in J(\lambda)\rangle$ is the associated parabolic subgroup; thus, $(W_\lambda, J(\lambda))$ is again a finite Coxeter system. For convenience in the sequel, we will usually identify $\lambda \in \Lambda$ with the subset $J(\lambda)$.

Fix a system $\{c_s \in \mathbb{Z}\}_{s\in S}$ of *index parameters*, i. e., integers c_s satisfying $c_s = c_t$ if s and t are W-conjugate. For $w \in W$, let $w = s_1 \cdots s_m$ be a reduced expression and put $q_w = q^{c_{s_1}} \cdots q^{c_{s_m}} \in \mathcal{Z}$. The definition of q_w is independent of the reduced expression chosen for w. If all $c_s = 1$, then $q_w = q^{\ell(w)}$. Put $\epsilon_w = (-1)^{\ell(w)}$.

The generic Hecke algebra $\widetilde{H}$ over $\mathcal{Z}$ is the algebra with $\mathcal{Z}$-basis $\{\tau_w\}_{w\in W}$, satisfying the relations (for $s \in S$, $w \in W$):–

$$(1.1)\qquad \tau_s\tau_w = \begin{cases} \tau_{sw} & \text{if } sw > w; \\ q_s\tau_{sw} + (q_s - 1)\tau_w & \text{if } sw < w. \end{cases}$$

Suppose that $\mathcal{Z}'$ is a commutative $\mathcal{Z}$-algebra and put $\widetilde{H}' = \widetilde{H}_{\mathcal{Z}'}$. Thus, $\widetilde{H}'$ has a basis $\tau_w \otimes 1$, $w \in W$, satisfying relations like those in (1.1). To simplify notation, continue to denote $\tau_w \otimes 1$ by τ_w; we will follow this same convention for other bases of $\widetilde{H}$ which arise below. Also, when no confusion results, let q_w denote the image of q_w in $\mathcal{Z}'$.

The algebra $\widetilde{H}'$ admits a $\mathcal{Z}'$-automorphism Φ (resp., $\mathcal{Z}'$-anti-automorphism ι) of order 2 defined on basis elements by $\Phi(\tau_w) = \epsilon_w q_w \tau_{w^{-1}}^{-1}$ (resp., $\iota(\tau_w) = \tau_{w^{-1}}$); see [L1; p.138]. Given an $\widetilde{H}'$-module $\widetilde{M}$, let $\widetilde{M}^\Phi$ denote the $\widetilde{H}'$-module obtained from $\widetilde{M}$ by twisting the action of $\widetilde{H}'$ on $\widetilde{M}$ by Φ. Similarly, if $\widetilde{M}$ is a left (resp., right) $\widetilde{H}'$-module, then $\widetilde{M}^\iota$ is the right (resp., left) $\widetilde{H}'$-module obtained from $\widetilde{M}$ by twisting the action by ι. In general, writing $\widetilde{M}^* = \mathrm{Hom}_{\mathcal{Z}'}(\widetilde{M}, \mathcal{Z}')$, we define a contravariant "duality" functor $\mathfrak{d}_{\widetilde{H}'} : \mathcal{C}_{\widetilde{H}'} \to \mathcal{C}^{\mathrm{op}}_{\widetilde{H}'}$ by setting, for $\widetilde{M}, \widetilde{N} \in \mathrm{Ob}(\mathcal{C}_{\widetilde{H}'})$ and $f \in \mathrm{Hom}_{\widetilde{H}'}(\widetilde{M}, \widetilde{N})$:

$$(1.2)\qquad \mathfrak{d}_{\widetilde{H}'}\widetilde{M} = (\widetilde{M}^*)^\iota \quad \text{and} \quad \mathfrak{d}_{\widetilde{H}'}f = \mathrm{Hom}_{\mathcal{Z}'}(f, \mathcal{Z}').$$

If $\widetilde{M}$ is a projective $\mathcal{Z}'$-module, $\mathfrak{d}^2_{\widetilde{H}'}\widetilde{M} \cong \widetilde{M}$. Whenever $\widetilde{M} \in \mathrm{Ob}(\mathcal{C}_{\widetilde{H}})$ is $\mathcal{Z}$-projective, there is an isomorphism

$$(1.3)\qquad (\mathfrak{d}_{\widetilde{H}} \otimes \mathcal{Z}')\widetilde{M}_{\mathcal{Z}'} = \mathfrak{d}_{\widetilde{H}}(\widetilde{M})_{\mathcal{Z}'} \cong \mathfrak{d}_{\widetilde{H}'}\widetilde{M}_{\mathcal{Z}'}$$

which is natural in $\widetilde{M}$. Since Φ and ι commute, $\mathfrak{d}_{\widetilde{H}'}(\widetilde{M}^\Phi) \cong \mathfrak{d}_{\widetilde{H}'}(\widetilde{M})^\Phi$.

For $\lambda \in \Lambda$, let $x_\lambda = \sum_{w\in W_\lambda} \tau_w$, $y_\lambda = \sum_{w\in W_\lambda} \epsilon_w q_w^{-1}\tau_w \in \widetilde{H}'_\lambda$. (If $\lambda \mapsto J \subseteq S$ under the surjection $\Lambda \to \mathcal{P}(S)$, we sometimes write x_J for x_λ and y_J for y_λ. For example, $x_\emptyset = y_\emptyset = 1$.) The q-permutation modules $x_\lambda\widetilde{H}'$, $y_\lambda\widetilde{H}'$ and $\widetilde{H}'x_\lambda$, $\widetilde{H}'y_\lambda$ play an important role. These modules are all free $\mathcal{Z}'$-modules. If $\widetilde{H}'_\lambda = \langle \tau_w \,|\, w \in W_\lambda\rangle$, $\tau_w x_\lambda = q_w x_\lambda = x_\lambda\tau_w$ and $\tau_w y_\lambda = \epsilon_w y_\lambda = y_\lambda\tau_w$, for $w \in W_\lambda$, and so $x_\lambda\widetilde{H}'$, $y_\lambda\widetilde{H}'$, etc. can be interpreted as induced modules (from $\widetilde{H}'_\lambda$); cf. [DPS1; (2.1.5)].

Let $\mathrm{tr} : \widetilde{H}' \to \mathcal{Z}'$, $\tau_w \mapsto \delta_{w1}$, be the $\mathcal{Z}'$-linear "trace" map. Then $\langle a, b\rangle = \mathrm{tr}(ab)$ defines a non-degenerate, symmetric, associative pairing on $\widetilde{H}'$ (see, e.g., [L1; (5.1.9)]). This pairing satisfies $\langle \tau_w, \tau_v\rangle = q_w\delta_{w,v^{-1}}$.

(1.4) Lemma. *Let $\lambda, \mu \in \Lambda$. Then:–*

(a) $(x_\lambda \widetilde{H}')^* \cong \widetilde{H}' x_\lambda$ and $(y_\lambda \widetilde{H}')^* \cong \widetilde{H}' y_\lambda$ in ${}_{\widetilde{H}'}\mathcal{C}$, where $(\Leftrightarrow)^* = \mathrm{Hom}_{\mathcal{Z}'}(\Leftrightarrow, \mathcal{Z}')$. Thus, $\mathfrak{d}_{\widetilde{H}'} x_\lambda \widetilde{H}' \cong x_\lambda \widetilde{H}'$ and $\mathfrak{d}_{\widetilde{H}'} y_\lambda \widetilde{H}' \cong y_\lambda \widetilde{H}'$. There is a non-degenerate form $(\ ,\ )$ defined on $x_\lambda \widetilde{H}'$ satisfying $(ah, b) = (a, bh^\iota)$ for $a, b \in x_\lambda \widetilde{H}'$, $h \in \widetilde{H}'$.

(b) $x_\lambda \widetilde{H}' = \{h \in \widetilde{H}' \mid \tau_s h = q_s h, \forall s \in \lambda\}$ and $y_\lambda \widetilde{H}' = \{h \in \widetilde{H}' \mid \tau_s h = \Leftrightarrow h, \forall s \in \lambda\}$.

(c) $(x_\lambda \widetilde{H}')^\Phi \cong y_\lambda \widetilde{H}'$ and $\mathrm{Hom}_{\widetilde{H}'}(y_\lambda \widetilde{H}', y_\mu \widetilde{H}') \cong \mathrm{Hom}_{\widetilde{H}'}(x_\lambda \widetilde{H}', x_\mu \widetilde{H}')$.

(d) If the subgroups W_λ and W_μ are W-conjugate, then $x_\mu \widetilde{H}' \cong x_\lambda \widetilde{H}'$ and $y_\mu \widetilde{H}' \cong y_\lambda \widetilde{H}'$.

(e) Both x_λ and y_λ are almost idempotents w.r.t. the left or right $\widetilde{H}'$-module $\widetilde{H}'$ (in the sense of (0.1)).

(f) Let $\mathcal{D}^0_{\lambda,\mu} = \{d \in W \mid \ell(udv) = \ell(u) + \ell(d) + \ell(v), \forall u \in W_\lambda, v \in W_\mu\}$.[3] Then $y_\lambda \widetilde{H}' x_\mu$ is a free $\mathcal{Z}'$-module with basis $\{y_\lambda \tau_d x_\mu\}_{d \in \mathcal{D}^0_{\lambda,\mu}}$. Similarly, $x_\mu \widetilde{H}' y_\lambda$ is a free $\mathcal{Z}'$-module with basis $\{x_\mu \tau_d y_\lambda\}_{d \in \mathcal{D}^0_{\mu,\lambda}}$. Both $x_\mu \widetilde{H}' y_\lambda$ and $y_\lambda \widetilde{H}' x_\mu$ are $\mathcal{Z}'$-direct summands of $\widetilde{H}'$.

(g) If $q_s + 1$ is not a zero divisor in $\mathcal{Z}'$, $\forall s \in \lambda$, then y_λ is an almost-idempotent (in the sense of (0.1)) w.r.t. the left modules $\widetilde{H}' x_\mu$, as well as for the right modules $x_\mu \widetilde{H}'$.

(h) Assume, for all $s \in S$, that $q_s + 1$ is not a zero divisor in $\mathcal{Z}'$. Then $\mathrm{Hom}_{\widetilde{H}'}(\widetilde{H}' y_\lambda, \widetilde{H}' x_\mu)$ is a free $\mathcal{Z}'$-module with basis $\{\phi_d\}_{d \in \mathcal{D}^0_{\lambda,\mu}}$ where $\phi_d(y_\lambda) = y_\lambda \tau_d x_\mu$. Similarly, $\mathrm{Hom}_{\widetilde{H}'}(y_\lambda \widetilde{H}', x_\mu \widetilde{H}')$ is a free $\mathcal{Z}'$-module with basis $\{\phi_d\}_{d \in \mathcal{D}^0_{\mu,\lambda}}$, where now $\phi_d(y_\lambda) = x_\mu \tau_d y_\lambda$.

Proof. (a) can be proved easily by using the pairing $\langle\ ,\ \rangle$ defined just before the statement of the lemma. For details, see [DPS1; (2.1.9)] for the first assertion and [DJ1; (4.4)] for the last assertion. For (b), see [Cu; (1.9)] or [DPS1; (2.1.6)]. Next, (c) is proved in [DJ2; (2.1)]. A proof for (d) can be found in [DJ1; (4.3)]. Also, (e) follows from (b), noting (0.2.2) is a consequence, while (f) follows from [DJ1; (4.1)] (or directly from [C; (2.7.5)]). (Another proof of (e) could be based on (a), rather than (b), using the fact that $x_\lambda \widetilde{H}'$ and $y_\lambda \widetilde{H}'$ are $\mathcal{Z}'$-direct summands of $\widetilde{H}'$.)

If $q_s + 1$ is not a zero divisor in $\mathcal{Z}'$ for $s \in \lambda$, the properties of distinguished double coset representatives given in [C; (2.7.4), (2.7.5)] easily imply that $y_\lambda \widetilde{H}' \cap \widetilde{H}' x_\mu = y_\lambda \widetilde{H}' x_\mu$. Now (g) follows from this fact using (e). Also, (e) and (0.1) then imply that

$$\mathrm{Hom}_{\widetilde{H}'}(\widetilde{H}' y_\lambda, \widetilde{H}' x_\mu) = y_\lambda \widetilde{H}' \cap \widetilde{H}' x_\mu = y_\lambda \widetilde{H}' x_\mu.$$

With this observation, (h) follows from (f). □

[3] $\mathcal{D}^0_{\lambda,\mu}$ is the set of distinguished W_λ, W_μ-double coset representatives with trivial intersection property. See [C; §2.7].

2. Hecke endomorphism algebras. Given $\lambda \in \Lambda$, write $\widetilde{T}_\lambda = x_\lambda \widetilde{H}$, so that, by (1.4c), $\widetilde{T}_\lambda^\Phi = y_\lambda \widetilde{H}$. For $\ ,\ \subseteq \Lambda$, put $\widetilde{T}(\ ,\ ) = \bigoplus_{\lambda\in\Gamma} \widetilde{T}_\lambda \in \mathrm{Ob}(\mathcal{C}_{\widetilde{H}})$, and form the *Hecke endomorphism algebra*

$$(2.1) \qquad \widetilde{A}(\ ,\ ,\mathcal{Z}') = \mathrm{End}_{\widetilde{H}'}(\widetilde{T}(\ ,\ )_{\mathcal{Z}'}) \cong \mathrm{End}_{\widetilde{H}'}(\widetilde{T}(\ ,\ )^\Phi_{\mathcal{Z}'})$$

for any commutative $\mathcal{Z}$-algebra $\mathcal{Z}'$. The isomorphism in (2.1) follows from (1.4c).

(2.2) Lemma. *Let $\mathcal{Z}'$ be a commutative $\mathcal{Z}$-algebra, and, if $\widetilde{M} \in \mathrm{Ob}(\mathcal{C}_{\mathcal{Z}})$, abbreviate $\widetilde{M}_{\mathcal{Z}'}$ by $\widetilde{M}'$. Let $\lambda, \mu \in \Lambda$, and $\ ,\ \subseteq \Lambda$. Then:–*

(a) $\mathrm{Hom}_{\widetilde{H}}(\widetilde{T}_\lambda, \widetilde{T}_\mu)' \cong \mathrm{Hom}_{\widetilde{H}'}(\widetilde{T}'_\lambda, \widetilde{T}'_\mu)$ *and* $\mathrm{Hom}_{\widetilde{H}}(\widetilde{T}^\Phi_\lambda, \widetilde{T}^\Phi_\mu)' \cong \mathrm{Hom}_{\widetilde{H}'}(\widetilde{T}^{\Phi\prime}_\lambda, \widetilde{T}^{\Phi\prime}_\mu)$.

(b) $\widetilde{A}(\ ,\ ,\mathcal{Z})' \cong \widetilde{A}(\ ,\ ,\mathcal{Z}')$.

(c) Any $\widetilde{H}'$-homomorphism $\widetilde{T}'_\mu \to \widetilde{T}'_\lambda$ is obtained by left multiplication by an element $h \in \widetilde{H}'$ satisfying $hx_\mu = x_\lambda h'$ for some $h' \in \widetilde{H}'$. Thus, the algebra $\widetilde{A}(\ ,\ ,\mathcal{Z}')$ is a homomorphic image of the algebra of all $\ ,\ \times\ ,\ $ matrices $(h_{\lambda,\mu})$ where $h_{\lambda,\mu} \in \widetilde{H}'$ satisfies $h_{\lambda,\mu} x_\mu \in x_\lambda \widetilde{H}'$.

Proof. The first assertion in (a) is well-known [DJ1; (3.3)]. (Another argument can be based on [DPS1; (2.3.4), (2.3.5)].) Next, (a) implies (b) from the definition (2.1). To prove (c), let $f : \widetilde{T}'_\mu = x_\mu \widetilde{H}' \to \widetilde{T}'_\lambda = x_\lambda \widetilde{H}'$ be a right $\widetilde{H}'$-module morphism. Then $f(x_\mu) = x_\lambda h'$ for some $h' \in \widetilde{H}'$. By (1.4b), we have $f(x_\mu) = hx_\mu$ for some $h \in \widetilde{H}'$ and it follows that f is left multiplication by h. Now the second assertion follows immediately from this fact. $\square$

In (2.1), we write $\widetilde{A}(\ ,\ )$ for $\widetilde{A}(\ ,\ ,\mathcal{Z})$, so that (2.2) implies that $\widetilde{A}(\ ,\ )' \cong \widetilde{A}(\ ,\ ,\mathcal{Z}')$ for any commutative $\mathcal{Z}$-algebra $\mathcal{Z}'$. For $\ ,\ \subseteq \Lambda$, let $e_\Gamma : \widetilde{T}(\Lambda)' \to \widetilde{T}(\ ,\ )'$ be the idempotent projection. Then $e_\Gamma \in \widetilde{A}(\Lambda)'$ and $\widetilde{A}(\ ,\ )' \cong e_\Gamma \widetilde{A}(\Lambda)' e_\Gamma$. Because the map $\Lambda \to \mathcal{P}(S)$ is surjective, there exists $\gamma \in \Lambda$ with $x_\gamma = 1$. Then $e = e_{\{\gamma\}}$ is an idempotent projection of $\widetilde{T}(\Lambda)'$ onto a summand isomorphic to $\widetilde{H}' = x_\gamma \widetilde{H}'$. (According to our convention, we could also write $x_\emptyset$ for x_γ here.) This implies that

$$(2.3) \qquad \begin{cases} (1)\ \widetilde{T}(\Lambda)' \cong \widetilde{A}(\Lambda)' e \text{ in } {}_{\widetilde{A}(\Lambda)'}\mathcal{C}; \\ (2)\ e\widetilde{A}(\Lambda)' e \cong \widetilde{H}' \text{ as algebras.} \end{cases}$$

We now show that $\mathfrak{d}_{\widetilde{H}'}$—see (1.2)—induces a duality on ${}_{\widetilde{A}(\Gamma)'}\mathcal{C}$. The following result is motivated by [CPS2; (1.2.1)].

(2.4) Lemma. *Let $\mathcal{Z}'$ be a commutative $\mathcal{Z}$-algebra. For any $\ ,\ \subseteq \Lambda$, there is a contravariant "duality" functor $\mathfrak{d}_{\widetilde{A}(\Gamma)'} : {}_{\widetilde{A}(\Gamma)'}\mathcal{C} \to {}_{\widetilde{A}(\Gamma)'}\mathcal{C}^{\mathrm{op}}$ satisfying:–*

(a) $\mathfrak{d}_{\widetilde{A}(\Gamma)'} \widetilde{T}(\ ,\ )' \cong \widetilde{T}(\ ,\ )'$; in fact, if $\widetilde{\phi} : \widetilde{T}(\ ,\ )^* \to \widetilde{T}(\ ,\ )$ is a $\mathcal{Z}$-isomorphism satisfying $\widetilde{\phi}(hf) = \widetilde{\phi}(f)h^\iota$ for all $f \in \widetilde{T}^*, h \in \widetilde{H}$, then there exists an anti-automorhism $\widetilde{\beta}$ of $\widetilde{A}(\ ,\ )$ such that $\widetilde{\beta}^2 = 1$ and $\widetilde{\phi}(fa^\beta) = a\widetilde{\phi}(f)$ for all $f \in \widetilde{T}^*, a \in \widetilde{A}(\ ,\ )$.

(b) $\mathfrak{d}^2_{\widetilde{A}(\Gamma)'} \widetilde{M} \cong \widetilde{M}$ for any $\widetilde{M} \in \mathrm{Ob}({}_{\widetilde{A}(\Gamma)'}\mathcal{C})$ which is $\mathcal{Z}'$-projective;

(c) If $\widetilde{M} \in \mathrm{Ob}({}_{\widetilde{A}(\Gamma)}\mathcal{C})$ is $\mathcal{Z}$-projective, then $(\mathfrak{d}_{\widetilde{A}(\Gamma)}(\widetilde{M}))' \cong \mathfrak{d}_{\widetilde{A}(\Gamma)'}(\widetilde{M}')$.

Proof. By (1.4a), fix an isomorphism $\widetilde{\phi} : \mathfrak{d}_{\widetilde{H}}(\widetilde{T}(\, ,\,)) \xrightarrow{\sim} \widetilde{T}(\, ,\,)$ of $\widetilde{H}$-modules. Such a $\widetilde{\phi}$ satisfies the condition "$\widetilde{\phi}(hf) = \widetilde{\phi}(f)h^{\iota}$" stated in (a). Define an anti-automorphism $\widetilde{\beta}$ of $\widetilde{A}(\, ,\,)$ by $\widetilde{\beta}(f) = \widetilde{\phi} \circ \mathfrak{d}_{\widetilde{H}}(f) \circ \widetilde{\phi}^{-1}$, $f \in \widetilde{A}(\, ,\,)$. For a commutative $\mathcal{Z}$-algebra $\mathcal{Z}'$, set $\widetilde{\beta}' = \widetilde{\beta}_{\mathcal{Z}'}$. By (1.2), one sees easily that $\widetilde{\beta}^2 = 1$. Given a left $\widetilde{A}(\, ,\,)'$-module $\widetilde{M}$, put $\mathfrak{d}_{\widetilde{A}(\Gamma)'}\widetilde{M} = \mathrm{Hom}_{\mathcal{Z}'}(\widetilde{M}, \mathcal{Z}')$ with the right action of $\widetilde{A}(\, ,\,)'$ converted to a left action by means of $\widetilde{\beta}'$. The required properties for $\mathfrak{d}_{\widetilde{A}(\Gamma)'}$ follow formally—see [CPS2; (1.2.1)] for (a),(b), while (c) follows from the definitions. □

As explained in [CPS2; (1.2.2a)], the above result recasts in the present set-up a familiar phenomenon in the theory of permutation groups, where the role of $\widetilde{A}(\, ,\,)$ is played by the centralizer algebra of a permutation module for a finite group, and $\widetilde{\beta}'$ corresponds to matrix transpose. In fact, by (2.2c), we have, if $f \in \widetilde{A}(\, ,\,)'$ is the image of the matrix $(h_{\lambda,\mu})$, then $\widetilde{\beta}'(f)$ is the image of the transpose of the matrix $({\iota}h_{\lambda,\mu})$

Despite its simple description, the module $\widetilde{X}$ ($= \widetilde{X}'$ for the case $\mathcal{Z}' = \mathcal{Z}$) introduced below in (2.5c) plays a central role in the representation theory of $\widetilde{A}(\Lambda)$.

(2.5) Theorem. *Let $\mathcal{Z}'$ be a commutative $\mathcal{Z}$-algebra, and write $\widetilde{A}' = \widetilde{A}(\Lambda)_{\mathcal{Z}'}$, $\widetilde{T}' = \widetilde{T}(\Lambda)_{\mathcal{Z}'}$, and $\widetilde{T}^{\Phi\prime} = \widetilde{T}(\Lambda)^{\Phi}_{\mathcal{Z}'}$. Then:–*

(a) For $\lambda \in \Lambda$, the left $\widetilde{A}'$-module $\widetilde{T}'y_\lambda$ satisfies $\mathfrak{d}_{\widetilde{A}'}\widetilde{T}'y_\lambda \cong \widetilde{T}'y_\lambda$. If, in addition, $q_s + 1$ is not a zero divisor in $\mathcal{Z}'$, $\forall s \in \lambda$, then $\widetilde{T}'y_\lambda \cong \mathrm{Hom}_{\widetilde{H}'}(\widetilde{T}^{\Phi\prime}_\lambda, \widetilde{T}') \in \mathrm{Ob}({}_{\widetilde{A}'}\mathcal{C})$.

(b) For $\lambda, \mu \in \Lambda$, $\mathrm{Hom}_{\widetilde{A}'}(\widetilde{T}'y_\lambda, \widetilde{T}'y_\mu) \cong \mathrm{Hom}_{\widetilde{H}'}(y_\mu\widetilde{H}', y_\lambda\widetilde{H}')$ in $\mathcal{C}_{\mathcal{Z}'}$.

(c) Put $\widetilde{X}' = \bigoplus_{\lambda\in\Lambda} \widetilde{T}'y_\lambda$. Then $\mathrm{End}_{\widetilde{A}'}(\widetilde{X}') \cong \widetilde{A}'^{\mathrm{op}}$.

Proof. We have $\widetilde{T}'y_\lambda \in \mathrm{Ob}({}_{\widetilde{A}'}\mathcal{C})$, since $\widetilde{T}'$ is an $(\widetilde{A}', \widetilde{H}')$-bimodule. Next, observe that the natural map $y_\lambda\widetilde{T}'^* \to (\widetilde{T}'y_\lambda)^*$, sending $y_\lambda f$ to the linear functional $ty_\lambda \mapsto f(ty_\lambda)$, $f \in \widetilde{T}'^*$, $t \in \widetilde{T}'$, is a well-defined isomorphism of right $\widetilde{A}'$-modules. (The surjectivity of this map is immediate from (1.4f).) Let $\widetilde{\phi} : \widetilde{T}^* \xrightarrow{\sim} \widetilde{T}$ be the $\mathcal{Z}$-isomorphism as in (2.4a). Since $y_\lambda^\iota = y_\lambda$, $\widetilde{\phi}$ defines (by restriction) an isomorphism of $y_\lambda\widetilde{T}^*$, viewed as a left $\widetilde{A}$-module by means of the anti-automorphism $\widetilde{\beta}$, to $\widetilde{T}y_\lambda$. Base-changing to $\mathcal{Z}'$ gives a similar isomorphism $y_\lambda\widetilde{T}'^* \cong \widetilde{T}y_\lambda$. Now the first assertion in (a) is clear.

By (1.4g), if $q_s + 1$ is not a zero divisor in $\mathcal{Z}'$, $\forall s \in \lambda$, y_λ is an almost-idempotent for any $x_\mu\widetilde{H}'$, and hence for $\widetilde{T}'$. Hence, $\mathrm{Hom}_{\widetilde{H}'}(\widetilde{T}^{\Phi\prime}_\lambda, \widetilde{T}') = \mathrm{Hom}_{\widetilde{H}'}(y_\lambda\widetilde{H}', \widetilde{T}') \cong \widetilde{T}'y_\lambda$, completing the proof of (a).

Using (2.3), (1.4e) and (0.1.1) give $\mathrm{Hom}_{\widetilde{A}'}(\widetilde{T}'y_\lambda, \widetilde{T}'y_\mu) \cong \mathrm{Hom}_{\widetilde{H}'}(\widetilde{H}'y_\lambda, \widetilde{H}'y_\mu)$. Now (b) follows from (1.4a).

If $f \in \mathrm{Hom}_{\widetilde{A}'}(\widetilde{T}'y_\lambda, \widetilde{T}'y_\mu)$, $g \in \mathrm{Hom}_{\widetilde{A}'}(\widetilde{T}'y_\mu, \widetilde{T}'y_\nu)$ map to $\bar{f} \in \mathrm{Hom}_{\widetilde{H}'}(y_\mu\widetilde{H}', y_\lambda\widetilde{H}')$, $\bar{g} \in \mathrm{Hom}_{\widetilde{H}'}(y_\nu\widetilde{H}', y_\mu\widetilde{H}')$, respectively, under the isomorphism given in (b), then gf is sent to $\bar{f}\bar{g}$. Thus, $\mathrm{End}_{\widetilde{A}'}(\widetilde{X}') \cong \mathrm{End}_{\widetilde{H}'}(\widetilde{T}^{\Phi\prime})^{\mathrm{op}} \cong \mathrm{End}_{\widetilde{H}'}(\widetilde{T}')^{\mathrm{op}} \cong \widetilde{A}'^{\mathrm{op}}$, and (c) follows. □

Note that (2.5a) holds if Λ is replaced by a subset , $\subseteq \Lambda$. We point out that the advantage of using $\widetilde{T}'y_\lambda$ instead of $\mathrm{Hom}_{\widetilde{H}'}(\widetilde{T}_\lambda'^{\Phi}, \widetilde{T}')$ as direct summands of $\widetilde{X}'$ is the following base change property: $\widetilde{T}'y_\lambda \cong (\widetilde{T}y_\lambda)_{\mathcal{Z}'}$, and hence, $\widetilde{X}' \cong \widetilde{X}_{\mathcal{Z}'}$, for *all* commutative $\mathcal{Z}$-algebras $\mathcal{Z}'$ without restriction on any q_s.

In (2.6) below, we denote either of the contravariant functors $\mathrm{Hom}_{\widetilde{H}'}(\Leftrightarrow, \widetilde{T}') : \mathcal{C}_{\widetilde{H}'} \to {}_{\widetilde{A}'}\mathcal{C}$ and $\mathrm{Hom}_{\widetilde{A}'}(\Leftrightarrow, \widetilde{T}') : {}_{\widetilde{A}'}\mathcal{C} \to \mathcal{C}_{\widetilde{H}'}$ by $(\Leftrightarrow)^\diamond$. Thus, $\widetilde{X}' \cong \widetilde{T}^{\Phi\prime\diamond}$ provided that $q_s + 1$ is not a zero divisor in $\mathcal{Z}'$, $\forall s \in S$. For any $\widetilde{A}'$ or $\widetilde{H}'$-module $\widetilde{M}$, there is a natural "evaluation map" $\mathrm{Ev}_{\widetilde{M}} : \widetilde{M} \to \widetilde{M}^{\diamond\diamond}$. (See [CPS2; (1.1)].)

(2.6) Corollary. *With the above notation, we have:–*

(a) The evaluation maps $\mathrm{Ev}_{\widetilde{T}_\lambda^{\Phi\prime}}$ *and* $\mathrm{Ev}_{\widetilde{T}_\lambda^{\Phi\prime\diamond}}$ *are isomorphisms, provided* $q_s + 1$ *is not a zero divisor in* $\mathcal{Z}'$, $\forall s \in \lambda$. *Thus, if* $q_s + 1$ *is not a zero divisor in* $\mathcal{Z}'$, $\forall s \in S$, $(\Leftrightarrow)^\diamond$ *induces an algebra isomorphism* $\tilde{f}' : \widetilde{A}^{\mathrm{op}\prime} = \mathrm{End}_{\widetilde{H}'}(\widetilde{T}^{\Phi\prime})^{\mathrm{op}} \to \mathrm{End}_{\widetilde{A}'}(\widetilde{T}^{\Phi\prime\diamond}) \overset{\mathrm{def}}{=} \widetilde{E}'$.

(b) Assume that $q_s + 1$ *is not a zero divisor in* $\mathcal{Z}'$, $\forall s \in S$, *and let* $\widetilde{F}' : {}_{\widetilde{E}'}\mathcal{C} \overset{\sim}{\to} {}_{\widetilde{A}'^{\mathrm{op}}}\mathcal{C}$ *be the equivalence of categories defined using pull-back through the isomorphism* $\tilde{f}'$ *in (a). Suppose that* $\widetilde{N}$ *is a right* $\widetilde{H}'$*-module such that* $\mathrm{Ev}_{\widetilde{N}}$ *is an isomorphism. Then, in* ${}_{\widetilde{A}'^{\mathrm{op}}}\mathcal{C}$,

$$\widetilde{F}'(\mathrm{Hom}_{\widetilde{A}'}(\widetilde{N}^\diamond, \widetilde{T}^{\Phi\prime\diamond})) \cong \mathrm{Hom}_{\widetilde{H}'}(\widetilde{T}^{\Phi\prime}, \widetilde{N}). \tag{2.6.1}$$

Proof. Taking $\mu = \emptyset$, we conclude from (2.5a,b), tracing through the maps involved, that $\mathrm{Ev}_{\widetilde{T}_\lambda^{\Phi\prime}}$ is an isomorphism. It then follows formally that $\mathrm{Ev}_{\widetilde{T}_\lambda^{\Phi\prime\diamond}}$ is an isomorphism. This proves (a). Then (b) follows by the functoriality of $(\Leftrightarrow)^\diamond$. (Observe in (2.6.1) that the natural left action of $\widetilde{A}'$ on $\widetilde{T}^{\Phi\prime}$ defines a right action of $\widetilde{A}'$ on $\mathrm{Hom}_{\widetilde{H}'}(\widetilde{T}^{\Phi\prime}, \widetilde{N})$, while the left $\widetilde{E}'$-module structure of $\widetilde{T}^{\Phi\prime\diamond}$ defines a left action of $\widetilde{E}'$ on $\mathrm{Hom}_{\widetilde{A}'}(\widetilde{N}^\diamond, \widetilde{T}^{\Phi\prime\diamond})$.) $\square$

3. Cell modules. In this section, we collect together a number of results from Kazhdan-Lusztig cell theory that will be needed later. The strong homological property described below in (3.5) will play a key role in what follows.

In [KL1] and [L2], certain bases $\{C_w\}_{w\in W}$ and $\{C'_w\}_{w\in W}$ were presented for the algebra $\widetilde{H}_0 = \widetilde{H} \otimes_{\mathcal{Z}} \mathbb{Z}[q^{1/2}, q^{-1/2}]$. Following [DPS1,2],[4] for $w \in W$, define $C_w^+ = q_w^{1/2} C'_w$ and $C_w^- = q_w^{-1/2} C_w$. Both $\{C_w^+\}_{w\in W}$ and $\{C_w^-\}_{w\in W}$ form a $\mathcal{Z}$-basis for $\widetilde{H}$. There are preorders $\leq_L$ and $\leq_R$ defined on W which satisfy:

$$\widetilde{H} C_w^\varepsilon \subseteq \sum_{y \leq_L w} \mathcal{Z} C_y^\varepsilon, \quad \text{and} \quad C_w^\varepsilon \widetilde{H} \subseteq \sum_{y \leq_R w} \mathcal{Z} C_y^\varepsilon \tag{3.1}$$

for each choice $\varepsilon = \pm$. In particular, $\tau_s C_w^+ = q_s C_w^+$ (resp., $\tau_s C_w^- = \Leftrightarrow C_w^-$) if $sw < w$, $s \in S$. Let $\leq_{LR}$ be the preorder generated by $\leq_L, \leq_R$. The equivalence classes in

[4] Although it would be simpler to work with the C'_w-basis (or the C_w-basis), and therefore with $\widetilde{H}_0$, stronger results are obtained using the Hecke algebra $\widetilde{H}$ over the smaller ring $\mathbb{Z}[q, q^{-1}]$; see, e. g., the discussion in [DPS2]. Some results below, such as (6.4) and the proof of (6.8), would simplify somewhat over $\mathbb{Z}[q^{1/2}, q^{-1/2}]$.

W defined by $\leq_L, \leq_R$ and $\leq_{LR}$ are the *left cells, right cells* and *two-sided cells*, respectively. For example, $w \sim_L y$ if and only if $w \leq_L y$ and $y \leq_L w$. Let Ω (resp., Ξ) be the set of left (resp., two-sided) cells in W. For $\omega \in \Omega$, $\omega^{-1} = \{y^{-1} \mid y \in \omega\}$ is a right cell. Also, the preorder $\leq_L$ (resp., $\leq_{LR}$) defines a poset structure on Ω (resp., Ξ).

Given $x \in W$, its *right-set* $\mathcal{R}(x)$ (resp., *left-set* $\mathcal{L}(x)$) is the subset of S consisting of those $s \in S$ such that $xs < x$ (resp., $sx < x$). It is well-known that, if $x \leq_R y$ (resp., $x \leq_L y$), then $\mathcal{L}(x) \supseteq \mathcal{L}(y)$ (resp., $\mathcal{R}(x) \supseteq \mathcal{R}(y)$) [KL1; (2.4)], [X; (1.20)]. Further, for any $\lambda \in \Lambda$, $\{C_y^+ \mid J(\lambda) \subseteq \mathcal{L}(y)\}$ is a $\mathcal{Z}$-basis for $\widetilde{T}_\lambda = \widetilde{H}x_\lambda$. Similarly, $\{C_y^+ \mid J(\lambda) \subseteq \mathcal{R}(y)\}$ is a $\mathcal{Z}$-basis for $\widetilde{H}x_\lambda$. See [DPS1; (2.3.5)].

Let $\omega \in \Omega$ be a left cell. Because of (3.1), ω determines a left cell module $\widetilde{E}_\omega$:

$$\widetilde{E}_\omega = (\sum_{y \leq_L w} \mathcal{Z}C_y^+)/(\sum_{y <_L w} \mathcal{Z}C_y^+) \in \mathrm{Ob}({}_{\widetilde{H}}\mathcal{C}) \quad (w \in \omega \text{ fixed}). \tag{3.2}$$

Then $\widetilde{S}_\omega = \mathrm{Hom}_{\mathcal{Z}}(\widetilde{E}_\omega, \mathcal{Z}) \in \mathrm{Ob}(\mathcal{C}_{\widetilde{H}})$ is the corresponding *dual left cell* module. The right cell $\gamma = \omega^{-1}$ defines, in the same way, a right cell module

$$\widetilde{K}_\gamma = (\sum_{y \leq_L w} \mathcal{Z}C_{y^{-1}}^+)/(\sum_{y <_L w} \mathcal{Z}C_{y^{-1}}^+) \in \mathrm{Ob}(\mathcal{C}_{\widetilde{H}}) \quad (w^{-1} \in \gamma \text{ fixed}). \tag{3.3}$$

Since $\iota(C_y^+) = C_{y^{-1}}^+$ (from the uniqueness in [KL1; (1.1)]), the definitions imply that

$$\mathfrak{d}_{\widetilde{H}}\widetilde{S}_\omega \cong \widetilde{K}_\gamma, \quad \gamma = \omega^{-1}. \tag{3.4}$$

By an $\widetilde{S}$-filtration of a right $\widetilde{H}$-module $\widetilde{M}$, we mean a filtration $\widetilde{F}_\bullet : 0 = \widetilde{F}_0 \subset \widetilde{F}_1 \subset \cdots \subset \widetilde{F}_t = \widetilde{M}$ by $\widetilde{H}$-submodules such that each nonzero section $\widetilde{F}_i/\widetilde{F}_{i-1}$ is isomorphic to some $\widetilde{S}_{\omega_i}$ for some $\omega_i \in \Omega$. For example, [DPS1; (2.3.9)] establishes that the modules $\widetilde{T}_\lambda = x_\lambda\widetilde{H}$, $\lambda \in \Lambda$, has an $\widetilde{S}$-filtration $\widetilde{F}_{\lambda\bullet}$ which satisfies, in addition, the strong homological property that

$$\mathrm{Ext}\,^1_{\widetilde{H}'}(\widetilde{T}'_\lambda/\widetilde{F}'_{\lambda i}, \widetilde{T}'_\mu) = 0, \quad \forall i,\, \mu \in \Lambda, \tag{3.5}$$

whenever $\mathcal{Z}'$ is a $\mathcal{Z}$-algebra which is an integral domain with the property that $\widetilde{H}_K$ is semisimple over the fraction field K of $\mathcal{Z}'$. Further, $\widetilde{F}_{\lambda\bullet}$ has bottom section $\widetilde{S}_{\omega_\lambda}$, where ω_λ is the left cell containing the longest word $w_{0,\lambda} \in W_\lambda$.

(3.6) Theorem. *Let , $\subseteq \Lambda$. Define* $\widetilde{\Delta}(\omega, ,) = \mathrm{Hom}_{\widetilde{H}}(\widetilde{S}_\omega, \widetilde{T}(,)) \in \mathrm{Ob}({}_{\widetilde{A}(\Gamma)}\mathcal{C})$, $\omega \in \Omega$. *Then:–*

(a) The $\mathcal{Z}$-module $\widetilde{\Delta}(\omega, ,)$ is free.

(b) The $\widetilde{A}(,)$-module $\widetilde{T}(,)$ has a $\{\widetilde{\Delta}(\omega, ,)\}_{\omega\in\Omega}$-filtration.

Proof. For (a), see [DPS1; (2.5.1)]. As described above, $\widetilde{H} = x_\emptyset\widetilde{H}$ has an $\widetilde{S}$-filtration $\widetilde{F}_\bullet = \widetilde{F}_{\emptyset\bullet}$ satisfying (3.5). Thus,

$$\mathrm{Ext}\,^1_{\widetilde{H}}(\widetilde{H}/\widetilde{F}_i, \widetilde{T}(,)) \cong \bigoplus_{\mu\in\Gamma} \mathrm{Ext}\,^1_{\widetilde{H}}(\widetilde{H}/\widetilde{F}_i, \widetilde{T}_\mu) = 0. \tag{3.6.1}$$

For an integer i, define $\widetilde{G}_i = \mathrm{Hom}_{\widetilde{H}}(\widetilde{H}/\widetilde{F}_i, \widetilde{T}(\,,\,))$. Then $\widetilde{T}(\,,\,) = \widetilde{G}_0 \supset \widetilde{G}_1 \supset \cdots \supset \widetilde{G}_t = 0$ for some t, so that $\widetilde{G}_\bullet$ is a filtration of $\widetilde{T}(\,,\,) = \mathrm{Hom}_{\widetilde{H}}(\widetilde{H}, \widetilde{T}(\,,\,))$. Furthermore, for any i, applying the functor $\mathrm{Hom}_{\widetilde{H}}(\Leftrightarrow, \widetilde{T}(\,,\,))$ to the short exact sequence $0 \to \widetilde{F}_i/\widetilde{F}_{i-1} \to \widetilde{H}/\widetilde{F}_{i-1} \to \widetilde{H}/\widetilde{F}_i \to 0$ and using (3.6.1) yields an exact sequence

$$0 \to \mathrm{Hom}_{\widetilde{H}}(\widetilde{H}/\widetilde{F}_i, \widetilde{T}(\,,\,)) \to \mathrm{Hom}_{\widetilde{H}}(\widetilde{H}/\widetilde{F}_{i-1}, \widetilde{T}(\,,\,)) \to \mathrm{Hom}_{\widetilde{H}}(\widetilde{F}_i/\widetilde{F}_{i-1}, \widetilde{T}(\,,\,)) \to 0$$

by the long exact sequence of $\mathrm{Ext}^{\bullet}_{\widetilde{H}}$. For some $\omega_i \in \Omega$, we have $\widetilde{F}_i/\widetilde{F}_{i-1} \cong \widetilde{S}_{\omega_i}$. Therefore,

$$\widetilde{G}_{i-1}/\widetilde{G}_i \cong \mathrm{Hom}_{\widetilde{H}}(\widetilde{S}_{\omega_i}, \widetilde{T}(\,,\,)) \cong \widetilde{\Delta}(\omega_i,\,,\,),$$

as required. □

Let $\mathcal{Z}'$ be a commutative $\mathcal{Z}$-algebra. To simplify notation, we will as before continue to denote the bases $\{C^{\varepsilon}_w \otimes 1\}_{w\in W}$ $(\varepsilon = \pm)$ for $\widetilde{H}' = \widetilde{H}_{\mathcal{Z}'}$ by $\{C^{\varepsilon}_w\}_{w\in W}$. In §6, we use the basis for $\widetilde{H}$ dual to the C^+_w-basis under the "trace form" $\langle\ ,\ \rangle$ introduced above (1.4):

(3.7) Lemma. *Let $\{D^+_w\}_{w\in W}$ be the dual basis for $\widetilde{H}'$ with respect to $\{C^+_w\}_{w\in W}$ in the sense that $\langle C^+_w, D^+_y\rangle = \delta_{w,y^{-1}}$, $\forall y, w \in W$. Then:–*

(a) If $C^+_x D^+_{y^{-1}} \neq 0$, then $y \leq_L x$.

(b) For any $y \in W$, $\widetilde{H}' D^+_y \subseteq \sum_{z \geq_L y} \mathcal{Z}' D^+_z$ and $D^+_y \widetilde{H}' \subseteq \sum_{z \geq_R y} \mathcal{Z}' D^+_z$.

Proof. For $\mathcal{Z}' = \mathcal{Z}$, (a) follows from [L1; (5.1.14)]. (In fact, this is an easy calculation using (3.1).) Also, (b) is immediate from (3.1). Now for a homomorphism $\mathcal{Z} \to \mathcal{Z}'$, the lemma follows by applying the base change functor $\Leftrightarrow \otimes_{\mathcal{Z}} \mathcal{Z}'$. □

4. Quasi-hereditary algebras. From §5 on, we will restrict to the Hecke algebras associated to symmetric groups and will mainly consider the q-Schur algebras as the Hecke endomorphism algebras defined in (2.1). Since q-Schur algebras are quasi-hereditary (see below), we briefly review the general theory of quasi-hereditary algebras in this section. For more details of the elementary theory, see [CPS1].

Let A be a finite dimensional algebra over a field k whose irreducible modules are absolutely irreducible. Let Λ^+ be a set indexing the representatives from the distinct isomorphism classes of irreducible (left) A-modules: for $\lambda \in \Lambda^+$, $L(\lambda) \in \mathrm{Ob}({}_A\mathcal{C})$ is the associated irreducible. Let $P(\lambda) \in \mathrm{Ob}({}_A\mathcal{C})$ (resp., $I(\lambda) \in \mathrm{Ob}({}_A\mathcal{C})$) be the projective cover (resp., injective envelope) of $L(\lambda)$. Suppose that Λ^+ has a fixed poset structure $\leq$ such that for $\lambda \in \Lambda^+$ there exists $\Delta(\lambda) \in \mathrm{Ob}({}_A\mathcal{C})$ satisfying the following two conditions:

(i) $L(\lambda)$ is the head of $\Delta(\lambda)$ and all other composition factors $L(\mu)$ satisfy $\mu < \lambda$;

(ii) $P(\lambda)$ has a filtration with top section $\Delta(\lambda)$ and lower sections $\Delta(\mu)$ for some $\mu > \lambda$.

Then ${}_A\mathcal{C}$ is a *highest weight category* (HWC) with poset Λ^+. The algebra A is *quasi-hereditary* if and only if ${}_A\mathcal{C}$ is a HWC for a poset structure on Λ^+. If ${}_A\mathcal{C}$ is a HWC, there exist $\nabla(\lambda) \in \mathrm{Ob}({}_A\mathcal{C})$, $\lambda \in \Lambda^+$, such that (i°) $\nabla(\lambda)$ has socle $L(\lambda)$ and all other composition factors $L(\mu)$ satisfy $\mu < \lambda$;

(ii°) $I(\lambda)$ has a filtration with bottom section $\nabla(\lambda)$ and higher sections $\nabla(\mu)$ for some $\mu > \lambda$.

Of course, these conditions are just dual to conditions (i), (ii) above. In case A needs to be explicitly mentioned, write $\Delta(\lambda, {}_A\mathcal{C})$, $P(\lambda, {}_A\mathcal{C})$, etc. for $\Delta(\lambda)$, $P(\lambda)$, etc.

The modules $\Delta = \{\Delta(\lambda)\}$ and $\nabla = \{\nabla(\lambda)\}$—called the *standard* and *costandard* modules, respectively, of the HWC ${}_A\mathcal{C}$—satisfy strong homological properties

$$(4.1)\ \text{ for } \lambda, \mu \in \Lambda^+, n \in \mathbb{Z}^+: \begin{cases} (1)\, \dim \operatorname{Ext}^n_A(\Delta(\lambda), \nabla(\mu)) = \delta_{n0}\delta_{\lambda\mu}; \\ (2)\, n > 0\ \&\ \operatorname{Ext}^n_A(\Delta(\lambda), \Delta(\mu)) \neq 0 \implies \lambda < \mu \\ (3)\, n > 0\ \&\ \operatorname{Ext}^n_A(\nabla(\lambda), \nabla(\mu)) \neq 0 \implies \lambda > \mu. \end{cases}$$

(This well known fact is immediate from the proof of [CPS1; (3.11)].) Any $M \in \operatorname{Ob}({}_A\mathcal{C})$ with both a Δ-filtration and a ∇-filtration is a *tilting module*, i. e., $M \in {}_A\mathcal{C}(\text{tilt}) \stackrel{\text{def}}{=} {}_A\mathcal{C}(\Delta) \bigcap {}_A\mathcal{C}(\nabla)$. If $M \in {}_A\mathcal{C}(\Delta)$ and $N \in {}_A\mathcal{C}(\nabla)$, the $n = 1$ case of (4.1(1)) implies that $\operatorname{Ext}^n_A(M, N) = 0$ for $n > 0$.

Ringel [R] has obtained some results on ${}_A\mathcal{C}(\text{tilt})$: For $\lambda \in \Lambda^+$, there exists a unique indecomposable $X(\lambda) \in {}_A\mathcal{C}(\text{tilt})$ such that λ is the maximal $\mu \in \Lambda^+$ for which $[X(\lambda) : L(\mu)] \neq 0$. In fact, $[X(\lambda) : L(\lambda)] = 1$. By (4.1(2)), $X(\lambda)$ has a Δ-filtration with bottom section $\Delta(\lambda)$ and "higher" sections $\Delta(\mu)$ for some $\mu < \lambda$. Similarly, $X(\lambda)$ has a ∇-filtration with top section $\nabla(\lambda)$ and "lower" sections $\nabla(\mu)$ for some $\mu < \lambda$. Every $X \in {}_A\mathcal{C}(\text{tilt})$ has a decomposition $X = \bigoplus_{\lambda \in \Lambda^+} X(\lambda)^{\oplus m_\lambda(X)}$. If each integer $m_\lambda(X) > 0$, then X is a *full tilting module*. In turn, the $X(\lambda)$, $\lambda \in \Lambda$, are often called the indecomposable *partial tilting modules* for ${}_A\mathcal{C}$. Given a full tilting module X, "the" *Ringel dual* $A^\star = \operatorname{End}_A(X)$ of A is quasi-hereditary. If Y is another full tilting module, the algebras $\operatorname{End}_A(X)$ and $\operatorname{End}_A(Y)$ are Morita equivalent—thus, ${}_A\mathcal{C}^\star \stackrel{\text{def}}{=} {}_{A^\star}\mathcal{C}$ (the Ringel dual of ${}_A\mathcal{C}$) is a HWC (with poset the opposite poset $(\Lambda^+, \leq^{\text{op}})$). (In other words, $\lambda \leq^{\text{op}} \mu$ if and only if $\mu \leq \lambda$.) Also, X is a full tilting module for $A^\star$ and $\operatorname{End}_{A^\star}(X) \cong A$. For $\lambda \in \Lambda^+$, $\Delta^\star(\lambda) = \operatorname{Hom}_A(\Delta(\lambda), X)$ is the Δ-object in ${}_A\mathcal{C}^\star$ corresponding to λ.[5]

Let $\,^+$ be an ideal in the poset Λ^+ of a HWC ${}_A\mathcal{C}$ (i. e., $\omega \leq \gamma \in \,^+$ implies that $\omega \in \,^+$). The full subcategory ${}_A\mathcal{C}[\,^+]$ of ${}_A\mathcal{C}$ consisting of objects having composition factors $L(\gamma)$, $\gamma \in \,^+$, is a HWC with poset $\,^+$. Its standard objects $\Delta(\gamma, {}_A\mathcal{C}[\,^+])$ and costandard objects $\nabla(\gamma, {}_A\mathcal{C}[\,^+])$, $\gamma \in \,^+$, are defined by: $\Delta(\gamma, {}_A\mathcal{C}[\,^+]) = \Delta(\gamma, {}_A\mathcal{C})$ and $\nabla(\gamma, {}_A\mathcal{C}[\,^+]) = \nabla(\gamma, {}_A\mathcal{C})$. For $\gamma \in \,^+$, $X(\gamma, {}_A\mathcal{C}) \in \operatorname{Ob}({}_A\mathcal{C}[\,^+])$, and so $X(\gamma, {}_A\mathcal{C})$ identifies with $X(\gamma, {}_A\mathcal{C}[\,^+])$.

The quotient category ${}_A\mathcal{C}(\Omega^+) = {}_A\mathcal{C}/{}_A\mathcal{C}[\,^+]$ is a HWC with poset $\Omega^+ = \Lambda^+ \backslash \,^+$ (using the induced poset structure $\leq$). For some idempotent $e \in A$, we have ${}_A\mathcal{C}(\Omega^+) \cong {}_{eAe}\mathcal{C}$. The exact functor $j^* : {}_A\mathcal{C} \to {}_{eAe}\mathcal{C}$, $M \mapsto j^*(M) = eM$, carries $L(\omega)$ (resp., $\Delta(\omega)$, $\nabla(\omega)$), for $\omega \in \Omega^+$, to the corresponding objects in ${}_{eAe}\mathcal{C}$. If

[5] The reader can also consult [CPS2; §3.4] for another treatment of these results.

$\gamma \in , ^+$, however, we have that $j^*L(\gamma) \cong j^*\Delta(\gamma) \cong j^*\nabla(\gamma) = 0$. Hence,

$$(4.2) \qquad \begin{cases} j^*{}_A\mathcal{C}(\Delta) \subseteq {}_{eAe}\mathcal{C}(\Delta) \\ j^*{}_A\mathcal{C}(\nabla) \subseteq {}_{eAe}\mathcal{C}(\nabla) \\ j^*{}_A\mathcal{C}(\mathrm{tilt}) \subseteq {}_{eAe}\mathcal{C}(\mathrm{tilt}). \end{cases}$$

(4.3) Lemma. *Let $\Omega^+ = \Lambda^+ \backslash , ^+$ be a coideal in the poset Λ^+ of the HWC ${}_A\mathcal{C}$. Then:–*

(a) For $\omega \in \Omega^+$, we have $j^*X(\omega) \cong X(\omega, {}_{eAe}\mathcal{C})$. Hence, $j^*{}_A\mathcal{C}(\mathrm{tilt}) = {}_{eAe}\mathcal{C}(\mathrm{tilt})$.

(b) We have:–

$$(4.3.1) \qquad {}_A\mathcal{C}(\Omega^+)^\star \cong {}_A\mathcal{C}^\star[\Omega^+],$$

where, on the right hand side, Ω^+ is regarded as an ideal in the opposite poset $(\Lambda^+, \leq^{\mathrm{op}})$.

Proof. By (4.2), $j^*X(\omega) \in {}_{eAe}\mathcal{C}(\mathrm{tilt})$ for $\omega \in \Omega^+$. Since $[j^*X(\omega) : j^*L(\omega)] = 1$ and ω is the maximal element $\tau \in \Omega^+$ such that $j^*L(\tau)$ is a composition factor of $j^*X(\omega)$, (a) follows if $j^*X(\omega)$ is indecomposable. By an (essentially elementary) homological argument [CPS2; (3.4.6.2)] using adjoints and ∇-filtrations, the restriction map $\mathrm{End}_A(X(\gamma)) \to \mathrm{End}_{eAe}(j^*X(\gamma))$ is surjective.[6] Since $X(\gamma)$ is indecomposable, $\mathrm{End}_A(X(\gamma))$ is a local algebra. Hence, $\mathrm{End}_{eAe}(j^*X(\gamma))$ is also local, so $j^*X(\gamma)$ is indecomposable. This proves (a).

Finally, (b) follows from [CPS2; (3.4.6)]. □

We conclude this section with the following result. It will be particularly useful to know that tilting modules satisfy the base change property (4.4.1); see (c) below. This property was originally suggested by a corresponding well-known property of projective modules (which tilting modules become, under appropriate functorial transformations).

(4.4) Lemma. *Let $\mathcal{Z} = \mathbb{Z}[q, q^{-1}]$ and let $\widetilde{A}$ be an arbitrary $\mathcal{Z}$-algebra which is finitely generated and projective as a $\mathcal{Z}$-module. Suppose $\widetilde{M}, \widetilde{N} \in \mathrm{Ob}({}_{\widetilde{A}}\mathcal{C})$ are $\mathcal{Z}$-projective. Then:–*

(a) Suppose $\mathrm{Ext}^i_{\widetilde{A}}(\widetilde{M}, \widetilde{N}) = 0$ for $i = 1, 2$. For any commutative $\mathcal{Z}$-algebra $\mathcal{Z}'$, we have:–

$$(4.4.1) \qquad \mathrm{Hom}_{\widetilde{A}}(\widetilde{M}, \widetilde{N})_{\mathcal{Z}'} \cong \mathrm{Hom}_{\widetilde{A}_{\mathcal{Z}'}}(\widetilde{M}_{\mathcal{Z}'}, \widetilde{N}_{\mathcal{Z}'}).$$

(b) Let $n \geq 1$ and suppose that $\mathrm{Ext}^n_{\widetilde{A}}(\widetilde{M}, \widetilde{N}) \neq 0$. Then there exists $\mathfrak{p} \in \mathrm{Supp}(\mathrm{Ext}^n_{\widetilde{A}}(\widetilde{M}, \widetilde{N}))$ such that, if $k = \mathcal{Z}_{\mathfrak{p}}/\mathfrak{p}\mathcal{Z}_{\mathfrak{p}}$, then $\mathrm{Ext}^n_{\widetilde{A}_k}(\widetilde{M}_k, \widetilde{N}_k) \neq 0$.

[6] We point out that line 9 of [CPS2; p.58] contains a misprint. The last isomorphism should be $\mathbf{j}_*\mathbf{j}^*T \cong j_*j^*T$.

(c) Suppose for each field k which is a $\mathcal{Z}$-algebra, $\widetilde{A}_k$ is quasi-hereditary and that $\widetilde{M}_k \in {}_{\widetilde{A}_k}\mathcal{C}(\Delta)$, $\widetilde{N}_k \in {}_{\widetilde{A}_k}\mathcal{C}(\nabla)$. (In particular, this holds if $\widetilde{M}_k, \widetilde{N}_k \in {}_{\widetilde{A}_k}\mathcal{C}(\mathrm{tilt})$.) Then (4.4.1) holds for any commutative $\mathcal{Z}$-algebra $\mathcal{Z}'$.

Proof. (a) Let $\widetilde{P}_\bullet \to \widetilde{M} \to 0$ be a resolution of $\widetilde{M}$ by projective $\widetilde{A}$-modules $\widetilde{P}_i$. The hypotheses of (a) imply that the complex

$$\begin{aligned} 0 \to \mathrm{Hom}_{\widetilde{A}}(\widetilde{M}, \widetilde{N}) \to \mathrm{Hom}_{\widetilde{A}}(\widetilde{P}_0, \widetilde{N}) &\to \mathrm{Hom}_{\widetilde{A}}(\widetilde{P}_1, \widetilde{N}) \\ \to \mathrm{Hom}_{\widetilde{A}}(\widetilde{P}_2, \widetilde{N}) &\to \mathrm{Hom}_{\widetilde{A}}(\widetilde{P}_3, \widetilde{N}) \end{aligned}$$

is exact. Also, because the ring $\mathcal{Z} = \mathbb{Z}[q, q^{-1}]$ has global dimension 2, [DPS2; (0.1)] implies that the terms in the above complex are $\mathcal{Z}$-projective. (The result [DPS2; (0.1)] is an elementary linear algebra argument based on a commutative algebra result of Auslander-Goldman [AG; Cor., p. 17].) Thus, the kernel $\widetilde{X}$ of the map $\mathrm{Hom}_{\widetilde{A}}(\widetilde{P}_2, \widetilde{N}) \to \mathrm{Hom}_{\widetilde{A}}(\widetilde{P}_3, \widetilde{N})$ is also $\mathcal{Z}$-projective (again since $\mathcal{Z}$ has global dimension 2.) It follows that the acyclic complex

$$(4.4.2) \qquad 0 \to \mathrm{Hom}_{\widetilde{A}}(\widetilde{M}, \widetilde{N}) \to \mathrm{Hom}_{\widetilde{A}}(\widetilde{P}_0, \widetilde{N}) \to \mathrm{Hom}_{\widetilde{A}}(\widetilde{P}_1, \widetilde{N}) \to \widetilde{X} \to 0$$

splits as a complex of $\mathcal{Z}$-modules (in the sense that the kernels and cokernels of the various maps are $\mathcal{Z}$-direct summands). Hence, (4.4.2) remains acyclic after applying the functor $-\otimes_{\mathcal{Z}} \mathcal{Z}'$. Since $\widetilde{P}_0$ and $\widetilde{P}_1$ are $\widetilde{A}$-projective, $\mathrm{Hom}_{\widetilde{A}}(\widetilde{Q}, \widetilde{N})_{\mathcal{Z}'} \cong \mathrm{Hom}_{\widetilde{A}_{\mathcal{Z}'}}(\widetilde{Q}_{\mathcal{Z}'}, \widetilde{N}_{\mathcal{Z}'})$ for $\widetilde{Q} = \widetilde{P}_0, \widetilde{P}_1$. Thus, $\mathrm{Hom}_{\widetilde{A}}(\widetilde{M}, \widetilde{N})_{\mathcal{Z}'} \cong \mathrm{Hom}_{\widetilde{A}_{\mathcal{Z}'}}(\widetilde{M}_{\mathcal{Z}'}, \widetilde{N}_{\mathcal{Z}'})$ as claimed in (4.4.1). This completes the proof of (a).

When $n = 1$, (b) is proved in [DPS2; (2.9)]. Because $\widetilde{M}$ is $\mathcal{Z}$-projective, (b) follows by induction on n, using dimension shifting.

To see (c), (4.1) implies that $\mathrm{Ext}^n_{\widetilde{A}_k}(\widetilde{M}_k, \widetilde{N}_k) = 0$ for all $n > 0$. Therefore, $\mathrm{Ext}^n_{\widetilde{A}}(\widetilde{M}, \widetilde{N}) = 0$ for all $n > 0$ by (b). Now we can apply (a) to conclude that (c) holds. □

An alternate argument for (4.4a) can be based on the convergent homology spectral sequence

$$E^2_{pq} = \mathrm{Tor}^{\mathcal{Z}}_p(\mathrm{Ext}^q_{\widetilde{A}}(\widetilde{M}, \widetilde{N}), \mathcal{Z}') \Rightarrow \mathrm{Ext}^{q-p}_{\widetilde{A}_{\mathcal{Z}'}}(\widetilde{M}_{\mathcal{Z}'}, \widetilde{N}_{\mathcal{Z}'})$$

in which $d^2_{pq} : E^2_{pq} \to E^2_{p-2,q-1}$. Assuming this fact, (a) follows, since $\mathcal{Z}$ has global dimension 2 so that $\mathrm{Tor}^{\mathcal{Z}}_p = 0$ for $p > 2$. The above spectral sequence is given in [DS; (2.9)] in a somewhat different context, but also follows (after reindexing) from the Künneth spectral sequence given in [Wi; (5.6.4)]. (The boundedness assumption required in [Wi; (5.6.4)] can be achieved by truncating the complex P there, or by observing that it is not necessary since $\mathcal{Z}$ has finite global dimension.)

5. q-Schur algebras. For the rest of this paper, let $W = \mathfrak{S}_r$, the symmetric group of degree r, and $S = \{(1,2), \cdots, (r-1, r)\}$. For a positive integer n, let $\Lambda(n, r)$ (resp. $\Lambda^+(n, r)$) be the set of compositions (resp., partitions) of r with n (resp. at most n non-zero) parts. (A composition of r with n parts is a sequence $\lambda = (\lambda_1, \cdots, \lambda_n)$

of non-negative integers with $\sum_i \lambda_i = r$; λ is a partition if $\lambda_1 \geq \lambda_2 \geq \cdots$.) Let $\Lambda^+(r) = \Lambda^+(r,r)$. For any $n \geq r$, $\Lambda^+(n,r) = \Lambda^+(r)$ and $\Lambda(r) = \Lambda(r,r)$. The set $\Lambda(n,r)$ has a poset structure $\trianglelefteq$ defined by setting $\lambda \trianglelefteq \mu$ if and only if $\lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i$ for all i. Then $\Lambda^+(n,r)$ is a coideal in $\Lambda^+(r)$ for all n, r. If $n \leq r$, any $\lambda \in \Lambda(n,r)$ can be regarded as an element in $\Lambda(r)$ by adding a string $n \Leftrightarrow r$ 0's to λ. With this identification, $\Lambda(n,r)$ is a coideal in $\Lambda(r)$.

There is a natural map J from $\Lambda(n,r)$ to the power set $\mathcal{P}(S)$ of S: For $\lambda \in \Lambda(n,r)$, let $\mathcal{Y}(\lambda)$ be the Young diagram of shape λ and let $\mathbf{t}^\lambda$ be the tableau of shape λ obtained by filling in the boxes in the first row of $\mathcal{Y}(\lambda)$ consecutively with the integers $1, 2, \cdots, \lambda_1$, the second row with the integers $\lambda_1 + 1, \cdots, \lambda_1 + \lambda_2$, etc. Then we define $J(\lambda)$ as the subset of S consisting of those s which stabilize the integers in the rows of $\mathbf{t}^\lambda$. Every subset of S has the form $J(\lambda)$ for some $\lambda \in \Lambda(n,r)$ if $n \geq r$. In the set-up of the previous sections, we worked with a pair $(\,,\,\Lambda)$ in which Λ is a poset, together with a surjective map $J : \,,\, \to \mathcal{P}(S)$, and $\,,\, \subseteq \Lambda$. In the present section, we will put $\,,\, = \Lambda(n,r)$, with $\Lambda = \,,\,$ if $n \geq r$ and with $\Lambda = \Lambda(r)$ if $n < r$. Now, the parabolic subgroup W_λ of W is precisely the row stabilizer of $\mathbf{t}^\lambda$. By rearranging terms, every composition $\lambda \in \Lambda(n,r)$ determines a unique partition $\lambda^+ \in \Lambda^+(n,r)$. Clearly, $\lambda \trianglelefteq \lambda^+$ and λ^+ is the minimal partition with this property. Also, W_λ and W_μ are W-conjugate if and only if $\lambda^+ = \mu^+$.

The generic algebra $\widetilde{H}$ over $\mathcal{Z} = \mathbb{Z}[q, q^{-1}]$ for $W = \mathfrak{S}_r$ is defined in (1.1) with each $c_s = 1$: thus, $q_s = q$ for all $s \in S$. Abbreviate $\widetilde{T}(\Lambda(n,r))$ as $\widetilde{T}(n,r)$. Explicitly, $\widetilde{T}(n,r) = \bigoplus_{\lambda \in \Lambda(n,r)} \widetilde{T}_\lambda \cong \bigoplus_{\lambda \in \Lambda^+(n,r)} \widetilde{T}_\lambda^{\oplus d_\lambda}$, where $d_\lambda = \#\{\mu \,|\, \mu^+ = \lambda\}$. (By (1.4d), $\widetilde{T}_\lambda \cong \widetilde{T}_{\lambda^+}$.) For $\lambda \in \Lambda(r)$, let $\lambda' \in \Lambda^+(r)$ be the dual partition: thus, $\lambda'_i = \#\{\lambda_j \geq i\}$. We require the following purely combinatorial result, closely related to (1.4h).

(5.1) Lemma. *(*[DJ1; (4.3)]*) Suppose that $\mathcal{Z}'$ is a commutative $\mathcal{Z}$-algebra in which $q+1$ is not a zero divisor. Then:–*

(a) For $\lambda, \mu \in \Lambda(r)$, if $\mathrm{Hom}_{\widetilde{H}'}(\widetilde{T}^{\Phi\prime}_{\lambda'}, \widetilde{T}'_\mu) \neq 0$, then $\lambda \trianglerighteq \mu$.

(b) For $\lambda \in \Lambda(r)$, $\mathrm{Hom}_{\widetilde{H}'}(\widetilde{T}^{\Phi\prime}_{\lambda'}, \widetilde{T}'_\lambda)$ is a free $\mathcal{Z}'$-module of rank 1.

We will now make use of the Kazhdan-Lusztig cell theory for W. First, there is a poset isomorphism $\alpha\colon (\Lambda^+(r), \trianglelefteq) \xrightarrow{\sim} (\Xi, \leq_{LR}^{\mathrm{op}})$. Explicitly, $\alpha(\lambda)$ is the two-sided cell $\xi \in \Xi$ containing the longest word $w_{0,\lambda} \in W_\lambda$; see [DPS2; §2]. Two left cell modules $\widetilde{E}_\omega$ and $\widetilde{E}_{\omega'}$ are isomorphic if and only if ω, ω' are contained in the same two-sided cell $\xi \in \Xi$ (and this occurs if and only if $\widetilde{E}_{\omega\mathbb{Q}(q)} \cong \widetilde{E}_{\omega'\mathbb{Q}(q)}$ are isomorphic irreducible modules).[7]

Thus, for $\xi \in \Xi$, let $\widetilde{S}_\xi = \widetilde{S}_\omega$ for any left cell $\omega \subseteq \xi$. If $\alpha(\lambda) = \xi$, we also denote $\widetilde{S}_\xi$ by $\widetilde{S}_\lambda$. We follow a similar convention, relative to the modules $\widetilde{\Delta}(\omega, ,\,)$ in (3.6). Thus, for $\,,\, = \Lambda(n,r)$ and $\lambda \in \,,^+ = \Lambda^+(n,r)$, $\widetilde{\Delta}(\lambda, ,\,) = \widetilde{\Delta}(\omega, ,\,)$ if $\omega \subseteq \alpha(\lambda)$ The $\widetilde{S}_{\lambda\mathbb{Q}(q)}$, $\lambda \in \Lambda^+(r)$, are the distinct irreducible (right) modules for the (split) semisimple algebra $\widetilde{H}_{\mathbb{Q}(q)}$. We record the following fact.

---

[7]This fact is proved in [KL1; (1.4)] for the generic Hecke algebra $\widetilde{H}_0$ over $\mathbb{Z}[q^{1/2}, q^{-1/2}]$. The assertion for $\widetilde{H}$ then follows from general principles (see [DPS2; (2.3)]).

(5.2) Lemma. *For $\lambda \in \Lambda^+(r)$, $\widetilde{T}_{\lambda\mathbb{Q}(q)} \cong \widetilde{S}_{\lambda\mathbb{Q}(q)} \oplus \bigoplus_{\mu \rhd \lambda} \widetilde{S}_{\mu\mathbb{Q}(q)}^{\oplus e_{\mu\lambda}}$ and $\widetilde{T}^{\Phi}_{\lambda'\mathbb{Q}(q)} \cong \widetilde{S}_{\lambda\mathbb{Q}(q)} \oplus \bigoplus_{\mu \lhd \lambda} \widetilde{S}_{\mu\mathbb{Q}(q)}^{\oplus f_{\mu\lambda}}$ for some positive integers $e_{\mu\lambda}, f_{\mu\lambda}$.*

Proof. The analogous decomposition of permutation modules for $\mathbb{Q}\mathfrak{S}_r$ is well-known (going back to Frobenius at the character level), so that the lemma follows from elementary "equal characteristic" Brauer theory; see the discussion for [DPS2; (2.6)]. □

For any commutative $\mathcal{Z}$-algebra $\mathcal{Z}'$, we call

$$\widetilde{S}_q(n,r,\mathcal{Z}') = \mathrm{End}_{\widetilde{H}'}(\widetilde{T}(n,r)') \cong \mathrm{End}_{\widetilde{H}'}(\widetilde{T}(n,r)^{\Phi\prime}), \tag{5.3}$$

the q-Schur algebra of bidegree (n,r). By (2.2b), $\widetilde{S}_q(n,r,\mathcal{Z}') \cong \widetilde{S}_q(n,r,\mathcal{Z})_{\mathcal{Z}'}$. Also, $\widetilde{S}_q(n,r,\mathcal{Z}')$ is Morita equivalent to $\mathrm{End}_{\widetilde{H}}(\widetilde{T}(\Lambda^+(n,r))'$; cf. (1.4d). For convenience, write $\widetilde{S}_q(n,r)$ for $\widetilde{S}_q(n,r,\mathcal{Z})$.

(5.4) Lemma. *For $\lambda \in \Lambda^+(n,r)$, we have $\mathrm{Hom}_{\widetilde{S}_q(n,r)}(\widetilde{\Delta}(\lambda), \widetilde{T}) \cong \widetilde{S}_\lambda$, where $\widetilde{T} = \widetilde{T}(n,r)$ and $\widetilde{\Delta}(\lambda) = \mathrm{Hom}_{\widetilde{H}}(\widetilde{S}_\lambda, \widetilde{T})$.*

Proof. As in (2.6), denote either of the functors $\mathrm{Hom}_{\widetilde{S}_q(n,r)}(\Leftrightarrow, \widetilde{T})$ or $\mathrm{Hom}_{\widetilde{H}}(\Leftrightarrow, \widetilde{T})$ by $(\Leftrightarrow)^\diamond$. As described in §3, $\widetilde{T}_\lambda$ has a $\widetilde{S}$-filtration $\widetilde{F}_{\lambda\bullet}$ satisfying the vanishing condition (3.5) and having bottom section $\widetilde{S}_\lambda$. By (5.2), the other sections $\widetilde{S}_\mu$ satisfy $\mu \rhd \lambda$. Therefore, $\widetilde{F}^\diamond_{\lambda\bullet}$ is a $\widetilde{\Delta}$-filtration of the projective $\widetilde{S}_q(n,r)$-module $\widetilde{T}^\diamond_\lambda$ with top section $\widetilde{\Delta}(\lambda)$. Hence, $\widetilde{S}^{\diamond\diamond}_\lambda \cong \widetilde{\Delta}(\lambda)^\diamond$ is an $\widetilde{H}$-submodule of $\widetilde{T}^{\diamond\diamond}_\lambda \cong \widetilde{T}_\lambda$. By naturality, the "evaluation map" $\widetilde{S}_\lambda \to \widetilde{S}^{\diamond\diamond}_\lambda$ factors through the isomorphism $\widetilde{T}_\lambda \cong \widetilde{T}^{\diamond\diamond}_\lambda$, and defines an inclusion $\widetilde{S}_\lambda \hookrightarrow \widetilde{\Delta}(\lambda)^\diamond$. Since $\widetilde{T}_\lambda/\widetilde{S}_\lambda$ is $\mathcal{Z}$-torsion free and $\widetilde{\Delta}(\lambda)^\diamond_K \cong \widetilde{S}_{\lambda K}$, the desired result follows from (5.2). □

For $\lambda \in \Lambda^+(n,r)$, write $\widetilde{\Delta}(\lambda) = \mathrm{Hom}_{\widetilde{H}}(\widetilde{S}_\lambda, \widetilde{T})$, where $\widetilde{T} = \widetilde{T}(n,r)$ as above. Put $\Delta(\lambda) = \widetilde{\Delta}(\lambda)_k$. By [DPS2; Theorem 1], for every field k which is a $\mathcal{Z}$-algebra, the module category ${}_{\widetilde{S}_q(n,r)_k}\mathcal{C}$ is a HWC with poset $(\Lambda^+(n,r), \unlhd)$ and with standard objects $\Delta(\lambda)$, $\lambda \in \Lambda^+(n,r)$. (This result is established, in fact, using little more than the machinery discussed in §3.) Let $\widetilde{\nabla}(\lambda) = \mathfrak{d}_{\widetilde{S}_q(n,r)}\widetilde{\Delta}(\lambda)$ and $\nabla(\lambda) = \widetilde{\nabla}(\lambda)_k$. By [DJ4; (4.11)], for any field k which is a $\mathcal{Z}$-algebra, $\mathfrak{d}_{\widetilde{S}_q(n,r)_k}$ is a strong duality in the sense that $\mathfrak{d}_{\widetilde{S}_q(n,r)_k} L(\lambda) \cong L(\lambda)$ for any irreducible $\widetilde{S}_q(n,r)_k$-module $L(\lambda)$.[8] Hence, by [CPS2; (1.2)], $\nabla(\lambda)$ is the ∇-object corresponding to λ for the HWC ${}_{S_q(n,r)}\mathcal{C}$. Of course, ${}_{\widetilde{S}_q(n,r)}\mathcal{C}(\mathrm{tilt})$ denotes the class of $\widetilde{S}_q(n,r)$-modules with both a $\widetilde{\Delta}$ and a $\widetilde{\nabla}$-filtration. By (3.6), $\widetilde{T}(n,r) \in {}_{\widetilde{S}_q(n,r)}\mathcal{C}(\widetilde{\Delta})$. Also, (2.4a) says that $\mathfrak{d}_{\widetilde{S}_q(n,r)}\widetilde{T}(n,r) \cong \widetilde{T}(n,r)$. We conclude:

(5.5) Proposition. $\widetilde{T}(n,r) \in {}_{\widetilde{S}_q(n,r)}\mathcal{C}(\mathrm{tilt})$. *For k as above,*

$$T(n,r) \in {}_{S_q(n,r)}\mathcal{C}(\mathrm{tilt}).$$

[8] From the point of view of the present paper, this fact is easy to see directly—see (7.3) below.

Further, by (4.1) and (4.4), we have

$$(5.6)\qquad \begin{cases} (1)\ \operatorname{Ext}^m_{\widetilde{S}_q(n,r)}(\widetilde{\Delta}(\lambda), \widetilde{\nabla}(\mu)) \cong \delta_{m0}\delta_{\lambda\mu}\mathcal{Z}; \\ (2)\ m>0\ \&\ \operatorname{Ext}^m_{\widetilde{S}_q(n,r)}(\widetilde{\Delta}(\lambda), \widetilde{\Delta}(\mu)) \neq 0 \implies \lambda < \mu; \\ (3)\ m>0\ \&\ \operatorname{Ext}^m_{\widetilde{S}_q(n,r)}(\widetilde{\nabla}(\lambda), \widetilde{\nabla}(\mu)) \neq 0 \implies \lambda > \mu. \end{cases}$$

Now we can prove:–

(5.7) Theorem. *We have* $\operatorname{End}_{\widetilde{S}_q(n,r)}(\widetilde{T}(n,r))_{\mathcal{Z}'} \cong \operatorname{End}_{\widetilde{S}_q(n,r)_{\mathcal{Z}'}}(\widetilde{T}(n,r)_{\mathcal{Z}'}$ *for any commutative* $\mathcal{Z}$*-algebra* $\mathcal{Z}'$*. Also,* $\dim \operatorname{End}_{\widetilde{S}_q(n,r)_k}(\widetilde{T}(n,r)_k)$ *is constant on residue fields* k *of* $\mathcal{Z}$*.*

Proof. By (5.5), $\widetilde{T}(n,r)_k$ is a tilting module for $\widetilde{S}_q(n,r)_k$ for every field k which is a $\mathcal{Z}$-algebra. (In case $\widetilde{H}_k$ is semisimple, this assertion is obvious!) The first assertion of the theorem follows from (4.4c).

The long exact sequence of Ext , (5.5) and (5.6) imply that

$$\operatorname{Ext}^1_{\widetilde{S}_q(n,r)}(\widetilde{\Delta}(\lambda), \widetilde{T}(n,r)) = 0\ \forall \lambda \in \Lambda^+(n,r).$$

Therefore, a $\widetilde{\Delta}$-filtration of the left $\widetilde{S}_q(n,r)$-module $\widetilde{T}(n,r)$ induces, after applying the functor $\operatorname{Hom}_{\widetilde{S}_q(n,r)}(\Leftrightarrow, \widetilde{T}(n,r))$ and using (5.4), a $\widetilde{S}$-filtration of the right $\widetilde{H}$-module

$$\operatorname{End}_{\widetilde{S}_q(n,r)}(\widetilde{T}(n,r)) = \operatorname{Hom}_{\widetilde{S}_q(n,r)}(\widetilde{T}(n,r), \widetilde{T}(n,r)).$$

Since any $\widetilde{S}_\lambda$ is $\mathcal{Z}$-free by definition, $\operatorname{End}_{\widetilde{S}_q(n,r)}(\widetilde{T}(n,r))$ is also $\mathcal{Z}$-free with rank d, say. So,

$$\dim \operatorname{End}_{\widetilde{S}_q(n,r)_k}(\widetilde{T}(n,r)_k) \cong \dim \operatorname{End}_{\widetilde{S}_q(n,r)}(\widetilde{T}(n,r))_k = d$$

for any field k which is a $\mathcal{Z}$-algebra. □

6. Quantum Weyl reciprocity, I. In this section, we first obtain the q-version of the double centralizer property (1) of the introduction. We maintain the notation of the sections 3 and 5.

Let γ_λ be the right cell in W containing the longest word $w_{0,\lambda}$ of W_λ. For $x \in \gamma_\lambda$, let ω_x be the left cell containing x. By [KL1; (1.4)], the ω_x are distinct for distinct x; the union $\bigcup_{x\in\gamma_\lambda}\omega_x$ is the two-sided Kazhdan-Lusztig cell ξ_λ containing $w_{0,\lambda}$. Using the basis $\{D^+_w\}_{w\in W}$ introduced in (3.7), we give an alternative description of the *submodule* $\widetilde{S}_\lambda$ of $\widetilde{T}_\lambda$.

(6.1) Lemma. *Let* $\lambda \in \Lambda^+(r)$*. For any* $x \in \gamma_\lambda$*, form the* $\mathcal{Z}$*-submodule* $\widetilde{M}_x = \sum_{y\in\omega_x}\mathcal{Z}C^+_x D^+_{y^{-1}}$ *of* $\widetilde{T}_\lambda = x_\lambda \widetilde{H}$ *and put* $\widetilde{S}_\lambda = M_{w_{0,\lambda}}$*. Then* $\widetilde{M}_x = \widetilde{S}_\lambda$ *for all* $x \in \gamma_\lambda$*. The set* $\{C^+_x D^+_{y^{-1}}\}_{y\in\omega_x}$ *forms a* $\mathcal{Z}$*-basis for* $\widetilde{S}_\lambda$ *and is part of a basis for* $\widetilde{T}_\lambda$ *as well.*

Proof. For $x \in \gamma_\lambda$, $\lambda(=J(\lambda))$ is equal to the left-set $\{s \in S \mid sx < x\}$ of x (see [KL1; (2.4)] for an easy argument), so $\tau_s C^+_x = qC^+_x$ if $s \in \lambda$. Hence, $\widetilde{M}_x \subseteq \widetilde{T}_\lambda$

by (1.4b). By (3.7), $\widetilde{M}_x$ is an $\widetilde{H}$-submodule of $\widetilde{T}_\lambda$. Given $h \in \widetilde{H}$, the matrix of h as a right operator on the dual left cell module $\widetilde{E}^*_{\omega_x}$ relative to the basis $\{\zeta_y\}_{y\in\omega_x}$ dual to that defined by the C^+_y, $y \in \omega_x$, is $(\langle hC^+_w, D^+_{y^{-1}}\rangle)_{(y,w)\in\omega_x\times\omega_x}$. Since $\langle hC^+_w, D^+_{y^{-1}}\rangle = \langle C^+_w, D^+_{y^{-1}}h\rangle$, the map $f : \widetilde{E}^*_{\omega_x} \to \widetilde{M}_x$, $\zeta_y \mapsto C^+_x D^+_{y^{-1}}$, defines a surjective $\widetilde{H}$-module homomorphism.[9] Also, $1 = \langle C^+_x, D^+_{x^{-1}}\rangle = \mathrm{tr}(C^+_x D^+_{x^{-1}})$, so $\widetilde{M}_x \neq 0$. But $\widetilde{E}^*_{\omega_x\mathbb{Q}(q)}$ is an irreducible $\widetilde{H}_{\mathbb{Q}(q)}$-module (see [DPS2; (2.3)]), so f must be an isomorphism onto its image. For any $x \in \gamma_\lambda$, $\widetilde{E}^*_{\omega_x} \cong \widetilde{S}_\lambda$.

By (5.2), the image $\mathrm{Im}\, f$ of f, viewed as a subspace of $\widetilde{T}_\lambda$, is contained in $\widetilde{S}_\lambda$. (Recall that $\widetilde{S}_\lambda$ identifies with the bottom section in a $\widetilde{S}$-filtration of $\widetilde{T}_\lambda$.) Since $\mathrm{End}_{\widetilde{H}}(\widetilde{S}_\lambda) \cong \mathcal{Z}$, $\mathrm{Im} f = r\widetilde{S}_\lambda$ for some $0 \neq r \in \mathcal{Z}$. Thus, $\mathcal{Z} = \mathrm{tr}(\widetilde{M}_x) = \mathrm{tr}(r\widetilde{S}_\lambda) \subseteq r\mathcal{Z}$, so r is a unit in $\mathcal{Z}$, and we can take $r = 1$. Finally, using the dual left cell filtration [DPS1; (2.3.7)], we see that $\widetilde{T}_\lambda/\widetilde{S}_\lambda$ is $\mathcal{Z}$-free. Therefore, every basis for $\widetilde{S}_\lambda$ is part of a basis for $\widetilde{T}_\lambda$, proving the last assertion. □

Let n be some positive integer. Since $\Lambda^+(n,r)$ is a coideal in $\Lambda^+(r)$, we can use the isomorphism $\alpha : (\Lambda^+(r), \trianglelefteq) \to (\Xi, \leq^{\mathrm{op}}_{LR})$ discussed defined in [DPS2] and discussed after (5.1) above to fix a listing $\xi_1, \cdots, \xi_m$ of Ξ satisfying $\xi_i \leq_{LR} \xi_j \implies i \leq j$ and such that $\xi_1 \bigcup \cdots \bigcup \xi_{m_0} = \alpha(\Lambda^+(n,r))$ for some integer m_0. Denote the ideal $\alpha(\Lambda^+(n,r))$ of Ξ by $\Xi(n,r)$.

Let $\widetilde{J}(n,r)$ be the $\mathcal{Z}$-span of the D^+_w, $w \notin \bigcup_{\xi\in\Xi(n,r)} \xi$. By (3.7b), $\widetilde{J}(n,r)$ is an ideal in $\widetilde{H}$; let $\widetilde{H}(n,r) = \widetilde{H}/\widetilde{J}(n,r)$.

(6.2) Theorem. *The natural map* $\psi : \widetilde{H}^{\mathrm{op}} \to \mathrm{End}_{\widetilde{S}_q(n,r)}(\widetilde{T}(n,r))$ *induces an isomorphism*

$$\widetilde{H}(n,r)^{\mathrm{op}} \cong \mathrm{End}_{\widetilde{S}_q(n,r)}(\widetilde{T}(n,r)).$$

(In particular, ψ is surjective.) Furthermore, if $\mathcal{Z}'$ is a commutative $\mathcal{Z}$-algebra, then

$$\psi_{\mathcal{Z}'} : \widetilde{H}(n,r)^{\mathrm{op}}_{\mathcal{Z}'} \to \mathrm{End}_{\widetilde{S}_q(n,r)_{\mathcal{Z}'}}(\widetilde{T}(n,r)_{\mathcal{Z}'})$$

is an isomorphism.

Proof. For $\lambda \in \Lambda^+(r)$, let (as before) $w_{0,\lambda}$ be the longest word in W_λ. As defined in [DPS2], $\alpha(\lambda)$ is the two-sided cell $\xi_\lambda \in \Xi$ containing $w_{0,\lambda}$. From the definition of the Kazhdan-Lusztig basis $\{C'_x\}$ for $\widetilde{H}_{\mathbb{Z}[q^{1/2},q^{-1/2}]}$ given in [KL1; ((1.1c)], $C^+_{w_{0,\lambda}} = q_w^{1/2} C'_{w_{0,\lambda}} = \sum_{y\leq w_{0,\lambda}} P_{y,w_{0,\lambda}} \tau_y$, where $P_{y,w_{0,\lambda}}$ is the Kazhdan-Lusztig polynomial for the pair $(y, w_{0,\lambda})$. By the standard formula [KL1; (2.3g)], we have that $P_{y,w_{0,\lambda}} = 1$ for all $y \leq w_{0,\lambda}$. Thus, $C^+_{w_{0,\lambda}} = x_\lambda$. It follows that any element in $\widetilde{T}_\lambda = x_\lambda \widetilde{H}$ is a $\mathcal{Z}$-linear combination of C^+_x satisfying $x \leq_R w_{0,\lambda}$. Therefore, if $y \in W$ does not lie in any $\xi \in \Xi(n,r)$ and $\lambda \in \Lambda^+(n,r)$, then (3.7a) imply that $\widetilde{T}_\lambda D^+_{y^{-1}} = 0$. By (1.4d), we conclude that $\psi(\widetilde{J}(n,r)) = 0$, yielding a homomorphism $\bar{\psi} : \widetilde{H}(n,r) \to \mathrm{End}_{\widetilde{S}_q(n,r)}(\widetilde{T}(n,r))$.

[9] Alternatively, the map f is obtained naturally from $\widetilde{H}^* \xrightarrow{\sim} \widetilde{H} \to C^+_x \widetilde{H}$, checking with (3.7) that the later is well-defined on the section of $\widetilde{H}^*$ defined by $\widetilde{S}_\lambda$.

Let $\mathcal{Z}'$ be a commutative $\mathcal{Z}$-algebra. The D_w^+, $w \in \xi$ for $\xi \in \Xi(n,r)$, define a $\mathcal{Z}'$-basis for $\widetilde{H}(n,r)_{\mathcal{Z}'}$. Suppose that $f = \sum_{w\in\xi,\xi\in\Xi(n,r)} a_w D_w^+$ acts as the zero operator on $\widetilde{T}(n,r)_{\mathcal{Z}'}$ with some $a_w \neq 0$. Choose a left cell $\omega \in \Omega$ minimal (w.r.t. $\leq_L$) for which $a_{w^{-1}} \neq 0$ with $w \in \omega$. For some $\lambda \in \Lambda^+(n,r)$, $\omega \subseteq \xi = \alpha(\lambda)$. There exists $x \in \gamma_\lambda$ so that $\omega = \omega_x$. Since x and $w_{0,\lambda}$ have the same left-set, we conclude that $C_x^+ \in \widetilde{T}'_\lambda$. (See the remarks above (3.2).) By (3.7a) and the minimality of ω, $0 = C_x^+ f = \sum_{y\in\omega_x} a_{y^{-1}} C_x^+ D_{y^{-1}}^+$. Since $\{C_x^+ D_{y^{-1}}^+\}_{y\in\omega_x}$ is linearly independent *over* $\mathcal{Z}'$ by (6.1), it follows that $a_{w^{-1}} = 0$, a contradiction. Hence, $\bar{\psi}_{\mathcal{Z}'}$ is injective.

Taking $\mathcal{Z}' = \mathbb{Q}(q)$, the semisimplicity of $\widetilde{H}_{\mathbb{Q}(q)}$ implies that $\widetilde{H}(n,r)$ has rank equal to dim $\mathrm{End}_{\widetilde{S}_q(n,r)_{\mathbb{Q}(q)}}(\widetilde{T}(n,r)_{\mathbb{Q}(q)})$. (Note that $\widetilde{H}(n,r)$ is $\mathcal{Z}$-free.) Hence, by (5.7) and the previous paragraph, $\bar{\psi}_k$ is an isomorphism for every field k which is a $\mathcal{Z}$-algebra. If $\widetilde{M} \to \widetilde{N}$ is a morphism of finitely generated $\mathcal{Z}$-modules which becomes an isomorphism upon passage to every residue field k of $\mathcal{Z}$, then an elementary commutative algebra argument establishes that $\widetilde{M} \to \widetilde{N}$ is an isomorphism of $\mathcal{Z}$-modules. Thus, $\psi : \widetilde{H}(n,r)^{\mathrm{op}} \to \mathrm{End}_{\widetilde{S}_q(n,r)}(\widetilde{T}(n,r))$ is an isomorphism, proving the first assertion. The second assertion follows from this and (5.7). □

The above result is the q-version of a theorem of De Concini–Procesi [dCP; (4.1)] for Schur algebras. (Another proof in that case is given in [D2; §2 Cor.].) As an application, we establish a double centralizer property (1), independent of the field k and the parameter q. Let $\widetilde{V}$ be a free $\mathcal{Z}$-module of rank n. For any positive integer r, there is a natural right action of $\widetilde{H}$ on $\widetilde{V}^{\otimes r}$ so that $\widetilde{V}^{\otimes r} \cong \widetilde{T}(n,r)$ [DD; (3.1.5)]. Let $\widetilde{U}_{q^{1/2}} = \widetilde{U}_{q^{1/2}}(\mathfrak{gl}_n)$ be the divided power $\mathcal{Z}$-form of the quantized enveloping algebra of $\mathfrak{gl}_n$. We have two natural algebra homomorphisms $\varphi : \widetilde{U}_{q^{1/2}} \to \mathrm{End}_{\widetilde{H}}(\widetilde{V}^{\otimes r})$ and $\psi : \widetilde{H}^{\mathrm{op}} \to \mathrm{End}_{\widetilde{U}_{q^{1/2}}}(\widetilde{V}^{\otimes r})$. (The map φ is defined in [Du], based on [BLM].)

(6.3) Theorem. *Both maps φ and ψ are surjective. Therefore, for any specialization of $\mathcal{Z}$ into a field k, we have*

$$\mathrm{im}(\varphi_k) = \mathrm{End}_{\widetilde{H}_k}(\widetilde{V}_k^{\otimes r}) \quad \text{and} \quad \mathrm{im}(\psi_k) = \mathrm{End}_{\widetilde{U}_{qk}}(\widetilde{V}_k^{\otimes r}).$$

Proof. The surjectivity of φ is proved in [Du; (3.4)] based on work of [BLM] over $\mathbb{Q}(q)$. With this, (6.2) and the discussion above on $\widetilde{S}_q(n,r)$ imply the surjectivity of ψ. The last assertion follows by base change. □

When $k = \mathbb{C}$ and $q \in \mathbb{C}$ is *not* a root of unity, we recover the quantized Weyl reciprocity established in [Ji]. See [Du; (1.2)]. When $k = \mathbb{C}$ and $q \in \mathbb{C}$ is a root of unity, then the fact that $\widetilde{H}(2,r)_{\mathbb{C}} \cong \mathrm{End}_{\widetilde{S}_q(2,r)_{\mathbb{C}}}(\widetilde{V}^{\otimes r})$ has been proved by Martin [M; §4] by different methods, while for general n, the kernel of $\psi_{\mathbb{C}}$ was explicitly described in [M; §3] in terms of Young symmetrizers. We emphasize, however, that results (6.2) and (6.3) are much stronger. We have proved that (6.2) holds over the ring $\mathcal{Z} = \mathbb{Z}[q, q^{-1}]$, and it behaves well under base change. The proof for the latter as well as the surjectivity of ψ requires the tilting module theory for quasi-hereditary algebras.

We next consider a filtration version of (2) in the introduction. For this, we require the $*$-operations on the elements of W. These operations are defined in [KL1; (4.1)] which gives all the needed properties.[10] The following result is implicit in [KL1; §§4,5]; an explicit proof is presented in [DPS2; (2.3)].

(6.4) Lemma. *(a) Let ω (resp., γ) be a left (resp., right) cell contained in the two-sided cell $\xi \in \Xi$. Then $\omega^* = \{w^* \mid w \in \omega\}$ (resp., ${}^*\gamma = \{{}^*w \mid w \in \gamma\}$) is a left (resp., right) cell in ξ. Every left (resp., right) cell in ξ can be obtained from ω (resp., γ) by applying a sequence of $*$-operations.*

(b) Let $w \in W$ and $h \in \widetilde{H}$. Suppose

$$hC_w^+ \equiv \sum_{x \sim_L w} \alpha_x(h,w)C_x^+ \mod \sum_{z <_L w} \mathcal{Z}C_z^+$$
$$C_w^+ h \equiv \sum_{y \sim_R w} \beta_y(h,w)C_y^+ \mod \sum_{z <_R w} \mathcal{Z}C_z^+.$$

Then

$$\alpha_{x^*}(h,w^*) = q_{w^*}^{1/2} q_w^{-1/2} q_{x^*}^{-1/2} q_x^{1/2} \alpha_x(h,w)$$
$$\beta_{{}^*y}(h,{}^*w) = q_{{}^*w}^{1/2} q_w^{-1/2} q_{{}^*x}^{-1/2} q_x^{1/2} \beta_y(h,w).$$

(c) For any left cell $\omega \in \Omega$,

$$C_w^+ + \sum_{z <_L w} \mathcal{Z}C_z^+ \mapsto q_w^{1/2} q_{w^*}^{-1/2} q^{1/2} C_{w^*}^+ + \sum_{z <_L w^*} \mathcal{Z}C_z^+, \ w \in \omega$$

defines an isomorphism $\widetilde{E}_\omega \xrightarrow{\sim} \widetilde{E}_{\omega^*}$ of left cell modules. Similarly, for any right cell γ,

$$C_w^+ + \sum_{z <_R w} \mathcal{Z}C_z^+ \mapsto q_w^{1/2} q_{{}^*w}^{-1/2} q^{1/2} C_{{}^*w}^+ + \sum_{z <_R {}^*w} \mathcal{Z}C_z^+, \ w \in \gamma$$

defines an isomorphism $\widetilde{K}_\gamma \xrightarrow{\sim} \widetilde{K}_{{}^*\gamma}$ of right cell modules.

For $\lambda \in \Lambda(r)$, let $\mathcal{D}_\lambda^+ = \{w \in W \mid sw < w, \forall s \in \lambda\}$. By [DPS1; (2.3.5)], $\widetilde{T}_\lambda$ has basis $\{C_w^+\}_{w \in \mathcal{D}_\lambda^+}$. Putting $C_w^\lambda = C_w^+$, we obtain a basis $\{C_w^\lambda\}_{\lambda \in \Lambda(n,r), w \in \mathcal{D}_\lambda^+}$ for $\widetilde{T}(n,r)$. As above, let $\omega_\lambda \in \Omega$ contain $w_{0,\lambda}$. Define

$$(6.5) \qquad \begin{cases} \widetilde{N}_\lambda = \mathrm{span}\{C_w^\mu \mid \mu \in \Lambda(n,r), w \in \mathcal{D}_\mu^+, w \leq_L w_{0,\lambda}\} \\ \widetilde{N}_\lambda^- = \mathrm{span}\{C_w^\mu \mid \mu \in \Lambda(n,r), w \in \mathcal{D}_\mu^+, w <_L w_{0,\lambda}\}. \end{cases}$$

By (2.2c), both $\widetilde{N}_\lambda$, $\widetilde{N}_\lambda^-$ are $\widetilde{S}_q(n,r)$-submodules of $\widetilde{T}(n,r)$.

(6.6) Lemma.[11] *Let $\lambda \in \Lambda^+(n,r)$. In (6.5), both $\widetilde{N}_\lambda$ and $\widetilde{N}_\lambda^-$ are $\widetilde{S}_q(n,r)$-submodules of $\widetilde{T}(n,r)$. In fact, $\widetilde{N}_\lambda/\widetilde{N}_\lambda^- \cong \widetilde{\Delta}(\lambda) = \mathrm{Hom}_{\widetilde{H}}(\widetilde{S}_\lambda, \widetilde{T}(n,r))$.*

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[10] Given $w \in W$, the definition of w^* depends on a choice of elements $s,t \in S$ such that st has order 3. It will not be necessary to explicitly mention s,t. Also, w^* is defined only if $\#\mathcal{R}(w) \cap \{s,t\} = 1$ where $\mathcal{R}(w) = \{s \in S \mid ws < w\}$. Similar remarks hold for ${}^*w$.

[11] It is clear that (6.5) and this lemma are valid in the context of $\widetilde{\Delta}(\omega_\lambda,,)$ in (3.6) for any Coxeter system.

Proof. Using (the $i = 1$ case of) (3.5), we see there is an exact sequence $0 \to \mathrm{Hom}_{\widetilde{H}}(\widetilde{T}_\lambda/\widetilde{S}_\lambda, \widetilde{T}(n,r)) \to \mathrm{Hom}_{\widetilde{H}}(\widetilde{T}_\lambda, \widetilde{T}(n,r)) \xrightarrow{\pi} \widetilde{\Delta}(\lambda) \to 0$ of $\widetilde{S}_q(n,r)$-modules. Observe that $x \in W$ has right-set $\mathcal{R}(x) = \{s \in S \,|\, xs < x\} \supseteq \lambda$ if and only if $x \leq_L w_{0,\lambda}$; see remarks above [DPS1; (2.3.6)]. Thus, any homomorphism $\widetilde{T}_\lambda \to \widetilde{T}_\mu$ is determined by the image of x_λ, which can be an arbitrary $\mathcal{Z}$-linear combination of terms C_w^+ for $w \in \mathcal{D}_\mu^+$ and $w \leq_L w_{0,\lambda}$. Hence, $\mathrm{Hom}_{\widetilde{H}}(\widetilde{T}_\lambda, \widetilde{T}(n,r)) \cong \widetilde{N}_\lambda$ as $\widetilde{S}_q(n,r)$-submodules of $\widetilde{T}(n,r)$. For any $\mu \in \Lambda(n,r)$, $\mathrm{Hom}_{\widetilde{H}}(\widetilde{T}_\lambda, \widetilde{T}_\mu) \cong \mathrm{Hom}_{\widetilde{H}_\mu}(x_\mu, \widetilde{H}x_\lambda)$. Those morphisms $\widetilde{T}_\lambda \to \widetilde{T}_\mu$ vanishing on $\widetilde{S}_\lambda$ identify with those maps $x_\mu \to \widetilde{H}x_\lambda$ with image in $\mathrm{span}\{C_y^+ \,|\, y \in \mathcal{D}_\mu^+, y <_L w_{0,\lambda}\}$. Thus, $\mathrm{Ker}\,\pi = \widetilde{N}_\lambda^-$. □

Let $\widetilde{R}^e = \widetilde{S}_q(n,r) \otimes \widetilde{H}(n,r)^{\mathrm{op}}$ and form the $\widetilde{R}^e$-module $\widetilde{\Upsilon}^e(\lambda) = \widetilde{\Delta}(\lambda) \otimes \mathfrak{d}_{\widetilde{H}}\widetilde{S}_\lambda$. We will study the $\widetilde{R}^e$-structure of $\widetilde{T}(n,r)$.

If $\mathbf{t}$ is a tableau, let $\lambda_{\mathbf{t}}$ be its shape. Let $\mathfrak{T}$ be the set of pairs $(\mathbf{r},\mathbf{s})$ of standard tableau such that $\lambda_{\mathbf{r}} = \lambda_{\mathbf{s}} \in \Lambda^+(r)$. Recall that $\mathbf{t}$ is standard if its entries, here chosen from $1, 2, \cdots, r$ (without repetition), increase strictly along rows and columns. We will use the Robinson-Schensted map $\mathfrak{r} : W \overset{\sim}{\Leftrightarrow} \mathfrak{T}$, $w \mapsto (\mathbf{r}(w), \mathbf{s}(w))$. This map is defined explicitly in [S; pp. 25–27]. (See also the recent book [F].)

By [BV; pp.171-2],

$$x \sim_L y \quad (\text{resp. } x \sim_R y) \Leftrightarrow \mathbf{s}(x) = \mathbf{s}(y) \quad (\text{resp. } \mathbf{r}(x) = \mathbf{r}(y)). \tag{6.7}$$

In the proof below, it will be useful to keep in mind the trivial fact that $\mathbf{r}(w^*) = \mathbf{r}(w)$ and $\mathbf{s}({}^*w) = \mathbf{s}(w)$, whenever w^* or ${}^*w$ are defined. For convenience, write $C^\lambda_{(\mathbf{r},\mathbf{s})}$ for $C^\lambda_{\mathfrak{r}^{-1}(\mathbf{r},\mathbf{s})}$. Also, let $\ell(\mathbf{r},\mathbf{s}) = \ell(\mathfrak{r}^{-1}(\mathbf{r},\mathbf{s}))$ and $q_{(\mathbf{r},\mathbf{s})} = q_{\mathfrak{r}^{-1}(\mathbf{r},\mathbf{s})}$.

(6.8) Theorem. *List $\Lambda^+(n,r)$ as $\nu_1, \cdots, \nu_m$, where $\nu_i \trianglerighteq \nu_j \implies i \leq j$. The $\widetilde{R}^e$-module $\widetilde{T}(n,r)$ has a $\widetilde{\Upsilon}^e$-filtration $0 = \widetilde{T}_0 \subset \widetilde{T}_1 \subset \cdots \subset \widetilde{T}_m = \widetilde{T}(n,r)$ in which $\widetilde{T}_i/\widetilde{T}_{i-1} \cong \widetilde{\Upsilon}^e(\nu_i) = \widetilde{\Delta}(\nu_i) \otimes \mathfrak{d}_{\widetilde{H}}\widetilde{S}_{\nu_i}$, for $i = 1, \cdots, m$.*

Proof. Let γ_i be the right cell containing w_{0,ν_i}, and consider the right cell module $\widetilde{K}_{\gamma_i} \cong \mathfrak{d}_{\widetilde{H}}\widetilde{S}_{\nu_i}$; see (3.3), (3.4). Define $\widetilde{T}_i = \mathrm{span}\{C_w^\lambda \,|\, \lambda \in \Lambda(n,r), w \in \mathcal{D}_\lambda^+, w \sim_L x \in \gamma_1 \cup \cdots \cup \gamma_i\}$. Since $\leq_{LR}^{\mathrm{op}} = \trianglelefteq$, (2.2c) and (3.1) imply that $\widetilde{T}_i$ is an $(\widetilde{S}_q(n,r), \widetilde{H})$-sub-bimodule of $\widetilde{T}$. The section $\widetilde{T}_i/\widetilde{T}_{i-1}$ has basis consisting of cosets $C_w^\lambda + \widetilde{T}_{i-1}$ for $w \sim_L x \sim_R w_{0,\nu_i}$, $w \in \mathcal{D}_\lambda^+$, and $\lambda \in \Lambda(n,r)$. By (6.7), these basis vectors can be described as $C^\lambda_{(\mathbf{r},\mathbf{s})} + \widetilde{T}_{i-1}$, letting $(\mathbf{r},\mathbf{s})$ run over all pairs of standard tableau of shape ν_i which satisfy $\mathfrak{r}^{-1}(\mathbf{r},\mathbf{s}) \in \mathcal{D}_\lambda^+$. Let $\mathbf{t}_i = \mathbf{r}(w_{0,\nu_i}) = \mathbf{s}(w_{0,\nu_i})$. Then $\widetilde{\Delta}(\nu_i)$ has a basis consisting of those $\bar{C}^\lambda_{(\mathbf{r},\mathbf{t}_i)} = C^\lambda_{(\mathbf{r},\mathbf{t}_i)} + \widetilde{N}^-_{\nu_i}$, where $\mathfrak{r}^{-1}(\mathbf{r},\mathbf{t}_i) \in \mathcal{D}_\lambda^+$ for $\lambda \in \Lambda(n,r)$. Also, the right cell module $\widetilde{K}_{\gamma_i}$ has a basis consisting of the $\bar{C}^+_{(\mathbf{t}_i,\mathbf{s})} = C^+_{(\mathbf{t}_i,\mathbf{s})} + \sum_{x <_R w_{0,\nu_i}} \mathcal{Z}C_x^+$.

For any w, the elements w^* and w have opposite parity. Hence, for a fixed i, (6.4a) implies that $\ell(\mathbf{r},\mathbf{t}_i) \Leftrightarrow \ell(\mathbf{r},\mathbf{s}) + \ell(\mathbf{t}_i,\mathbf{s}) \equiv \ell(\mathbf{t}_i,\mathbf{t}_i) \bmod 2$. So, there exists $\epsilon_i \in \{0,1\}$ such that $\ell(\mathbf{r},\mathbf{t}_i) \Leftrightarrow \ell(\mathbf{r},\mathbf{s}) + \ell(\mathbf{t}_i,\mathbf{s}) + \epsilon_i \equiv 0 \bmod 2$, for all standard tableau $\mathbf{r},\mathbf{s}$ of shape ν_i. The map $\widetilde{\Upsilon}^e(\nu_i) \xrightarrow{f} \widetilde{T}_i/\widetilde{T}_{i-1}$ defined by

$$\bar{C}^\lambda_{(\mathbf{r},\mathbf{t}_i)} \otimes \bar{C}^+_{(\mathbf{t}_i,\mathbf{s})} \mapsto q^{1/2}_{(\mathbf{r},\mathbf{t}_i)} q^{-1/2}_{(\mathbf{r},\mathbf{s})} q^{1/2}_{(\mathbf{t}_i,\mathbf{s})} q^{\epsilon_i/2} C^\lambda_{(\mathbf{r},\mathbf{s})} + \widetilde{T}_{i-1}$$

is a linear isomorphism. Using (6.4b), we see that f is an $(\widetilde{S}_q(n,r),\widetilde{H})$-bimodule map. $\square$

(6.9) Remarks. (a) The duality functors $\mathfrak{d}_{\widetilde{H}}$ and $\mathfrak{d}_{\widetilde{S}_q(n,r)}$ defined in (1.2) and (2.4) formally induce a duality functor $\mathfrak{d}_{\widetilde{R}^e}$ on the category of $\widetilde{R}^e$-modules; see [CPS2; (1.2.2c)]. Clearly, $\mathfrak{d}^e\widetilde{T}(n,r)\cong\widetilde{T}(n,r)$. Hence, by (6.8), $\widetilde{T}(n,r)$ also has a filtration with sections $\mathfrak{d}^e\widetilde{\Upsilon}^e(\lambda)\cong\widetilde{\nabla}(\lambda)\otimes\widetilde{S}_\lambda$, $\lambda\in\Lambda^+(n,r)$.

(b) For any field k which is a $\mathcal{Z}$-algebra, (6.8) implies that $\widetilde{T}(n,r)_k$ has a $R^e=S_q(n,r)\otimes H(n,r)^{\rm op}$-filtration with sections $\Delta(\lambda)\otimes\mathfrak{d}_HS_\lambda$, $\lambda\in\Lambda^+(n,r)$.

(c) The $\widetilde{\Delta}(\lambda)_{\mathbb{Q}(q)}$, $\lambda\in\Lambda^+(n,r)$, are the distinct irreducible modules for the semisimple algebra $\widetilde{S}_q(n,r)_{\mathbb{Q}(q)}$. Also, $\mathfrak{d}_{\widetilde{H}_{\mathbb{Q}(q)}}\widetilde{S}_{\lambda\mathbb{Q}(q)}\cong\widetilde{S}_{\lambda\mathbb{Q}(q)}$. Hence, (6.8) agrees with the decomposition (2) in the introduction.

7. Young modules and tilting q-Schur algebras. Maintain the notation of the previous two sections. We first give a direct development of the basic theory of Young modules. These modules were first developed for Hecke algebras of type A by Dipper and James in [DJ2]. For example, they prove (7.4c) and (part of) (7.6c). However, using the results of the present paper, we are able to establish these results (and more) very quickly. Then we apply this work to describe explicitly the partial tilting modules $X(\lambda)$ for the q-Schur algebras $S_q(n,r)$ and determine the Ringel dual of $S_q(n,r)$. Since this paper was written, Donkin has given us a copy of his paper [D4] which also calculates the Ringel dual of $S_q(n,r)$ in the special case $n\geq r$.

For *the rest of this section*, fix a field k which is a $\mathcal{Z}$-algebra. To avoid trivialities, we concentrate on the case in which the algebra $\widetilde{H}_k$ is *not* semisimple—hence, we assume $1+q+\cdots+q^{r-1}=0$ in k (see, e. g., [DPS1; (4.2.2)]). There exists a $\mathcal{Z}$-algebra $\mathcal{O}$ with the following properties two: (i) $\mathcal{O}$ is a discrete valuation ring with residue field k and fraction field K; (ii) the image of q in K is not a root of unity. Thus, $\widetilde{H}_K$ is a split semisimple algebra. Note that $q+1\neq 0$ in the domain $\mathcal{O}$.

Denote $\widetilde{S}_q(n,r)_k$ (resp., $\widetilde{H}_k$, $\widetilde{T}(n,r)_k$, $\widetilde{T}_{\lambda k}$, etc.) by $S_q(n,r)$ (resp., H, $T(n,r)$, T_λ, etc.). The Krull-Schmidt theorem holds for finitely generated $\widetilde{H}_{\mathcal{O}}$-modules. Also, if $\widetilde{T}(n,r)_{\mathcal{O}}=\bigoplus_i\widetilde{Y}_i^{\oplus n_i}$ is a decomposition of $\widetilde{T}(n,r)$ into distinct, indecomposable $\widetilde{H}_{\mathcal{O}}$-summands $\widetilde{Y}_i$, then $T(n,r)=\bigoplus_iY_i^{\oplus n_i}$ is a decomposition of $T(n,r)$ into distinct indecomposable H-summands $Y_i=\widetilde{Y}_{ik}$. (Use "Heller's theorem", see. e. g., [CPS2; (1.5.6)], and (2.2b).)

By (5.1b), $\mathrm{Hom}_{\widetilde{H}'}(\widetilde{T}^{\Phi\prime}_{\lambda'},\widetilde{T}'_\lambda)\cong\mathcal{Z}'$ if $\lambda\in\Lambda(r)$ and $q+1$ is not a zero divisor in the commutative $\mathcal{Z}$-algebra $\mathcal{Z}'$. Thus, taking $\mathcal{Z}'=\mathcal{O}$, for $\lambda\in\Lambda^+(r)$, there exist unique indecomposable $\widetilde{H}_{\mathcal{O}}$-summands $\widetilde{Y}^\natural_\lambda$ of $\widetilde{T}^\Phi_{\lambda'\mathcal{O}}$ and $\widetilde{Y}_\lambda$ of $\widetilde{T}_{\lambda\mathcal{O}}$ such that

$$\mathrm{Hom}_{\widetilde{H}_{\mathcal{O}}}(\widetilde{Y}^\natural_\lambda,\widetilde{Y}_\lambda)\cong\mathrm{Hom}_{\widetilde{H}_{\mathcal{O}}}(\widetilde{T}^\Phi_{\lambda'\mathcal{O}},\widetilde{T}_{\lambda,\mathcal{O}})\cong\mathcal{O}. \tag{7.1}$$

Then $\widetilde{Y}_\lambda$ (resp., $\widetilde{Y}^\natural_\lambda$) is the *Young module* (resp., *twisted Young module*) associated to λ. Let $Y^\natural_\lambda=\widetilde{Y}^\natural_{\lambda k}$ and $Y_\lambda=\widetilde{Y}_{\lambda k}$.

(7.2) Lemma. *(a) For $\lambda \in \Lambda^+(r)$, we have $\widetilde{Y}_{\lambda K} \cong \widetilde{S}_{\lambda K} \oplus \bigoplus_{\mu \rhd \lambda} \widetilde{S}_{\mu K}^{\oplus n_{\mu\lambda}}$ and $\widetilde{Y}^\natural_{\lambda K} \cong \widetilde{S}_{\lambda K} \oplus \bigoplus_{\mu \lhd \lambda} \widetilde{S}_{\mu K}^{\oplus m_{\mu\lambda}}$.*

(b) The $\widetilde{Y}_\lambda$, $\lambda \in \Lambda^+(n,r)$, are the distinct indecomposable $\widetilde{H}_{\mathcal{O}}$-summands of $\widetilde{T}(n,r)_{\mathcal{O}}$.

(c) For $\lambda \in \Lambda^+(r)$, let $\widetilde{P}(\lambda) = \mathrm{Hom}_{\widetilde{H}_{\mathcal{O}}}(\widetilde{Y}_\lambda, \widetilde{T}_{\mathcal{O}})$. Each $\widetilde{P}(\lambda)$ is a projective indecomposable $\widetilde{S}_q(r,r)_{\mathcal{O}}$-module. The modules $P(\lambda) = \widetilde{P}(\lambda)_k \cong \mathrm{Hom}_H(Y_\lambda, T)$, $\lambda \in \Lambda^+(r)$, are the projective indecomposable $S_q(r,r)$-modules.

Proof. Because $\widetilde{H}_K$ is a split semisimple algebra, the irreducible $\widetilde{H}_{\mathbb{Q}(q)}$ and $\widetilde{H}_K$-modules correspond bijectively $\widetilde{S}_{\lambda\mathbb{Q}(q)} \mapsto \widetilde{S}_{\lambda K}$. (This assertion follows immediately from elementary Brauer theory for algebras over a regular ring of Krull dim. ≤ 2; cf. [DPS1; (1.1.3)], or use [B; (1.9.6)].) Thus, the decomposition (5.2) holds with $\mathbb{Q}(q)$ replaced throughout by K. Now (a) follows from this fact.

By (5.1), the $\widetilde{Y}_\lambda$ are distinct for distinct $\lambda \in \Lambda^+(r)$. By general principles, the $\widetilde{P}(\lambda) = \mathrm{Hom}_{\widetilde{H}_{\mathcal{O}}}(\widetilde{Y}_\lambda, \widetilde{T}(n,r)_{\mathcal{O}})$, $\lambda \in \Lambda^+(n,r)$, are the distinct indecomposable, projective $\widetilde{S}_q(n,r)_{\mathcal{O}}$-modules. Since $\widetilde{S}_q(n,r)_K$ is split semisimple, the functor $\Leftrightarrow\otimes_{\mathcal{O}} k$ takes distinct projective indecomposable $\widetilde{S}_q(n,r)_{\mathcal{O}}$-modules to distinct projective indecomposable $S_q(n,r)$-modules (see [CR; Ex. 16, p. 142]), proving (c). Since $\#\Lambda^+(n,r)$ is the number of distinct irreducible $S_q(n,r)$-modules, (b) follows.[12] □

We remark that a version of (7.2b) over fields is implicit in [DJ2; (2.6),(3.11)] and [DJ4; (8.8)]. (As noted in [DPS2], it is easy to check that the modules $\widetilde{S}_\lambda$, $\widetilde{S}_{\lambda k}$, $\lambda \in \Lambda^+(r)$, in this paper identify with the Specht modules as considered in [DJ1,2].)

We consider next how the duality functors (1.2) and (2.4) behave.

(7.3) Lemma. *(a) For $\lambda \in \Lambda^+(r)$, $\mathfrak{d}_{\widetilde{H}_{\mathcal{O}}}\widetilde{Y}_\lambda \cong \widetilde{Y}_\lambda$ and $\mathfrak{d}_H Y_\lambda \cong Y_\lambda$. Also, $\mathfrak{d}_{\widetilde{H}_{\mathcal{O}}}\widetilde{Y}^\natural_\lambda \cong \widetilde{Y}^\natural_\lambda$ and $\mathfrak{d}_H Y^\natural_\lambda \cong Y^\natural_\lambda$.*

(b) For $\lambda \in \Lambda^+(r)$, $\widetilde{Y}^{\natural\Phi}_\lambda \cong \widetilde{Y}_{\lambda'}$ and $Y^{\natural\Phi}_\lambda \cong Y_{\lambda'}$.

(c) $\mathfrak{d}_{S_q(n,r)}$ is a strong duality on ${}_{S_q(n,r)}\mathcal{C}$, i. e., $\mathfrak{d}_{S_q(n,r)}L(\lambda) \cong L(\lambda)$ for all irreducible $S_q(n,r)$-modules $L(\lambda)$.

Proof. By (1.4a), $\mathfrak{d}_{\widetilde{H}_{\mathcal{O}}}\widetilde{T}_{\lambda\mathcal{O}} \cong \widetilde{T}_{\lambda\mathcal{O}}$ and $\mathfrak{d}_{\widetilde{H}_{\mathcal{O}}}\widetilde{T}^\Phi_{\lambda\mathcal{O}} \cong \widetilde{T}^\Phi_{\lambda\mathcal{O}}$. Hence, $\mathfrak{d}_{\widetilde{H}_{\mathcal{O}}}\widetilde{Y}_\lambda$ is an indecomposable summand of $\widetilde{T}_{\lambda\mathcal{O}}$ which has a nonzero homomorphism to $\widetilde{T}^\Phi_{\lambda'\mathcal{O}}$. Thus,

$$\mathrm{Hom}_{\widetilde{H}_K}(\mathfrak{d}_{\widetilde{H}_K}\widetilde{Y}_{\lambda K}, \widetilde{T}^\Phi_{\lambda' K}) \cong \mathrm{Hom}_{\widetilde{H}_{\mathcal{O}}}(\mathfrak{d}_{\widetilde{H}_{\mathcal{O}}}\widetilde{Y}_{\lambda\mathcal{O}}, \widetilde{T}^\Phi_{\lambda'\mathcal{O}})_K \neq 0.$$

Since $\widetilde{H}_K$ is a semisimple algebra, it follows that $\mathrm{Hom}_{\widetilde{H}_K}(\widetilde{T}^\Phi_{\lambda' K}, \mathfrak{d}_{\widetilde{H}_K}\widetilde{Y}_{\lambda K}) \not 0$, so $\mathrm{Hom}_{\widetilde{H}_{\mathcal{O}}}(\widetilde{T}^\Phi_{\lambda'\mathcal{O}}, \mathfrak{d}_{\widetilde{H}_{\mathcal{O}}}\widetilde{Y}_\lambda) \neq 0$ also. Therefore, $\mathfrak{d}_{\widetilde{H}_{\mathcal{O}}}\widetilde{Y}_\lambda \cong \widetilde{Y}_\lambda$. Similarly, $\mathfrak{d}_{\widetilde{H}_{\mathcal{O}}}\widetilde{Y}^\natural_\lambda \cong \widetilde{Y}^\natural_\lambda$. This proves two of the isomorphisms in (a); the other two follow by base change and (1.3).

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[12] Using (7.2c), it can be shown that $n_{\mu\lambda} = m_{\mu'\lambda'}$. Similarly, in (5.2), $e_{\mu\lambda} = f_{\mu'\lambda'}$.

Applying $\mathfrak{d}_{\widetilde{H}_\mathcal{O}}$ and then Φ to a nonzero morphism $\widetilde{Y}_\lambda^\natural \to \widetilde{Y}_\lambda$, (a) and (1.4c) imply that $\widetilde{Y}_\lambda^\Phi$ is an indecomposable summand of $\widetilde{T}_{\lambda\mathcal{O}}^\Phi$ for which there exists a nonzero morphism to $\widetilde{T}_{\lambda'\mathcal{O}}$. Hence, $\widetilde{Y}_\lambda^\Phi \cong \widetilde{Y}_{\lambda'}^\natural$. Applying Φ again gives the first assertion in (b). Now base change gives the second assertion in (b).

Finally, since the duality $\mathfrak{d}_H$ fixes each Y_λ, [CPS2; (1.2.1c)] states that $\mathfrak{d}_{S_q(n,r)}$ is a strong duality, so (c) holds. □

The following result provides an essential key for understanding the partial tilting modules $X(\lambda)$ for q-Schur algebras. The proof makes crucial use of the isomorphisms $\widetilde{T}y_\lambda \cong \mathrm{Hom}_{\widetilde{H}}(\widetilde{T}_\lambda^\Phi, \widetilde{T})$ given in (2.5a) and the filtration results (6.8). Compare [DJ1; (4.12)] and [DJ3; (3.5)] for parts (b) and (c).

(7.4) Proposition. *Let $\widetilde{T} = \widetilde{T}(r,r)$ and put $\widetilde{V}(\lambda) = \mathrm{Hom}_{\widetilde{H}}(\widetilde{T}_{\lambda'}^\Phi, \widetilde{T})$ and $\widetilde{X}(\lambda) = \mathrm{Hom}_{\widetilde{H}_\mathcal{O}}(\widetilde{Y}_\lambda^\natural, \widetilde{T}_\mathcal{O})$ for $\lambda \in \Lambda^+(r)$. Then:–*

(a) $\widetilde{V}(\lambda) \in {}_{\widetilde{S}_q(r,r)}\mathcal{C}(\mathrm{tilt})$. (See above (5.5) for a definition.)

(b) $\widetilde{T}_{\lambda'}^\Phi$ has a $\widetilde{H}$-module filtration with top section $\widetilde{S}_\lambda$ and lower sections $\widetilde{S}_\mu$ for $\mu \lhd \lambda$.

(c) $\mathfrak{d}_{\widetilde{H}}\widetilde{S}_\lambda \cong \widetilde{S}_{\lambda'}^\Phi$.

(d) There is an isomorphism

$$\mathrm{Hom}_{\widetilde{S}_q(r,r)_\mathcal{O}}(\widetilde{X}(\lambda), \widetilde{T}_\mathcal{O}) \xrightarrow{\sim} \widetilde{Y}_\lambda^\natural \tag{7.4.1}$$

of $\widetilde{H}_\mathcal{O}$-modules. Also, $\widetilde{X}(\lambda) \in {}_{\widetilde{S}_q(r,r)_\mathcal{O}}\mathcal{C}(\mathrm{tilt})$. Finally, $\widetilde{X}(\lambda)_k \in {}_{S_q(r,r)}\mathcal{C}(\mathrm{tilt})$ has highest weight λ.

Proof. By (2.5a), $\widetilde{V}(\lambda) \cong \widetilde{T}y_{\lambda'}$. Consider the $\widetilde{\Upsilon}^e$-filtration $\widetilde{T}_\bullet$ of $\widetilde{T}$ defined in (6.8). By its construction, it can be refined to a $\widetilde{\Delta}$-filtration $\widetilde{F}_\bullet$ in which every $\widetilde{F}_i$ is spanned by certain Kazhdan-Lusztig basis elements C_w^μ, $\mu \in \Lambda(r)$ and $w \in \mathcal{D}_\mu^+$. If $\widetilde{F}_i/\widetilde{F}_{i-1} \cong \widetilde{\Delta}(\nu)$, $\nu \in \Lambda^+(r)$, then $\widetilde{F}_i/\widetilde{F}_{i-1}$ has basis $\{C_w^\mu + \widetilde{F}_{i-1} \mid \mu \in \Lambda(r), w \sim_L x$, for some fixed $x \sim_R w_{0,\nu}\}$.

For $\lambda \in \Lambda^+(r)$, let $\mathcal{D}_\lambda = \{w \in W \mid sw > w, \forall s \in J(\lambda)\}$. Then $C_w^+ y_\lambda \neq 0$ if and only if $w^{-1} \in \mathcal{D}_\lambda$. If $C_w^+ y_\lambda \neq 0$, then $w^{-1} \in \mathcal{D}_\lambda$, since otherwise, there exists $s \in J(\lambda)$ so that $ws < w$. Then $C_w^+\tau_s = qC_w^+$, while $\tau_s y_\lambda = -y_\lambda$, contradicting $q+1 \neq 0$. Conversely, any $C_w^+ = \tau_w +$ terms involving τ_v for $\ell(v) < \ell(w)$, so, for $w^{-1} \in \mathcal{D}_\lambda$, $C_w^+ y_\lambda = \epsilon_{w_{0,\lambda}} q_{w_{0,\lambda}}^{-1} \tau_{ww_{0,\lambda}} +$ terms involving τ_u, $\ell(u) < \ell(ww_{0,\lambda})$. Hence, the elements $C_w^+ y_\lambda$, $w^{-1} \in \mathcal{D}_\lambda$, are $\mathcal{Z}$-linearly independent.

Thus, $\widetilde{F}_i y_\lambda$ has a basis consisting of various products $C_w^\mu y_\lambda$, for $\mu \in \Lambda(r)$, $w \in \mathcal{D}_\mu^+ \cap \mathcal{D}_\lambda^{-1}$. Hence, any non-zero $\widetilde{F}_i y_\lambda/\widetilde{F}_{i-1}y_\lambda$ is $\mathcal{Z}$-torsion free. For such i, $\widetilde{F}_i/\widetilde{F}_{i-1} \xrightarrow{\sim} \widetilde{F}_i y_\lambda/\widetilde{F}_{i-1}y_\lambda$, since each $\widetilde{\Delta}(\lambda)_{\mathbb{Q}(q)}$ is an irreducible $\widetilde{S}_q(r,r)_{\mathbb{Q}(q)}$-module. Therefore, $\widetilde{V}(\lambda') = \widetilde{T}y_\lambda$ has a $\widetilde{\Delta}$-filtration. By (2.5a), $\mathfrak{d}_{\widetilde{S}_q(r,r)}\widetilde{T}y_\lambda \cong \widetilde{T}y_\lambda$; so $\widetilde{T}y_\lambda$ also has a $\widetilde{\nabla}$-filtration. This proves (a).

Since $\widetilde{T}$ has a $\widetilde{\nabla}$-filtration, (5.6(1)) implies $\mathrm{Ext}^1_{\widetilde{S}_q(r,r)}(\widetilde{M}, \widetilde{T}) = 0$ for any $\widetilde{M} \in {}_{\widetilde{S}_q(n,r)}\mathcal{C}(\widetilde{\Delta})$. Hence, if $\widetilde{F}_\bullet$ is a $\widetilde{\Delta}$-filtration of $\widetilde{V}(\lambda)$, then $\mathrm{Hom}_{\widetilde{S}_q(r,r)}(\widetilde{F}_\bullet, \widetilde{T})$ is a filtration of $\mathrm{Hom}_{\widetilde{S}_q(r,r)}(\widetilde{V}(\lambda), \widetilde{T})$ with sections $\widetilde{S}_\nu \cong \mathrm{Hom}_{\widetilde{S}_q(r,r)}(\widetilde{\Delta}(\nu), \widetilde{T})$ (by (5.4)) for every ν for which $\widetilde{\Delta}(\nu)$ is a section of $\widetilde{F}_\bullet$. By (2.6a), $\widetilde{T}^\Phi_{\lambda'} \cong \mathrm{Hom}_{\widetilde{S}_q(r,r)}(\widetilde{V}(\lambda), \widetilde{T})$. Thus, (5.2) implies that $\widetilde{\Delta}(\lambda)$ appears as a section in $\widetilde{V}(\lambda)$ and all other sections $\widetilde{\Delta}(\mu)$ satisfy $\mu \lhd \lambda$. Also, (5.6(2)) says that $\widetilde{F}_\bullet$ can be assumed to have bottom section $\widetilde{\Delta}(\lambda)$. Clearly, (b) follows from these observations.

Let ϕ_λ generate the free rank one $\mathcal{Z}$-module $\mathrm{Hom}_{\widetilde{H}}(\widetilde{T}^\Phi_{\lambda'}, \widetilde{T}_\lambda)$; cf. (5.1b). By (b) and (5.2), $\mathrm{Im}\,\phi_\lambda \cong \widetilde{S}_\lambda$. By (1.4b), $\mathfrak{d}_{\widetilde{H}}(\phi^\Phi_\lambda) = \phi_{\lambda'}$. Since $\mathrm{Im}\,\phi_{\lambda'} \cong \widetilde{S}_{\lambda'}$, (c) follows.

In proving (d), denote $\widetilde{\Delta}(\lambda)_\mathcal{O}$, $\widetilde{\nabla}(\lambda)_\mathcal{O}$ simply by $\widetilde{\Delta}(\lambda)$, $\widetilde{\nabla}(\lambda)$. Now $\widetilde{X}(\lambda)$ is a direct summand of $\mathrm{Hom}_{\widetilde{H}_\mathcal{O}}(\widetilde{T}^\Phi_{\lambda'\mathcal{O}}, \widetilde{T}_\mathcal{O}) \in {}_{\widetilde{S}_q(r,r)_\mathcal{O}}\mathcal{C}(\mathrm{tilt})$, so, $\mathrm{Ext}^1_{\widetilde{S}_q(r,r)}(\widetilde{X}(\lambda), \widetilde{\nabla}(\mu))$ and $\mathrm{Ext}^1_{\widetilde{S}_q(r,r)}(\widetilde{\Delta}(\mu), \widetilde{X}(\lambda))$ vanish $\forall \mu \in \Lambda^+(r)$ by (5.6(1,2)). By "Donkin's criterion" [CPS2; (4.5.1)], $\widetilde{X}(\lambda) \in {}_{\widetilde{S}_q(r,r)_\mathcal{O}}\mathcal{C}(\mathrm{tilt})$. The isomorphism

$$\widetilde{T}^\Phi_{\lambda'\mathcal{O}} \cong \mathrm{Hom}_{\widetilde{S}_q(r,r)_\mathcal{O}}(\widetilde{V}(\lambda)_\mathcal{O}, \widetilde{T}_\mathcal{O})$$

immediately implies (7.4.1). (Observe that, in the notation just above (2.6), if $\mathrm{Ev}_{\widetilde{M}}$ is an isomorphism, then $\mathrm{Ev}_{\widetilde{N}}$ is also an isomorphism for any direct summand $\widetilde{N}$ of $\widetilde{M}$.) Thus, by (7.2a), λ is the maximal ν for which $\widetilde{\Delta}(\nu)_\mathcal{O}$ is a section of $\widetilde{X}(\lambda)$. □

(7.5) Remark. If we localize $\mathcal{Z}$ to $\mathcal{Z}'$ in which $q+1$ is invertible, then the analog of (7.4a) over $\mathcal{Z}'$ is much easier to prove: One can show

$$\mathrm{Ext}^1_{\widetilde{H}'}(\widetilde{S}'_\lambda, \widetilde{T}') = 0, \tag{7.5.1}$$

using the arguments from [CPS2; (1.5.2)] and [DPS1; (1.2.13)]. Then one gets a filtration of $\widetilde{V}(\lambda)'$ by modules $\widetilde{\Delta}(\lambda)'$, and then duality may be applied to complete the proof.

If an $\widetilde{H}$-module $\widetilde{M}$ has a $\widetilde{S}$-filtration $\widetilde{F}_\bullet$, then the multiplicity of any $\widetilde{S}_\lambda$ as a section in $\widetilde{F}_\bullet$ equals $[\widetilde{M}_K : \widetilde{S}_{\lambda K}]$ (which in turn equals $[\widetilde{M}_{\mathbb{Q}(q)} : \widetilde{S}_{\lambda\mathbb{Q}(q)}]$). Hence, the multiplicity is independent of the filtration chosen; we denote it by $[\widetilde{M} : \widetilde{S}_\lambda]$. Similarly, for a $\widetilde{S}_q(n,r)$-module $\widetilde{M}$ with a $\widetilde{\Delta}$-filtration, the multiplicity $[\widetilde{M} : \widetilde{\Delta}(\lambda)]$ of $\widetilde{\Delta}(\lambda)$ as a section of a $\widetilde{\Delta}$-filtration of $\widetilde{M}$ is well-defined. Similar remarks apply for $\widetilde{H}_\mathcal{O}$ and $\widetilde{S}_q(n,r)_\mathcal{O}$.

(7.6) Proposition. *For $\lambda \in \Lambda^+(r)$, let $\widetilde{P}(\lambda) = \mathrm{Hom}_{\widetilde{H}_\mathcal{O}}(\widetilde{Y}_\lambda, \widetilde{T}_\mathcal{O})$ be as in (7.2). Then:–*

(a) Each $\widetilde{P}(\lambda)$ is in ${}_{\widetilde{S}_q(r,r)_\mathcal{O}}\mathcal{C}(\widetilde{\Delta})$.

(b) For $\lambda \in \Lambda^+(r)$, $\widetilde{Y}_\lambda$ (resp., Y_λ) has a $\widetilde{S}$-filtration (resp., S-filtration) with bottom section $\widetilde{S}_{\lambda\mathcal{O}}$ (resp., S_λ) and higher sections $\widetilde{S}_{\mu\mathcal{O}}$ (resp., S_μ) for $\mu \rhd \lambda$. Also, for any $\mu \in \Lambda^+(r)$,

$$[\widetilde{Y}_\lambda : \widetilde{S}_{\mu\mathcal{O}}] = [\widetilde{P}(\lambda) : \widetilde{\Delta}(\mu)_\mathcal{O}] = [P(\lambda) : \Delta(\mu)], \tag{7.6.1}$$

where $P(\lambda) = \widetilde{P}(\lambda)_k$.

(c) Similarly, for $\lambda \in \Lambda^+(r)$, $\widetilde{Y}_\lambda^\natural$ (resp., $Y_\lambda^\natural$) has a $\widetilde{S}$-filtration with top section $\widetilde{S}_{\lambda\mathcal{O}}$ (resp., S_λ) and lower sections $\widetilde{S}_{\mu\mathcal{O}}$ (resp., S_μ) for $\mu \trianglelefteq \lambda$.

(d) Given $\lambda, \mu \in \Lambda^+(r)$, there is an equality

$$(7.6.2) \qquad [\widetilde{X}(\lambda) : \widetilde{\Delta}(\mu)_\mathcal{O}] = [\widetilde{Y}_\lambda^\natural : \widetilde{S}_{\mu\mathcal{O}}]$$

of multiplicities relative to any $\widetilde{\Delta}$-filtration of $\widetilde{X}(\lambda)$ and $\widetilde{S}$-filtration of $\widetilde{Y}_\lambda^\natural$.

Proof. (a) follows from the quasi-heredity of $\widetilde{S}_q(r,r)$ (or, as we have done earlier, use [CPS2; (4.5.1)]). Since (5.5) implies that $\widetilde{T} = \widetilde{T}(r,r) \in {}_{\widetilde{S}_q(r,r)}\mathcal{C}(\widetilde{\nabla})$, (5.4) and (5.6(1)) imply each section $\widetilde{\Delta}(\mu)_\mathcal{O}$ of $\widetilde{P}(\lambda)$ contributes a section $\widetilde{S}_{\mu\mathcal{O}} = \mathrm{Hom}_{\widetilde{S}_q(r,r)_\mathcal{O}}(\widetilde{\Delta}(\mu)_\mathcal{O}, \widetilde{T}_\mathcal{O})$ in a filtration of $\widetilde{Y}_\lambda \cong \mathrm{Hom}_{\widetilde{S}_q(r,r)_\mathcal{O}}(\widetilde{P}(\lambda), \widetilde{T}_\mathcal{O})$, proving (b).

Finally, both (c), (d) follow immediately from (7.4d). □

Recall from §4 that, given a HWC ${}_A\mathcal{C}$ with poset Λ^+, there is an associated partial tilting module $X(\lambda)$ defined for every $\lambda \in \Lambda^+$. Our next result gives an explicit description of these modules for q-Schur algebras.

(7.7) Theorem. *For $\lambda \in \Lambda^+(n,r)$, $\widetilde{X}(\lambda)_k = \mathrm{Hom}_{\widetilde{H}_\mathcal{O}}(\widetilde{Y}_\lambda^\natural, \widetilde{T}(n,r)_\mathcal{O})_k$ identifies with the partial tilting module $X(\lambda)$ of highest weight λ for the HWC ${}_{S_q(n,r)}\mathcal{C}$.*

Proof. By (4.3a), we can assume $r = n$. Since $\widetilde{Y}_\lambda^\natural$ is a direct summand of $\widetilde{T}_\mathcal{O}^\Phi = \widetilde{T}(r,r)_\mathcal{O}^\Phi$, the evaluation map $\widetilde{Y}_\lambda^\natural \to \mathrm{Hom}_{\widetilde{S}_q(r,r)_\mathcal{O}}(\mathrm{Hom}_{\widetilde{H}_\mathcal{O}}(\widetilde{Y}_\lambda^\natural, \widetilde{T}_\mathcal{O}), \widetilde{T}_\mathcal{O})$ is an isomorphism by (2.6a). Thus, the indecomposability of $\widetilde{Y}_\lambda^\natural$ implies that of $\widetilde{X}(\lambda)$. Now [CPS2; (1.5.6b)] and (4.4c) imply that $\widetilde{X}(\lambda)_k$ is indecomposable. The theorem follows by (7.4d). □

As an easy consequence of our results, we describe the Ringel dual of $S_q(n,r)$. Let $\Lambda^+(n,r)'$ be the ideal in $\Lambda^+(r)$ consisting of all dual partitions λ', $\lambda \in \Lambda^+(n,r)$. Also, following (2.5), let

$$(7.8) \qquad \widetilde{X} = \bigoplus_{\lambda \in \Lambda(r,r)} \widetilde{T}(r,r)y_\lambda \cong \mathrm{Hom}_{\widetilde{H}}(\widetilde{T}(r,r)^\Phi, \widetilde{T}(r,r)).$$

By (7.7), $\widetilde{X}_k$ is a full tilting module for $S_q(r,r)$ for any field k which is a $\mathcal{Z}$-algebra. So, if $\widetilde{E} = \mathrm{End}_{\widetilde{S}_q(r,r)}(\widetilde{X})$, then $\widetilde{E}_k \cong \mathrm{End}_{S_q(r,r)}(\widetilde{X}_k)$ by (4.4c). As discussed in §4, the Ringel dual ${}_{S_q(r,r)}\mathcal{C}^\star$ of ${}_{S_q(r,r)}\mathcal{C}$ identifies with ${}_{\widetilde{E}_k}\mathcal{C}$. It is a highest weight category with weight poset $(\Lambda^+(r), \trianglelefteq^{\mathrm{op}})$ and standard objects $\Delta^\star(\lambda) = \mathrm{Hom}_{S_q(r,r)}(\Delta(\lambda), \widetilde{X}_k)$. Now we can prove the following important result.

(7.9) Theorem. *Let k be a field which is a $\mathcal{Z}$-algebra.*

(a) Then, for $\widetilde{X}$ as in (7.8), we have $\mathrm{End}_{S_q(n,r)}(\widetilde{X}_k) \cong S_q(n,r)$. Moreover, there is an equivalence $G : {}_{S_q(r,r)}\mathcal{C}^\star \xrightarrow{\sim} {}_{S_q(r,r)}\mathcal{C}$ of highest weight categories such that $G(L^\star(\lambda)) \cong L(\lambda')$, $\lambda \in \Lambda^+(r)$. (Here $L^\star(\lambda)$ denotes the simple object in ${}_{S_q(r,r)}\mathcal{C}^\star$ corresponding to $\lambda \in \Lambda^+(r)$.)

(b) For $n > r$, ${}_{S_q(n,r)}\mathcal{C}^\star \cong {}_{S_q(n,r)}\mathcal{C}$.

(c) For $n < r$, ${}_{S_q(n,r)}\mathcal{C}^\star \cong {}_{S_q(r,r)}\mathcal{C}[\Lambda^+(n,r)']$.

Proof. We will use the notation immediately above the statement of the theorem. By (2.6a), there is an isomorphism $\widetilde{f} : \widetilde{S}_q(r,r)^{\rm op} \xrightarrow{\sim} \widetilde{E}$. Let $\widetilde{\beta}$ be the anti-automorphism in the proof of (2.4), and form the isomorphism $\widetilde{g} = \widetilde{f} \circ \widetilde{\beta} : \widetilde{S}_q(r,r) \xrightarrow{\sim} \widetilde{E}$. Then $g = \widetilde{g}_k : S_q(r,r) \to \widetilde{E}_k$ is an algebra isomorphism, proving the first assertion of (a).

Next, as in (2.6b), $\widetilde{f}$ defines a category equivalence $\widetilde{F} : {}_{\widetilde{E}}\mathcal{C} \xrightarrow{\sim} {}_{\widetilde{S}_q(r,r)^{\rm op}}\mathcal{C}$. For $\lambda \in \Lambda^+(r)$, let $\widetilde{N} = \widetilde{S}_\lambda$ in (2.6b), so $\widetilde{N}^\diamond = \widetilde{\Delta}(\lambda)$. By (5.4) and its proof, the hypothesis of (2.6b) holds for $\widetilde{N}$. If we put $\widetilde{\Delta}^\star(\lambda) = \mathrm{Hom}_{\widetilde{S}_q(r,r)}(\widetilde{\Delta}(\lambda), \widetilde{X})$ and define $\Delta^\star(\lambda)$ similarly, then $\widetilde{\Delta}^\star(\lambda)_k \cong \Delta^\star(\lambda)$ by (4.4c). By (2.6.1), $\widetilde{F}(\widetilde{\Delta}^\star(\lambda)) \cong \mathrm{Hom}_{\widetilde{H}}(\widetilde{T}^\Phi, \widetilde{S}_\lambda)$, where $\widetilde{T} = \widetilde{T}(r,r)$. Thus, the isomorphism $\widetilde{g}$ defines, by pull-back, a category equivalence $\widetilde{G} : {}_{\widetilde{E}}\mathcal{C} \xrightarrow{\sim} {}_{\widetilde{S}_q(r,r)}\mathcal{C}$ in which $\widetilde{G}(\widetilde{\Delta}^\star(\lambda)) \cong \mathrm{Hom}_{\widetilde{H}}(\widetilde{T}^\Phi, \widetilde{S}_\lambda)$. In this isomorphism, the natural right action of $\widetilde{S}_q(r,r)$ on $\mathrm{Hom}_{\widetilde{H}}(\widetilde{T}^\Phi, \widetilde{S}_\lambda)$, through its left action on $\widetilde{T}^\Phi$, is converted to a left action by means of $\widetilde{\beta}$. Now

$$\begin{aligned} \mathrm{Hom}_{\widetilde{H}}(\widetilde{T}^\Phi, \widetilde{S}_\lambda) &\cong \mathrm{Hom}_{\widetilde{H}}(\widetilde{S}_\lambda^*, \widetilde{T}^{\Phi *}) \cong \mathrm{Hom}_{\widetilde{H}}(\mathfrak{d}_{\widetilde{H}}\widetilde{S}_\lambda, \widetilde{T}^\Phi) \\ &\cong \mathrm{Hom}_{\widetilde{H}}(\widetilde{S}_{\lambda'}^\Phi, \widetilde{T}^\Phi) \cong \mathrm{Hom}_{\widetilde{H}}(\widetilde{S}_{\lambda'}, \widetilde{T}) \\ &\cong \widetilde{\Delta}(\lambda') \end{aligned}$$

as left $\widetilde{S}_q(r,r)$-modules. In the above display, the second isomorphism follows from (2.4a), and the third isomorphism follows from (7.4c). Now the isomorphism g above defines, by pull-back, an equivalence $G : {}_{\widetilde{E}_k}\mathcal{C} \xrightarrow{\sim} {}_{S_q(r,r)}\mathcal{C}$. We have $G(\Delta^\star(\lambda)) \cong G(\widetilde{\Delta}^\star(\lambda)_k) \cong \widetilde{G}(\widetilde{\Delta}^\star(\lambda))_k \cong \widetilde{\Delta}(\lambda')_k \cong \Delta(\lambda')$. It follows $G(L^\star(\lambda)) \cong L(\lambda')$. Since $(\Lambda^+(r), \trianglelefteq^{\rm op}) \xrightarrow{\sim} (\Lambda^+(r), \trianglelefteq)$, $\lambda \mapsto \lambda'$, is a poset isomorphism, G is an equivalence of highest weight categories. So, (a) is proved.

Next, (b) follows since for $n > r$, $S_q(n,r)$ is Morita equivalent to $S_q(r,r)$.

In the notation just above (2.3), we have $S_q(n,r) = e_{\Lambda^+(n,r)} S_q(r,r) e_{\Lambda^+(n,r)}$. Thus, ${}_{S_q(n,r)}\mathcal{C} \cong {}_{S_q(r,r)}\mathcal{C}(\Lambda^+(n,r))$, so (4.3.1) implies that

$${}_{S_q(n,r)}\mathcal{C}^\star \cong {}_{S_q(r,r)}\mathcal{C}^\star[\Lambda^+(n,r)],$$

where, on the right-hand side, $\Lambda^+(n,r)$ is an ideal in the poset $(\Lambda^+(r), \trianglelefteq^{\rm op})$. Now (c) follows since the equivalence G defined in (a) carries ${}_{S_q(r,r)}\mathcal{C}^\star[\Lambda^+(n,r)]$ to ${}_{S_q(r,r)}\mathcal{C}[\Lambda^+(n,r)']$. □

(7.10) Remarks. (a) The determination of the Ringel dual of the Schur algebra $S(n,r)$ (i. e., the algebra $S_q(n,r)$ when $q = 1$ in k) is due to Donkin [D1], using algebraic group/Hopf algebra methods. Donkin has also announced [D3; p.236] a result similar to (the first part of) (7.9a) for q-Schur algebras. (In fact, the proof, which is quite different from our methods, has now been circulated in preprint form [D4; §4.1].) [CPS2; §5.2] gave a different, more combinatorial development of tilting modules for $S(n,r)$, valid under the assumption that char $k \neq 2$. The above arguments, besides being short and independent of any representation theory of quantum groups, extend those methods to q-Schur algebras. The methods also give integral versions of the classical $q=1$ results, and, specializing to $q=1$, give efficient proofs of those results.

(b) In characteristic 2, when $q=1$, the treatment here confirms and deepens the tilting module description proposed in [CPS2; (5.2.8)], via reduction of an integral intertwining module, and does the same for q-Schur algebras over any field in which $q+1=0$; see (7.8). These exceptional cases are quite important for the nondescribing characteristic representation theory of the finite general linear groups. It is interesting to note that the tilting module description (2.5c) remains "the same" over any field or $\mathcal{Z}$-algebra, though that is not true of the intertwining module description.

(c) In (8.6c) below, we will present another description of the Ringel dual of $\widetilde{S}_q(n,r)_k$ in some cases.

8. Quantum Weyl reciprocity, II. In this section, we maintain the notation of the previous section. In particular, $\widetilde{H}$ is the generic Hecke algebra over $\mathcal{Z} = \mathbb{Z}[q,q^{-1}]$ for the $W = \mathfrak{S}_r$ and the field k is a $\mathcal{Z}$-algebra so that $H = \widetilde{H}_k$ is *not* semisimple. Thus, the image of q is a primitive lth root of 1 for some integer l satisfying $1 \leq l \leq r$. If $l=1$, then $0 < \text{char}\, k = p \leq r$. As at the beginning of §7, fix a local triple $(\mathcal{O}, k, K)$, where the discrete valuation ring $\mathcal{O}$ is a $\mathcal{Z}$-algebra and $\widetilde{H}_K$ is a (split) semisimple algebra.

A partition λ is (row) a-regular ($a \in \mathbb{Z}^+$) if λ contains no part λ_i which is repeated a or more times. Let $\Lambda^+(n,r)_{a-\text{reg}}$ be the subset of a-regular partitions. Put

$$(8.1) \qquad \Lambda^+(n,r)_{\text{reg}} = \begin{cases} \Lambda^+(n,r)_{l-\text{reg}} & \text{if } l > 1; \\ \Lambda^+(n,r)_{p-\text{reg}} & \text{if } l = 1, \end{cases} \qquad \text{and } \Lambda^+(r)_{\text{reg}} = \Lambda^+(r,r)_{\text{reg}}.$$

(8.2) Lemma. *(a) For $\lambda \in \Lambda^+(r)$, the restriction of the pairing $(\ ,\ )$ in (1.4a) to $S_\lambda \subseteq T_\lambda$ is non-zero if and only if $\lambda \in \Lambda^+(r)_{\text{reg}}$.*

(b) For $\lambda \in \Lambda^+(r)_{\text{reg}}$, let D_λ be the head of S_λ. The algebra H has $\#\Lambda^+(r)_{\text{reg}}$ distinct irreducible modules. Each irreducible H-module is isomorphic to some D_λ.

(c) For $\lambda \in \Lambda$, Y_λ (resp., $\widetilde{Y}_\lambda$) is projective as an H-module (resp., $\widetilde{H}_{\mathcal{O}}$-module) if and only if $\lambda' \in \Lambda^+(r)_{\text{reg}}$.

Proof. It has already been remarked (above (7.3)) that S_λ identifies with the Specht module associated to λ in the sense of [DJ1]. For this reason, (a) follows from [DJ1; p. 42]. (b) and (c) are also proved in [DJ1; (7.7)], but, using (a), follow from general principles, see [CPS2; (4.5.3)]. □

(8.3) Proposition. *(a) For any $\lambda, \mu \in \Lambda^+(r)$, the number of occurrences $[X(\lambda) : \Delta(\mu)]$ of $\Delta(\mu)$ as a section in a Δ-filtration of $X(\lambda)$ equals $[\Delta(\mu') : L(\lambda')]$.*

(b) If $\lambda \in \Lambda^+(r)_{\text{reg}}$, then $[X(\lambda) : \Delta(\mu)] = [S_\mu : D_\lambda]$.

Proof. By (7.6.2), $[X(\lambda):\Delta(\mu)] = [\widetilde{X}(\lambda):\widetilde{\Delta}(\mu)_{\mathcal{O}}] = [\widetilde{Y}^{\natural}_{\lambda}:\widetilde{S}_{\mu\mathcal{O}}]$. But (7.3a,b) and (7.4c) imply that $[\widetilde{Y}^{\natural}_{\lambda}:\widetilde{S}_{\mu\mathcal{O}}] = [\widetilde{Y}_{\lambda'}:\widetilde{S}_{\mu'}]$, which in turn equals $[P(\lambda'):\Delta(\mu')]$ by (7.6.1). By Brauer-Humphreys reciprocity, $[P(\lambda'):\Delta(\mu')] = [\nabla(\mu'):L(\lambda')]$ which also equals $[\Delta(\mu'):L(\lambda')]$ since $S_q(r,r)$ has a strong duality. If $\lambda \in \Lambda^+(r)_{\text{reg}}$, then $Y^{\natural}_{\lambda} \cong Y^{\Phi}_{\lambda'}$ is projective by (8.2c). By elementary Brauer theory (see [DPS1; (1.1.3)]), $[\widetilde{Y}^{\natural}_{\lambda}:\widetilde{S}_{\mu}] = [S_{\mu}:D_{\lambda}]$. □

(8.4) Remarks. (a) Using (4.3a), we could have taken $\lambda,\mu \in \Lambda^+(n,r)$ in (8.3). We could also obtain the first assertion in (8.3a) from (7.9a), but have given the above argument since it is very concrete.

(b) Taking $q=1$ in (8.3) yields relations between the decomposition numbers for symmetric groups and those for Weyl modules for Schur algebras. In particular, (a) is in [D1], and (b) is in [E1]. Putting (a) and (b) together, gives a result of James [J1] on embedding the decomposition matrix of $\mathfrak{S}_r$ into that for $S(r,r)$.

(c) Erdmann [E2] considered whether *every* decomposition number for $S(n,r)$ occurs as a decomposition number for some symmetric group $\mathfrak{S}_{r'}$. She proved that

(8.4.1) $$\text{for } \lambda,\mu \in \Lambda^+(n,r): \quad [\Delta(\mu):L(\lambda)] = [S_{t(\mu')}:D_{t(\lambda')}],$$

where $t(\mu') = p\mu' + (p-1)(n-1, n-2, \cdots, 1, 0) \in \Lambda^+(pr + (p-1)\binom{n}{2})_{p-\text{reg}}$. The proof of (8.4.1) follows from the $q=1$ case of (8.3) using the $GL_n(k)$-isomorphisms

(8.4.2) $$X(\lambda)^{(1)} \otimes St \cong X(t(\lambda)), \qquad \Delta(\lambda)^{(1)} \otimes St \cong \Delta(t(\lambda)).$$

Here St is the Steinberg module, and $M^{(1)}$ denotes the Frobenius twist of M.

(d) Let $q \in k$ be a primitive lth root of unity, $l > 1$. There is *no* evident quantization of (8.4.1), since now the Frobenius morphism $\text{Fr}: GL_{n,q}(k) \to GL_n(k)$ goes from the quantum group $GL_{n,q}(k)$ to the classical algebraic group $GL_n(k)$. (See [PW; §7].) Thus, although there exist isomorphisms as in (8.4.2), the twisted modules $X(\lambda)^{(1)}$ and $\Delta(\lambda)^{(1)}$ are the pull-backs through Fr of the tilting module $X(\lambda)$ and Weyl module $\Delta(\lambda)$ for $GL_n(k)$, and *not* for $GL_{n,q}(k)$.

(e) Suppose in (8.4) that $\operatorname{char} k = 0$, so that $l > 1$. By (8.3) and (6.4), the decomposition matrix D for H embeds into the decomposition matrix D_q for $U_q(\mathfrak{sl}_n)$. When l is odd at least, D_q is described in terms of inverse Kazhdan-Lusztig polynomials, using [KL3], [KT]. Another approach to determining D in terms of the crystal bases for the affine quantum enveloping algebra $U_q(\widehat{\mathfrak{gl}}_n)$ is described in [LLT]. Perhaps, our remark explains the "in principle" comment [LLT; p. 205].

Now assume that

(8.5) $$\begin{cases} n < l, & \text{if } l > 1; \\ n < p, & \text{if } l = 1. \end{cases}$$

Since n is the Coxeter number of $GL_n(k)$ (or the quantun general linear group $GL_{n,q}(k)$), (8.5) covers all cases (except $n = l$ or p, which we omit for simplicity) relevant for the Lusztig conjecture relating the characters of $U_q(\mathfrak{sl}_n)$ with those for

$GL_n(k)$, when k has positive characteristic, etc. James [J2; §4] has formulated a conjecture (in the case $n = r$ and k has positive characteristic p) asserting that, when $r < lp$, the irreducible $S_q(r,r)$-modules arise by "reduction mod p" from the corresponding irreducible modules for the complex q-Schur algebra $S_q(r,r)_{\mathbb{C}}$ at a primitive lth root of unity in $\mathbb{C}$. Clearly, this conjecture can be made for $n < r$ (and its validity for $n = r$ implies the validity for $n < r$). Also, if $l = p$, the condition $r < lp$ is similar in spirit to the weight restrictions in the modular Lusztig conjecture (for the irreducible characters of reductive groups G). Thus, the conditions $n < l = p$ and $r < lp$ represent an overlap between the Lusztig and James conjectures.[13]

We prove the following result, first observed by Erdmann for $q = 1$ [E1] (building on work of Donkin). We will make use of the algebras $\widetilde{H}(n,r)$ defined above (6.2). Also, we set $H(n,r) = \widetilde{H}(n,r)_k$.

(8.6) Theorem. *(a) As an $S_q(n,r)$-module, $T(n,r) = \bigoplus_{\lambda\in\Lambda^+(n,r)_{\rm reg}} X(\lambda)^{\oplus r_\lambda}$ where each $r_\lambda > 0$.*

(b) Assume (8.5) holds. Then $T(n,r)$ is a full tilting module for $S_q(n,r)$. In particular, $H(n,r)^{\rm op} \cong \mathrm{End}_{S_q(n,r)}(T(n,r))$ (see (6.2)) is quasi-hereditary, and ${}_{H(n,r)^{\rm op}}\mathcal{C}$ identifies with the Ringel dual of ${}_{S_q(n,r)}\mathcal{C}$.

(c) If (8.5) holds, then ${}_{H(n,r)^{\rm op}}\mathcal{C}$ is a HWC with poset $(\Lambda^+(n,r), \trianglelefteq^{\rm op})$, standard objects S_λ, partial tilting modules Y_λ, and projective indecomposable modules $Y_\lambda^\natural$, $\lambda \in \Lambda^+(n,r)$.

Proof. We first prove (a). First, by (7.7), the partial tilting module $X(\lambda)$ is isomorphic to $\mathrm{Hom}_H(\widetilde{Y}_\lambda^\natural, \widetilde{T}(n,r)_\mathcal{O})_k$. If $\lambda \in \Lambda^+(n,r)_{\rm reg}$, then (8.2c) and (7.3b) imply that $Y_\lambda^\natural$ is projective, so $\mathrm{Hom}_H(Y_\lambda^\natural, T(n,r)) \cong \mathrm{Hom}_{\widetilde{H}_\mathcal{O}}(\widetilde{Y}_\lambda^\natural, \widetilde{T}(n,r)_\mathcal{O})_k$. Therefore, in this case, $X(\lambda)$ is a direct summand of $T(n,r)$.

Since $T(n,r)$ is a faithful $H(n,r)$-module, the irreducible $H(n,r)$-modules are precisely those D_λ which appear as an H-composition factor of $T(n,r)$. Thus, using (8.3b), it follows that $\mathrm{Irr}(H(n,r)) = \{D_\lambda \mid \lambda \in \Lambda^+(n,r)_{\rm reg}\}$. Since $H(n,r) = \mathrm{End}_{S_q(n,r)}(T(n,r))$ (by (6.2) and (4.4)), $\mathrm{Irr}(H(n,r))$ has cardinality equal to the number of distinct (up to isomorphism), indecomposable summands of the $S_q(n,r)$-module $T(n,r)$. In view of the previous paragraph, (a) is established.

If (8.5) holds, then $\Lambda^+(n,r)_{\rm reg} = \Lambda^+(n,r)$, so that (b) follows from (a). Then by §4, ${}_{H(n,r)^{\rm op}}\mathcal{C}$ is a HWC with poset $(\Lambda^+(n,r), \trianglelefteq)$ and standard objects

$$\Delta^\star(\lambda) = \mathrm{Hom}_{S_q(n,r)}(\Delta(\lambda), T(n,r)) \cong S_\lambda.$$

(The second isomorphism here follows from (5.4), (4.4) and the discussion above and below (5.5).) Since Ringel's theory establishes that $T(n,r)$ is a full tilting module for $H(n,r)^{\rm op}$, it follows that the Y_λ, $\lambda \in \Lambda^+(n,r)$, are the distinct partial tilting modules for $H(n,r)$. Finally, any $Y_\lambda^\natural$, $\lambda \in \Lambda^+(n,r)$, is a projective H-module and an $H(n,r)$-module; thus, it is also a projective $H(n,r)$-module. Because the $Y_\lambda^\natural$ are distinct for distinct λ, (c) follows. □

[13] For further discussion of the James conjecture, see [GH] and [CPS3]. In particular, these papers discuss the fact that the conjecture is true "generically".

We remark that the proof of (8.6a) above has used Theorem 6.2. However, another proof, independent of (6.2), exists. We may first prove the result for $n \geq r$ then apply (4.3a) and (7.7). (See [E1; 4.2].) However, (8.6b,c) require (6.2) in an essential way.

Now let $\Lambda^e = \Lambda^+(n,r) \times \Lambda^+(n,r)$ and define $(\lambda,\mu) \leq^e (\tau,\sigma)$ if and only if $\lambda \trianglelefteq \tau$ and $\mu \trianglerighteq \sigma$.

(8.7) Corollary. *Assume that (8.5) holds. Then the algebra* $R^e = S_q(n,r) \otimes H(n,r)^{\mathrm{op}}$ *is a quasi-hereditary algebra. The module category* ${}_{R^e}\mathcal{C}$ *is a HWC with poset* $(\Lambda^e, \leq^e)$ *and standard objects* $\Delta(\lambda) \otimes S_\mu$, $(\lambda,\mu) \in \Lambda^e$.

Proof. At least for an algebraically closed field, the result is immediate from (8.6) and [W], where tensor products of quasi-hereditary algebras are studied. We give a different proof, applying [DPS2; (1.6)] to the algebra $\widetilde{R}^e = \widetilde{S}_q(n,r)_{\mathcal{O}} \otimes \widetilde{H}(n,r)^{\mathrm{op}}_{\mathcal{O}}$ and the poset Λ^e. First, each $\widetilde{\Delta}(\lambda,\mu)_{\mathcal{O}} = \widetilde{\Delta}(\lambda)_{\mathcal{O}} \otimes \widetilde{S}_{\mu\mathcal{O}} \in \mathrm{Ob}({}_{\widetilde{R}^e}\mathcal{C})$ is a finitely generated projective $\mathcal{O}$-module. Let $\widetilde{P}(\lambda,\mu) = \widetilde{P}(\lambda) \otimes \widetilde{Y}^\natural_\mu$, where $\widetilde{P}(\lambda)$ is the projective cover in ${}_{\widetilde{S}_q(n,r)_{\mathcal{O}}}\mathcal{C}$ of the irreducible $S_q(n,r)$-module $L(\lambda)$. Then $\widetilde{P}(\lambda,\mu)$ is a projective $\widetilde{R}^e$-module. In the Grothendieck group of $\widetilde{S}_q(n,r)_K$-modules, we have $[\widetilde{P}(\lambda,\mu)_K] = [\widetilde{\Delta}(\lambda,\mu)_K] + \sum_{(\sigma,\tau) >^e (\lambda,\mu)} m_{(\sigma,\tau),(\lambda,\mu)} [\widetilde{\Delta}(\sigma,\mu)_K]$. Each $\widetilde{\Delta}(\lambda,\mu)_K \cong \widetilde{\Delta}(\lambda)_K \otimes \widetilde{S}_{\mu K}$ is absolutely irreducible for $\widetilde{R}^e_K = \widetilde{S}_q(n,r)_K \otimes \widetilde{H}(n,r)^{\mathrm{op}}_K$. Finally, $\bigoplus_{\delta \in \Lambda^e} \widetilde{P}(\delta)$ is clearly a projective generator for ${}_{\widetilde{R}^e}\mathcal{C}$. Thus, the hypotheses of [DPS2; (1.6)] holds, and (d) follows. $\square$

(8.8) Remarks. (a) We emphasize that, even without the assumption (8.5), Theorem 6.8 gives a filtration of tensor space by $S_q(n,r) \otimes H(n,r)^{\mathrm{op}}$-modules $\Delta(\lambda) \otimes \mathfrak{d}_H S_\lambda$, $\lambda \in \Lambda^+(n,r)$, in analogy with the classical decomposition (2) described at the beginning of the paper. In the presence of (8.5), this filtration is even by modules which make sense in terms of quasi-hereditary algebras. However, note that the $\Delta(\lambda) \otimes \mathfrak{d}_H S_\lambda$ are *not* (generally) standard modules for $S_q(n,r) \otimes H(n,r)^{\mathrm{op}}$. The standard modules are the modules $\Delta(\lambda) \otimes S_\lambda$ by (8.7).

(b) We mention without detailed proof one further aspect of Weyl reciprocity, namely the correspondence between Weyl and Specht modules. One requires here the set-up of (7.5), where we localize $\mathcal{Z}$ to $\mathcal{Z}'$ in which $q+1$ is invertible. Then there is a contravariant equivalence of exact categories

$$ {}_{\widetilde{S}_q(n,r)'}\mathcal{C}(\widetilde{\Delta}') \xrightarrow{\sim} \mathcal{C}_{\widetilde{H}'}(\widetilde{S}')^{\mathrm{op}}. $$

The proof is obtained using the approach given in [CPS2; (4.6.4)]. A version at $q = 1$ of this result is due to Erdmann [E1] over a field, where stronger assumptions ($p > n$ is sufficient) are required.

Acknowledgment. This paper was written while the first (resp., second) author was on leave from the University of New South Wales (resp., the University of Virginia) at the University of Virginia (resp., Northwestern University). Thanks are due to the University of Virginia, the University of Chicago, and Northwestern University for their hospitality during the preparation of this paper.

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