Maximal Submodules and the Second Loewy Layer of Standard Modules

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This paper is currently in preliminary form, with a bibliography and corresponding references still to be made explicit.

Introduction

The paper begins in §1 with a foundational discussion of a new notion, that of a semistandard filtration in a highest weight category. The main result is Theorem 1, which says that "multiplicities" of standard modules in such filtration are well-defined. In §2, we specialize to the case of semistandard filtrations of maximal submodules of standard modules. The main result is Theorem 2, which under a simple Kazhdan-Lusztig theory hypothesis, characterizes standard module multiplicities in terms of Ext^1 between irreducible modules, or equivalently, second Loewy layers of standard modules. This theorem was really the starting point of this paper. The author found it while trying to explain the mysterious frequency of "0 or 1" answers for semistandard module multiplicities in maximal submodules of standard modules.

The next section, §3, applies the theory of §2 to obtain some general inequalities on the behavior of Ext^1 in the presence of a suitable exact functor. This theory is used in §4 to attack some well-known issues about Ext^n and parity conditions involving standard and irreducible modules with singular high weights in the presence of the Lusztig conjecture for representations of semisimple algebraic groups in positive characteristic. Some indirect evidence is given for an expected parity behavior for weights on a wall, and this behavior is formalized as Conjecture 1. We also offer a suggested possible direction for its future proof. The usefulness of such parity behavior is exhibited by Theorem 3, which shows Conjecture 1 implies some new results relating Ext^n been standard and irreducible modules in the singular weight cases to Ext^n for related modules with nonsingular high weights. Proposition 8 proves a special case of the theorem, without the use of Conjecture 1.

1 Semistandard Filtrations in Highest Weight Categories

Let A be a quasihereditary algebra over a field k. Thus, there is a poset Λ, whose elements are called weights, indexing the irreducibles modules as $\{L(\lambda)\}_{\lambda\in L}$, with distinct weights indexing distinct irreducible modules, and every irreducible module occurring as some $L(\lambda)$, up to isomorphism. There are also standard modules $\Delta(\lambda)$, and constant modules $\nabla(\lambda)$, for each $\lambda \in \Lambda$.

The standard module $\Delta(\lambda)$ has head $L(\lambda)$ and all other composition factors $L(\mu)$ satisfy $\mu < \lambda$. For additional axioms and further properties, the reader is referred to [CPS Crelle].

One important property is that projective modules have *standard filtrations*. We will define this notion only for finite-dimensional modules, and, throughout this paper assume that all modules under consideration are finite-dimensional, unless otherwise indicated. A standard filtration of a module M is a filtration

$$(*) \qquad 0 \leq F_0 \leq F_1 \leq \cdots \leq F_n = M,$$

in which each quotient F_i/F_{i-1} is a direct sum of standard modules $\bigoplus_{\lambda \in \Lambda} \Delta(\lambda)^{m_i(\lambda)}$. (Each $\Delta(\lambda)$ appears as a summand $m_i(\lambda)$ times.) Standard properties of quasihereditary algebras allow us some rearranging of where the standard modules appear. We will only consider standard filtrations on which some of this rearranging has been done, and always *assume*

$(**)$ If $m_i(\lambda) \neq 0$ for a given index i and weight λ, then λ is maximal among all weights ν in Λ with $m_j(\nu) \neq 0$ for some $j \geq i$.

Notice this definition allows $m_i(\lambda)$ to be nonzero for multiple values of i. We could require

$(***)$ For any $\lambda \in \Lambda$, $m_i(\lambda) \neq 0$ for at most one index i.

We will call standard filtrations with this additional property $(***)$ *strict* standard filtrations. But we do not require it for our notion of standard filtration (though we do require $(**)$). The reason is that some exact functors, such as "translation to a wall or facet" in algebraic group theory, do, under mild conditions, preserve standard filtrations, but not strictness (see §3).

We now introduce a new but very natural notion, that of a *semistandard* filtration of a module M. This is a filtration $(*)$ of M in which

$(\dagger)$ There is a surjective homomorphism

$$\bigoplus_{\lambda \in \Lambda} \Delta(\lambda)^{m_1(\lambda)} \twoheadrightarrow F_i/F_{i-1},$$

for some choice of multiplicities $m_i(\lambda)$ satisfying the $(**)$ condition that λ must be maximal among all ν with $m_j(\nu) \neq 0$ and $j \geq i$, whenever $m_i(\lambda) \neq 0$ for a given index i and weight λ. (In particular, $m_i(\lambda) \neq 0$ implies that no composition factor $L(\nu)$ of M/F_{i-1} has weight $\nu > \lambda$.)

If, in addition, $(***)$ is satisfied, we will call the given semistandard filtration *strict*.

Recall that a partial order $\preceq$ is a *refinement* of a partial order $\leq$ if $\lambda \leq \mu$ always implies $\lambda \preceq \mu$, for $\lambda, \mu \in \Lambda$. The partial order on Λ can always be replaced by any refinement, keeping the same standard and costandard modules. Obviously, we can define standard and semistandard filtrations with respect to any refinement $\preceq$, using the same definitions.

Proposition 1. *For each refinement $\preceq$ of $\leq$, every A module has at least one semistandard filtration. Such a filtration may be chosen so as to be strict.*

Proof. This is clear from the corresponding facts for projective modules, where the filtrations may be chosen as standard. Q.E.D.

Actually, there is always one *canonical* way to choose a semistandard filtration: Let Λ_1 be the set of maximal weights in $\Gamma_1 = \Lambda$, and recursively define Λ_{j+1} as the set of maximal weights in $\Gamma_{j+1} = \Lambda - \bigcup_{i=1}^{i} \Lambda_j$. Eventually, Λ_{j+1} is empty for all $j \geq n$, for some n. Take n minimal with this property, and, for $i = 1, 2, \ldots, n$, put $F_0 = 0$, and, for $1 \leq i \leq n$, put

$$F_i = \text{smallest submodule of } M \text{ such that all composition factors of } M/F_i \text{ are of the form } L(\lambda) \text{ with } \lambda \in \Gamma_{i+1}.$$

Call this filtration, which is certainly well-defined (as a filtration, irrespective of the semistandard property), the *canonical* filtration of M. Clearly, every projective module has a standard filtration which is of this form. That is, its canonical filtration is standard. So we just need to show canonical filtrations are well-behaved under homomorphic images, to see that they are all semistandard.

Proposition 2. *Let $\varphi : M \to M'$ be a surjective homomorphism of A-modules. If $\{F_i\}_{i=0}^n$ is the canonical filtration of M, then $\{\varphi(F_i)]\}_{i=0}^n$ is the canonical filtration of $\varphi(M) = M'$.*

Proof. Fix an index i. Clearly, $\varphi(F_i)$ has the property that all composition factors $L(\lambda)$ of $M'/\varphi(F_i)$ have the property $\lambda \in \Gamma_{i+1}$. So $\varphi(F_i) \supseteq F'_i$, the i^{th} term of the canonical filtration of M'. Now each section F_j/F_{j-1} of F_i, $j \leq i$ has a head with composition factors $L(\nu)$, $\nu \in \Lambda_j = \Gamma_j - \Gamma_{j+1}$. Since $\Lambda_j \cap \Gamma_{i+1} \subseteq \Lambda_j \cap \Gamma_{j+1} = \emptyset$, the head of the homomorphic image

$$(\varphi(F_j) + F'_i)/(\varphi(F_{j-1}) + F'_i)$$

of $\varphi(F_j/F_{j-1})$ has only composition factors $L(\nu)$, $\nu \notin \Gamma_{i+1}$. Hence the displayed homomorphic image is zero for each $j \leq i$. It follows that $\varphi(F_i) \subseteq F'_i$. Since this holds for each i, the proposition is proved. Q.E.D.

Corollary. *The canonical filtration of any A-module M' is (strict) semistandard.*

Proof. Just apply the previous proposition to any surjection $P \to M'$, with P playing the role of M. Note, of course, that $\varphi(F_i)/\varphi(F_{i-1})$ is a homomorphic image of F_i/F_{i-1}.

It is instructive to give a second proof of the corollary, which is more direct: Each Γ_i is an ideal in Λ, thus the category of A-modules with all composition factors of the form $L(\nu)$, $\nu \in \Gamma_i$, is the category of modules for a quasihereditary algebra A_i, with the same standard modules $\Delta(\nu)$, for those $\nu \in \Gamma_i$. So, by induction on Γ, it is sufficient to show that F'_1 is a homomorphic image of a direct sum of copies of standard modules $\Delta(\lambda)$ with $\lambda \in \Lambda_1$. However, no simple component $L(\nu)$ of the head of F''_1 can have $\nu \in \Gamma_2$, by maximality of the quotient M'/F''_1. Thus, the head of F'_1 is a direct sum of modules of the form $L(\lambda)$, $\lambda \in \Lambda_1 = \Lambda - \Gamma_2$. Since $\Delta(\lambda)$ is projective for such a λ (which is maximal in Λ), the required surjection follows, and induction completes the proof. Q.E.D.

Remark 1. The canonical filtration, and linear order variations, are very natural for computer calculations in a Lie theory setting: To determine F_1 for a module M, one simply looks for all maximal vectors associated to maximal weights in Λ. Having found these, one sets F_1 equal to the module these maximal vectors generate. Then one passes to M/F_1 and repeats the process with a smaller weight poset (Γ_2 in the notation above), repeating the process as often as necessary to exhaust Λ.

Such a program was implemented, by the author and undergraduate students, for algebraic groups of type A_4 with k algebraically closed of characteristic 5 or 7, in the special case where M is the radical of a standard module $\Delta(\lambda)$ with restricted high weight. In this case one uses as weight poset Λ all dominant weights less than λ that are linked [Jantzen] to λ. It is not necessary to keep track of all weight spaces in the various modules M/F_i, but only those of the form ν or $\nu+\alpha$ to where $\nu \in \Lambda$ and α is a fundamental root. (The weights of the form $\nu+\alpha$ are needed in searching for maximal vectors of weight ν). Actually, this description is somewhat over-simplified, in that our current implementation works with the restricted Lie algebra and "baby Verna modules" instead of Weyl modules (the standard modules for the algebraic group), and a true Weyl module implementation has not yet been achieved. A few details of the work may be found in [Scott 2003] and on the author's website: www.math.virginia.edu/∼lls2l, along with discussions of other programs. The determinations obtained of canonical filtration were very revealing, and led to the results in this paper. See especially §2 below.

We now address multiplicities in semistandard filtrations. Suppose $(*)$ is a semistandard filtration of M, and the numbers $m_i(\lambda)$ are chosen minimal in (†) for each i. *Given an index i and a weight λ, the unique minimal $m_i(\lambda)$ which works in* (†) *is the multiplicity of $L(\lambda)$ in the head of F_i/F_{i-1}*: Clearly the value of $m_i(\lambda)$ must be at least this multiplicity. Moreover, for any values of $m_i(\nu)$, $\nu \in \Lambda$, the surjection (†) implies that the head of F_i/F_{i-1} consists of irreducible modules $L(\nu)$ with $m_i(\nu) \neq 0$. All such ν are maximal as weights of composition factors of M/F_{i-1}, so that $\Delta(\nu)$ is projective in the category of modules with composition factors $L(\omega)$, $\omega \leq \tau$, for some τ with $L(\tau)$ a composition factor of M/F_{i-1}. So *there is a surjection*

$$(\dagger\dagger) \qquad \bigoplus_{\lambda \in \Lambda} \Delta^{m_i(\lambda)}(\lambda) \longrightarrow F_i/F_{i-1},$$

with $m_i(\lambda)$ equal to the multiplicity of $L(\lambda)$ in the head of F_i/F_{i-1}. Such values $m_i(\lambda)$ are certainly minimal over all possibilities for $m_i(\lambda)$ in (†), in that any instance of (†) has values of $m_i(\lambda)$ at least as large, for each λ, as the values in (††).

It is also worth mentioning that, for a fixed i and λ with $m_i(\lambda) \neq 0$, if $j \geq i$, then $m_i(\lambda()$ in (††) is the multiplicity of $L(\lambda)$ in all of F_j/F_{j-1} (not just in the head). If not, $L(\lambda)$ would occur as a composition factor of $\Delta(\nu)$ for some $\nu \neq \lambda$ with $m_j(\nu) \neq 0$. This would contradict the maximality of λ.

As previously noted, we allow, for a given λ, $m_i(\lambda) \neq 0$ for more than index i. We now set, for any semistandard filtration $(*)$ of a module M,

$$[M : \Delta(\lambda)] = \sum_i m_i(\lambda),$$

where $m_i(\lambda)$ are given as in (††), and call this number $[M : \Delta(\lambda)]$ the (*semistandard*) *multiplicity* of $\Delta(\lambda)$ in M.

Theorem 1. *For each $\lambda \in \Lambda$, the multiplicity $[M : \Delta(\lambda)]$ is independent of the chosen semistandard filtration. Moreover, we have*

$$[M : \Delta(\lambda)] = \dim_{k_\lambda} \ Hom_A(M, \nabla(\lambda)),$$

where k_λ is the skew field $End_A(L(\lambda)) \cong End_A(\nabla(\lambda))$.

Proof. For each λ, choose an index i in a given semistandard filtration (λ) with $m_i(\lambda) \neq 0$ in (††) and i minimal with that property. The preceding discussion shows:

(1) $\nabla(\lambda)$ is injective in a category of A-modules containing all composition factors of M/F_{i-1};

(2) $m_j(\lambda)$ is the multiplicity of $L(\lambda)$ in F_j/F_{j-1}, for all $j \geq i$. In particular, $[M : \Delta(\lambda)]$, as computed from the given filtration, is the multiplicity of $L(\lambda)$ in M/F_{i-1}.

Combining (1) and (2), we find that $[M : \Delta(\lambda)]$, as computed from the given filtration, is $\dim_{k_\lambda} \mathrm{Hom}_A(M/F_{i-1}, \nabla(\lambda))$. To complete the proof of the theorem, it suffices to show $\mathrm{Hom}_A(F_{i-1}, \nabla(\lambda)) = 0$. For this, it is sufficient to show $\mathrm{Hom}_A(F_j/F_{j-1}, \nabla(\lambda)) = 0$ for all $j \leq i-1$. However, $m_j(\lambda) = 0$ in (††) for such a j, so that $\mathrm{Hom}_A(\bigoplus_{\nu\in\Lambda} \Delta(\lambda)^{m_j(\nu)}, \nabla(\lambda)) = 0$. Since (††) is a surjection, it follows that $\mathrm{Hom}_A(F_j/F_{j-1}, \nabla(\lambda)) = 0$. This completes the proof. Q.E.D.

2 Maximal Submodules of Standard Modules

We now apply the theory of the previous section to the case $M = \mathrm{rad}\ \Delta(\lambda)$, the unique maximal submodule of a standard module $\Delta(\lambda)$. The weight λ will remain fixed throughout this section. However, we note that all composition factors $L(\gamma)$ of M have weights in the poset ideal $\Gamma = \{\gamma \in \Lambda | \gamma < \lambda\}$. In particular, $[M : \Delta(\gamma)] = 0$ unless $\gamma \in \Gamma$.

Proposition 3. *For any $\gamma \in \Lambda$, we have*

$$[M : \Delta(\gamma)] = dim_{k_\gamma} \ Hom_A(M, \nabla(\gamma)) = dim_{k_\gamma} \ Ext^1_A(L(\lambda), \nabla(\gamma)).$$

Proof. The first equality follows from Theorem 1. For the second, use the long exact sequence of $\mathrm{Ext}_A(-, \nabla(\gamma))$ for the exact sequence

$$0 \to M \to \Delta(\lambda) \to L(\lambda) \to 0,$$

and note that $\mathrm{Ext}^1_A(\Delta(\lambda), \nabla(\gamma)) = 0$. Also, the map

$$\mathrm{Hom}_A(\Delta(\lambda), \nabla(\gamma)) \to \mathrm{Hom}_A(M, \nabla(\gamma))$$

is always zero. Thus,

$$\mathrm{Hom}_A(M, \nabla(\lambda)) \cong \mathrm{Ext}^1_A(L(\lambda), \nabla(\gamma)),$$

and the proposition is proved. Q.E.D.

The reader familiar with Kazhdan-Lusztig theories will immediately recognize the right-hand dimension in the proposition as a quantity computable from Kazhdan-Lusztig polynomials, in the presence of such a theory, in standard situations arising in algebraic and quantum groups, and in Lie algebraic representation theory. See [CPS AKL]. Moreover, in these situations, one has the additional property:

$$(\star) \qquad \mathrm{Ext}^1_A(L(\lambda), \nabla(\gamma)) \cong \mathrm{Ext}^1_A(L(\lambda), L(\gamma)),$$

for all $\gamma < \lambda$. We will say $L(\lambda)$ *has the simple* KL *property* provided $(\star)$ holds for λ.

Theorem 2. *Suppose $L(\lambda)$ has the simple KL property, and let $M = rad\ \Delta(\lambda)$, as before. Then*

$$\text{(a)} \qquad [M : \Delta(\gamma)] = [head(M) : L(\gamma)],$$

the multiplicity of $L(\gamma)$ in the second Loewy layer $rad\ \Delta(\lambda)/rad^2\Delta(\lambda) = head(M)$ of $\Delta(\lambda)$, for each $\gamma \in \Lambda$. In addition, the natural maps

$$head(F_i/F_{i-1}) \to head(M/F_{i-1})$$

are injective, for any semistandard filtration $0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n = M$ and any index $i = 1, \ldots, n$. There is a compatible isomorphism

$$\text{(b)} \qquad \bigoplus_{i=1}^{n} head(F_i/F_{i-1}) \cong head(M).$$

Finally, if $\gamma < \lambda$, then

$$\text{(c)} \qquad [M : \Delta(\gamma)] = dim_{k_\gamma}\ Ext^1_a(L(\lambda), L(\gamma)),$$

while $[M : \Delta(\gamma)] = 0$, if $\gamma < \lambda$ does not hold.

Proof. Apply the long exact sequence in the proof of the proposition, but replacing $\mathrm{Ext}_A(-, \nabla(\gamma))$ with $\mathrm{Ext}_A(-, L(\gamma))$. This gives

$$\mathrm{Hom}_A(M, L(\gamma)) \cong \mathrm{Ext}^1_A(L(\lambda), L(\gamma)),$$

if $\gamma < \lambda$ (which implies $\mathrm{Ext}^1_A(\Delta(\lambda), L(\gamma)) = 0$). Moreover, this isomorphism is compatible with the isomorphism

$$\mathrm{Hom}_A(M, \nabla(\gamma)) \cong \mathrm{Ext}^1_A(L(\lambda), \nabla(\gamma))$$

in the proposition. That is, the map $L(\gamma) \subseteq \nabla(\gamma)$ induces a commutative square in which the remaining two sides are the isomorphism

$$\mathrm{Ext}^1_A(L(\lambda), L(\gamma)) \cong \mathrm{Ext}^1_A(L(\lambda), \nabla(\gamma)),$$

and the map

$$\mathrm{Hom}_A(M, L(\gamma)) \to \mathrm{Hom}_A(M, \nabla(\gamma)).$$

It follows that the latter map is also an isomorphism. Together with the previous proposition, this proves part (a) of the theorem.

For part (b), observe that the filtration of M by the F_i induces a filtration of head$(M) = M/\text{rad } M$ by the images $(F_i + \text{rad } M)/\text{rad } M$. The sections $(F_i + \text{rad } M)/(F_{i-1} + \text{rad } M)$ may be identified with the images of the natural maps

$$\text{head}(F_i/F_{i-1}) \to \text{head}(M/F_{i-1}).$$

In particular, the total number of occurrences $[M : \Delta(\gamma)]$ of $L(\gamma)$ in any head(F_i/F_{i-1}) is at least as great as its multiplicity $[\text{head}(M) : L(\gamma)]$ as a composition factor of head(M), with equality for all r (which occurs, by (a)), forcing

$$\text{head}(F_i/F_{i-1}) \cong (F_i + \text{rad } M)/(F_{i-1} + \text{rad } M),$$

for each i. Equivalently, the maps

$$\text{head}(F_i/F_{i-1}) \to \text{head}(M/F_{i-1}),$$

are all injective. Let us compose each of these maps with a splitting head$(M/F_{i-1}) \to$ head(M) of the natural projections head$(M) \to$ head(M/F_{i-1}). The compositions are maps from head(F_i/F_{i-1}) to head(M), and the sum of compositions is a map

$$\text{(B)} \qquad \bigoplus_{i=1}^{n} \text{head}(F_i/F_{i-1}) \to \text{head}(M).$$

When this latter map is composed with any natural projection head$(M) \to$ head(M/F_j) and (at the other end) with the inclusion head$(F_j/F_{j-1}) \subseteq \bigoplus_{i=1}^{n}$ head(F_i/F_{i-1}), the natural map

$$\text{head}(F_j/F_{j-1}) \to \text{head}(M/F_{j-1})$$

is obtained, by construction. This is the compatibility required in (b). Also, we obtain that (B) is surjective, since each element of each section $(F_j + \text{rad } M)/(F_{j-1} + \text{rad } M) \cong$ head(F_j/F_{j-1}) is the image under the natural projection $M/\text{rad } M \to M/(F_{j-1} + \text{rad } M)$ of an element of the image of (B). (Note that this latter projection is precisely the map head$(M) \to$ head(M/F_{j-1}) used above.) This proves (b).

Part (c) is just a restatement of $(\star)$, the previous proposition, and the discussion above it. Q.E.D.

Remark 2. Part (c) of the above theorem is a key part of the motivation of the research in this paper. When running the computer program described in Remark 1 for Weyl module radicals M, the author observed empirically that, extremely often, but not always, either $[M : \Delta(\nu)] = 1$ or $[M : \Delta(\nu)] = 0$ (though the notion $[M : \Delta(\nu)]$ had not yet been formulated—what was observed directly were dimensions of spaces of high weight vectors in modules M/F_{i-1}). Efforts to understand this led eventually to part (c) of Theorem 2. Thus, the unexpected observations were due to two things:

(1) That Lusztig's conjecture held for the algebraic groups under consideration (thus giving property ($\star$); see [CPS AKL]), and

(2) Guralnick's principle (empirical) that H^1 and Ext^1 groups for finite groups with coefficients in irreducible modules are, generally, extremely small. (See [Scott 2003] for further discussion and [CPS 2006] for some new, positive, results.)

3 Exact Functors and Semistandard Filtrations

We suppose in this section that, in addition to the quasihereditary algebra A of §1, we have a second quasihereditary algebra $\overline{A}$ with weight poset $\overline{\Lambda}$. Both algebras are assumed to have the same ground field k, and all their irreducible modules are assumed to be absolutely irreducible. We let $\mathcal{C}$ be a category equivalent to mod A, and $\overline{\mathcal{C}}$ to mod $\overline{A}$. The definitions and results of the previous two sections of course carry over to $\mathcal{C}$ and $\overline{\mathcal{C}}$, and we will refer to their objects as "modules." We write simply $\Delta(\nu)$ for the standard module $\Delta_{\overline{\mathcal{C}}}(\nu)$, $\nu \in \overline{\Lambda}$, when it is clear what the meaning is from context. It is convenient to assume the weight posets Λ and $\overline{\Lambda}$ are disjoint, so that $\Delta(\gamma)$ is unambiguous, if $\nu \in \overline{\Lambda}$ or $\nu \in \Lambda$. We will use a similar convention for irreducible modules $L(\gamma)$, and costandard modules $\nabla(\nu)$.

We also assume we have a map $\lambda \mapsto \overline{\lambda}$ from Λ to $\overline{\Lambda} \cup \{\emptyset\}$, and that $\emptyset$ is not an element of $\overline{\Lambda}$. We put $L(\emptyset) = \Delta(\emptyset) = \nabla(\emptyset) = 0$.

Finally, we suppose we have an exact functor $T : \mathcal{C} \to \overline{\mathcal{C}}$ with the properties

$$T(\Delta(\lambda)) \cong \Delta(\overline{\lambda}),$$

and

$$\text{(C)} \qquad T(\nabla(\lambda)) \cong \nabla(\overline{\lambda}) \qquad (\lambda \in \Lambda).$$

It follows that $T(L(\lambda)) \cong L(\overline{\lambda})$. These conditions force natural "versions" of the poset orderings on Λ, $\overline{\Lambda}$ (keeping the same standard and costandard modules) to be compatible, in the sense that, for $\lambda, \mu \in \Lambda$,

$$\text{(D)} \qquad \lambda \leq \mu \text{ implies } \overline{\lambda} \leq \overline{\mu}, \qquad \text{if } \overline{\lambda} \neq \emptyset, \quad \overline{\mu} \neq \emptyset.$$

We will simply assume this compatibility condition, to keep flexibility for the possible choices of poset orderings on λ and μ.

Lemma 1. *Suppose $\preceq$ is a linear order on $\overline{\Lambda}$ which refines $\leq$ on $\overline{\Lambda}$. Then there is a linear order $\preceq$ on Λ which is compatible with that on $\overline{\Lambda}$, in the sense of (D) and refines $\leq$ on Λ.*

Proof. Let $[\lambda]$ denote the inverse image of $\overline{\lambda}$ under the map $\Lambda \to \overline{\Lambda} \cup \{\emptyset\}$. Linear order each of the disjoint sets $[\lambda]$ individually, compatibly with $\leq$. Complete the definition of $\preceq$ on $\Lambda^{\#} = \{\lambda \in \Lambda | \overline{\lambda} \neq \emptyset\}$,

$$\lambda \preceq \mu \text{ iff either } \overline{\lambda} \prec \overline{\mu} \text{ or } \overline{\lambda} = \overline{\mu} \text{ and } \lambda \prec \mu$$

($\lambda, \mu \in \Lambda^{\#}$). The order $\preceq$ is clearly linear on $\Lambda^{\#}$, and compatible with $\leq$ there (refining it). Any linear order on a subposet λ' of a finite poset Λ, and compatible there with the poset order, extends to a linear order on Λ compatible with (refining) the poset order. (The proof is an easy induction, adding one element at a time to Λ'.) Extend $\preceq$ from $\Lambda^{\#}$ to Λ accordingly, taking $\Lambda' = \Lambda^{\#}$. Q.E.D.

Proposition 4. *Let M be any module in $\mathcal{C}$. Then for any weight $\lambda \in \Lambda$,*

$$[T(M) : \Delta(\overline{\lambda})] \leq \sum_{\overline{\mu}=\overline{\lambda}} [M : \Delta(\mu)].$$

Also, if $T(L(\lambda)) \neq 0$, then

$$[T(M) : \Delta(\overline{\lambda})] \geq [M : \Delta(\lambda)].$$

Proof. Looking ahead to the proposition below and the previous lemma, we can find a semistandard filtration $0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n = M$ of M such that $0 = T(F_0) \subseteq T(F_1) \subseteq \cdots \subseteq T(F_n) = T(M)$ is a semistandard filtration of $T(M)$. Choose surjective homomorphisms

$$\bigoplus_{\nu \in \Lambda} \Delta(\nu)^{m_i(\nu)} \longrightarrow F_i/F_{i-1}$$

as in display (††) in §1. Thus, $[M : \Delta(\lambda)] = \sum_{i=1}^{n} m_i(\lambda)$. We obtain a surjective homomorphism

$$\bigoplus_{\nu \in \Lambda} \Delta(\overline{\nu})^{m_i(\nu)} \longrightarrow T(F_i/F_{i-1}) = T(F_i)/T(F_{i-1}),$$

from which it follows that

$$[\text{head}(T(F_i/F_{i-1})) : L(\overline{\lambda}] \leq \sum_{\overline{\mu}=\overline{\lambda}} m_i(\mu).$$

Adding these inequalities over all $i = 1, \ldots, n$, we get

$$[T(M) : \Delta(\overline{\lambda})] \leq \sum_{i} \sum_{\overline{\mu}=\overline{\lambda}} m_i(\mu) = \sum_{\overline{\mu}=\overline{\lambda}} \sum_{i} m_i(\mu) = \sum_{\overline{\mu}=\lambda} [M : \Delta(\mu)].$$

This proves the first part of the proposition.

Note also that, $T(L(\lambda)) = L(\overline{\lambda})$, then

$$[\text{head}(T(F_i/F_{i-1})) : L(\overline{\lambda})] \geq [\text{head}(T(F_i/F_{i-1})) : L(\lambda)].$$

Now summing over i gives the second part. Q.E.D.

Proposition 5. *Suppose the poset order Λ is linear* (which by Lemma 1, we may assume by refining the orders on Λ and $\overline{\Lambda}$, retaining compatibility). *If $0 = F_1 \subseteq F_2 \subseteq \cdots \subseteq F_n = M$ is a semistandard stratification of an A-module M, then $0 = T(F_1) \subseteq T(F_2) \subseteq \cdots \subseteq T(F_n) = T(M)$ is a semistandard stratification of $T(M)$.*

Proof. Without loss, we may assume $F_i/F_{i-1} \neq 0$ for each $i = 1, 2, \ldots, n$. Let $\lambda_i \in \Lambda$ be a weight such that $\text{head}(F_i/F_{i-1}) : L(\lambda_i)] \neq 0$. Then λ_i is maximal among weights ν with $[M/F_{i-1} : (L(\nu)] \neq 0$. Since we have assumed a linear order on Λ, the weight λ_i is uniquely determined. Hence, there is a surjection $\Delta(\lambda_i)^{m(\lambda_i)} \to F_i/F_{i-1}$, and a consequent surjection $\Delta(\overline{\lambda})_i^{m(\lambda_i)} \to T(F_i)/T(F_{i-1})$. Since $\lambda_i \geq \lambda_j$ for all $j \geq i$, by linearity, we have $\overline{\lambda}_i \geq \overline{\lambda}_j$ by compatibility. It follows that the filtration of $T(M)$ by the $T(F_i)$'s is semistandard. Q.E.D.

As a consequence of Proposition 2, we have the following inequalities on Ext^1 groups:

Corollary. *Let $\lambda, \nu \in \Lambda$ with $T(L(\lambda)) \neq 0$ and $\overline{\nu} \neq \emptyset$. Then*

$$dim_k Ext^1_{\mathcal{C}}(L(\lambda), \nabla(\nu)) \leq dim_k Ext^1_{\overline{\mathcal{C}}}(L(\overline{\lambda}), \nabla(\overline{\nu}) \leq \sum_{\overline{\mu}=\overline{\nu}} dim_k Ext^1_{\mathcal{C}}(L(\lambda), \nabla(\mu).$$

Proof. Let $M = \text{rad } \Delta(\lambda)$. Thus $T(M) \cong \text{rad } \Delta(\overline{\lambda})$, since $T(L(\lambda)) \neq 0$ by hypothesis (and, so, $T(L(\lambda)) \cong L(\overline{\lambda})$). By Proposition 2,

$$\dim_k \text{Ext}^1_{\mathcal{C}}(L(\lambda), \nabla(\mu_1) = [M : \Delta(\mu)] \quad (\text{any } \mu \in \Lambda),$$

and

$$\dim_k \text{Ext}^1_{\overline{\mathcal{C}}}(L(\overline{\lambda}), \nabla(\overline{\nu}) = [T(M) : \Delta(\overline{\nu})] \quad \text{for any } \overline{\nu} \in \overline{\Lambda}.$$

The inequalities of the corollary now follow immediately from those of Proposition 3. Q.E.D.

Remark 3. If T is a quotient functor associated to a poset coideal $\Omega = \overline{\Lambda}$ of weights in Λ, then any $\overline{\nu} \in \overline{\Lambda}$ is the image of at most one μ in Λ, and so we have equality

$$\text{Ext}^1_{\mathcal{C}}(L(\lambda), \nabla(\nu)) \cong \text{Ext}^1_{\overline{\mathcal{C}}}(L(\overline{\lambda}), \nabla(\overline{\nu}),$$

a well-known result which may be seen directly, and which also holds for all Ext^n, $n \geq 0$. If T is a functor "translation to a facet" in, say, an algebraic groups context, then the right-hand inequality is obtainable by use of the adjoint translation away from the facet. Again, this is a result for all Ext^n, $n \geq 0$. The left-hand inequality is not so obvious, but can be proved directly, without Proposition 3, by a method similar to that used to prove Proposition 5 below.

One valuable aspect of the corollary is perhaps the unified perspective it gives to each of these two cases. Also, when used with the proposition below, we are able to establish some evidence in the translation case for a result well known in at least one quotient functor case. This will all play out in the next section, where we discuss a new conjecture.

Proposition 6. *Under the same hypothesis is the above corollary, but assuming also $T(L(\nu)) \neq 0$, we have*

$$dim_k \ Ext^1_{\mathcal{C}}(L(\lambda), L(\nu)) \leq dim_k \ Ext^1_{\overline{\mathcal{C}}}(L(\overline{\lambda}), L(\overline{\nu})).$$

Proof. Let $m = \dim_k \text{Ext}^1_{\mathcal{C}}(L(\lambda), L(\nu))$. If $m = 0$, there is nothing to prove. If $m \neq 0$, then we have either $\lambda > \nu$ or $\lambda < \nu$, as is well known.

Let us assume the first case holds. Then there is an extension

$$0 \to L(\nu)^m \to E \to L(\lambda) \to 0,$$

in which $\mathrm{head}(E) \cong L(\lambda)$. Such an E is a homomorphic image of $\Delta(\lambda)$. Since $T(L(\lambda)) \cong L(\overline{\lambda})$, and $T(L(\nu)) \cong L(\overline{\nu})$, with $\overline{\lambda}, \overline{\nu} \in \overline{\Lambda}$, the module $T(E)$ gives rise to an extension

$$0 \to L(\overline{\nu})^m \to T(E) \to L(\overline{\lambda}) \to 0.$$

Also, $T(E)$ is a homomorphic image of $T(\Delta(\lambda)) \cong \Delta(\overline{\lambda})$. Thus, $\mathrm{head}(T(E)) \cong L(\overline{\lambda})$. It follows easily that

$$\dim_k \mathrm{Ext}^1_{\overline{\mathcal{C}}}(L(\overline{\lambda}), L(\overline{\nu})) \geq m.$$

In the case $\lambda \prec \nu$, we may construct an extension

$$0 \to L(\nu) \to E' \to L(\lambda)^m \to 0,$$

for which the socle of E' is isomorphic to $L(\nu)$. Such an E' embeds in $\nabla(\nu)$. Since $T(\nabla(\nu)) = \nabla(\overline{\nu})$, we again obtain

$$\dim_k \mathrm{Ext}^1_{\overline{\mathcal{C}}}(L(\overline{\lambda}), L(\overline{\nu})) \geq m.$$

This completes the proof in all cases. Q.E.D.

4 Translation to a Wall

We continue the notation of the previous section, but specialize much further. In particular, we assume that Λ is a finite ideal of dominant regular weights for a semisimple and simply connected algebraic group G over an algebraically closed field k of characteristic $p > 0$, and that $\mathcal{C}$ is the category of finite-dimensional G-modules with composition factors all of the form $L(\lambda)$, $\lambda \in \Lambda$. We pick an orbit $W_p \cdot \overline{\tau}$ of the affine Weyl group, with $\overline{\tau}$ on a wall (a facet of codimension 1), and let $\overline{\Lambda}$ be the set of dominant weights in $W_p \cdot \overline{\tau}$, which are in the closure of some alcove containing an element of Λ. (The existence of such elements forces $p \geq h$, the Coxeter number. For $G = \mathrm{SL}(n, k)$, the Coxeter number h is n.) The ordering $\leq$ used in both Λ and $\overline{\Lambda}$ is the "up-arrow" order $\uparrow$, which guarantees that $\overline{\Lambda}$ is an ideal, and the compatibility (D) of §3. See [Jantzen], especially II, 6.5(2) and 6.5(4), for this ordering, further details on alcove geometry, and the definitions of translation functors. The category $\overline{\mathcal{C}}$ is taken to be the category of finite-dimensional G-modules with composition factors $L(\tau)$, $\tau \in \overline{\Lambda}$.

If $\lambda \in \Lambda$, the closure of the alcove containing λ contains at most one weight in $\overline{\Lambda}$, and we call it $\overline{\lambda}$ if there is such a weight. If not, we set $\overline{\lambda} = \emptyset$, as in the previous section. (This only occurs if λ is near the boundary of the dominant region, and the intersection of $W_p \cdot \overline{\tau}$ with the closure of that alcove consists of a non-dominant weight.)

The functor T is defined on each block $\mathcal{B}$ of $\mathcal{C}$ as follows: All the composition factors $L(\gamma)$ of such a block have their parameterizing dominant weight γ in a fixed orbit $W_p \cdot \lambda^-$. We may take λ^- and τ^- in the closure of the same alcove $\mathcal{C}$. (We prefer the "negative dominant" alcove containing the weight -2ρ, the negative sum of all roots. However, [Jantzen] uses the alcove containing the weight 0.) Then T is defined on $\mathcal{B}$ as the translation functor $T_{\lambda^-}^{\tau^-}$. In general, a module M in $\mathcal{C}$ is a direct sum of modules in the various blocks $\mathcal{B}$, and $T(M)$ may be defined as the sum of various $T_{\lambda^-}^{\tau^-}$. In all our results of interest, the modules in $\mathcal{C}$ under consideration will be indecomposable.

The following properties of T follow from those of the individual translation functors $T_{\lambda^-}^{\tau^-}$. They do not depend on our assumption that the facet containing τ^- is a wall. (This will be useful in understanding many of the comments we make in this section, though we continue to assume elsewhere that τ^- belongs to a wall—unless otherwise noted.)

(4.1) We have $T(\Delta(\lambda)) \cong T(\Delta(\overline{\lambda}))$, $T(\nabla(\lambda)) \cong \nabla(\overline{\lambda})$. Also, $T(L(\lambda)) \neq \emptyset$ iff $\overline{\lambda}$ belongs to the "upper closure" of the alcove containing λ ($\lambda \in \Lambda$).

The "upper closure" is defined in [Jantzen]. If $\lambda = w \cdot \lambda^- \in \Lambda$, and $w \in W_p$, then $\overline{\lambda}$ belongs to the upper closure of the alcove containing λ if and only if there is a simple reflection with $\lambda < \lambda s$, where λs is defined as $ws \cdot \lambda^-$, and $\overline{\lambda}$ lies on the wall in the intersection of the closures of the two alcoves containing λ and λs, respectively. (This statement remains the same if one replaces the alcove $\mathcal{C}$, which we take to contain -2ρ, with that containing 0, though the labeling of s may change.) Equivalently, $\overline{\lambda} = w \cdot \tau^-$ and the length $\ell(w)$ of the element w of W_p is minimal among all y in W_p with $\overline{\lambda} = y \cdot \tau^-$. This characterization of the upper closure follows from [Jantzen, p.]. It is valid for τ^- in any type facet, provided $\overline{\lambda}$ is in the closure of an alcove containing a dominant weight.[1]

The various $T_{\lambda^-}^{\tau^-}$, viewed as functors on the category of (finite-dimensional) rational G-modules, all have (left and right) adjoints $T_{\tau^-}^{\lambda^-}$. This is true for τ^- in any facet in the closure of the alcove containing λ^-. The situation where τ^- belongs to a wall is particular simple:

(4.2) If $\lambda \in \Lambda$ and $\lambda < \lambda s$ for a simple reflection s, then there are natural exact sequences (nonsplit) of G-modules

$$0 \to \Delta(\lambda s) \to T_{\tau^-}^{\lambda^-}(\Delta(\overline{\lambda}) \to \Delta(\lambda) \to 0,$$

and

$$0 \to \nabla(\lambda) \to T_{\tau^-}^{\lambda^-}\nabla(\overline{\lambda}) \to \nabla(\lambda s) \to 0.$$

The maps at each end arise from adjunction.

We now introduce an additional hypothesis.

Hypothesis. The Lusztig character formula holds for each irreducible module $L(\lambda)$ with $\lambda \in \Lambda$.

This formula asserts, with λ^- as above, and $\lambda = w \cdot \lambda^-$ dominant

$$\text{(4.3)} \qquad \operatorname{ch} L(\lambda) = \sum_{\substack{y \cdot \lambda^- \text{ dominant} \\ y\cdot\lambda^- \leq \lambda}} (-1)^{\ell(w)-\ell(y)} P_{y,w}(1) \operatorname{ch} \Delta(y \cdot \lambda^-).$$

Here $P_{y,w}$ denotes a Kazhdan-Lusztig polynomial for the affine Weyl group. The formula is conjectured to hold whenever λ satisfies $\langle \lambda + \rho, \check{\alpha_0} \rangle \leq p(p - h + 2)$ where α_0 is the maximal

[1] It is our view that the notion of "upper closure" should be redefined so at to make this assertion true for all alcoves.

short root, and h is the Coxeter number. It is known to be true for any p sufficiently large, size unknown and depending on the root system [AJS]. Also, when it is known, for all λ in an ideal Λ of regular dominant weights, then an even more powerful assertion can be proved, called the "homological Lusztig character formula" in [CPS 2006], in which each $\dim \mathrm{Ext}^n_{\mathcal{C}}(L(\lambda), \nabla(\nu))$, $\lambda, \nu \in \Lambda$, is interpreted as a coefficient in a Kazhdan-Lusztig polynomial. This goes back to work of Vogan [Vogan] and Andersen [Andersen], and was used in [CPS AKL] to calculate $\dim \mathrm{Ext}^n_{\mathcal{C}}(L(\lambda), L(\nu))$, $n \geq 0$, and $\lambda, \nu \in \Lambda$. The main tool in this latter work was an axiomatized version of the Vogan-Andersen work, which CPS called an *abstract Kazhdan-Lusztig theory.*

Such a theory exists for a highest weight category, such as $\mathcal{C}$, with finite weight poset Λ if there is a "length" function $\ell : \Lambda \to \mathbb{Z}$ such that

(4.4) For each $w \geq 0$ and $\lambda, \mu \in \Lambda$,

$$\begin{aligned} \mathrm{Ext}^n_{\mathcal{C}}(L(\lambda), \nabla(\nu)) &\neq 0 \text{ implies } n \equiv \ell(\lambda) - \ell(\mu) \bmod 2, \text{ and} \\ \mathrm{Ext}^n_{\mathcal{C}}(\Delta(\nu), \nabla(\lambda)) &\neq 0 \text{ implies } n \equiv \ell(\lambda) - \ell(\mu) \bmod 2. \end{aligned}$$

For the present $\mathcal{C}$ and $\lambda = w \cdot \lambda^- \in \Lambda$, with $w \in W_p$, we take $\ell(\lambda) = \ell(w)$. As shown by Andersen (see [CPS AKL]), when the Lusztig character formula (4.3) holds—our hypothesis—then the Kazhdan-Lusztig theory property (4.4) also holds.

If we consider $\overline{\mathcal{C}}$ and $\overline{\Lambda}$, then the Lusztig character formula (4.3) holds for $L(\tau)$, if $\tau \in \overline{\Lambda}$, and $\tau = w \cdot \tau^-$, with $w \in W_p$ chosen of minimal length for this equation, and with $y \cdot \tau^-$ represented through an element of minimal length. See, for instance, [Scott Newton]. Thus, it is very natural to define $\ell(\tau) = \ell(w)$, if w is of minimal length with $w \cdot \tau^- = \tau$. One can then ask if (4.4) holds.

Unfortunately, this is not known. However, we will provide some evidence in this section for its validity, and so we make a conjecture.

Conjecture 1. *Assume the Hypothesis. Then the category $\overline{\mathcal{C}}$ has a Kazhdan-Lusztig theory* (4.4)*, with the indicated length function and notation.*

It is also reasonable to ask if the analogue of Conjecture 1 is true if τ^- is taken from a smaller facet than a wall.

Next, we establish results which give evidence for Conjecture 1.

Proposition 7. *Assume the Hypothesis. Suppose $\tau, \eta \in \overline{\Lambda}$ with $\ell(\tau) \not\equiv \ell(\eta)$ mod 2. Then the maps*

$$\begin{aligned} Ext^1_{\overline{\mathcal{C}}}(L(\tau), L(\eta)) &\to Ext^1_{\overline{\mathcal{C}}}(L(\tau), \nabla(\eta), \\ \text{and } Ext^1_{\overline{\mathcal{C}}}(L(\tau), L(\eta)) &\to Ext^1_{\overline{\mathcal{C}}}(\Delta(\tau), L(\eta), \end{aligned}$$

are surjective.

There is a completely equivalent version of the proposition:

Proposition 7′. *Assume the Hypothesis. Suppose $\tau, \eta \in \overline{\Lambda}$ with $\ell(\tau) \not\equiv \ell(\eta)$ mod 2. If τ, η are equal or not related in the poset order, then $Ext^1_{\overline{\mathcal{C}}}(L(\tau), L(\eta) = 0$. Otherwise,*

$$\begin{aligned} Ext^1_{\overline{\mathcal{C}}}(L(\tau), L(\eta)) &\cong Ext^1_{\overline{\mathcal{C}}}(L(\tau), \nabla(\eta), \text{ if } \tau > \eta, \\ \text{and } Ext^1_{\overline{\mathcal{C}}}(L(\tau), L(\eta)) &\cong Ext^1_{\overline{\mathcal{C}}}(\Delta(\tau), L(\eta), \text{ if } \tau < \eta. \end{aligned}$$

We will establish Propositions 7 and 7′, along with an additional result:

Proposition 8. *Assume the hypothesis of Proposition 7′, and suppose $\tau = \overline{\lambda}$, $\eta = \overline{\gamma}$, where $T(L(\lambda)) \neq 0$, and $T(L(\gamma)) \neq 0$, and $\gamma \in W_p \cdot \lambda$. Then*

$$Ext^1_{\mathcal{C}}(L(\lambda), L(\gamma)) \cong Ext^1_{\overline{\mathcal{C}}}(L(\overline{\lambda}), L(\overline{\gamma})).$$

Proof. We apply Proposition 6 and the corollary to Proposition 5. Choose λ minimal in it's orbit under the dot action of W_p with $\overline{\lambda} = \tau$. This guarantees $T(L(\lambda)) = L(\overline{\lambda})$, and λ, so chosen, is unique in its orbit [Jantzen, II, 7.15]. Choose γ similarly with $\overline{\gamma} = \eta$. Then $\ell(\lambda) = \ell(\overline{\lambda})$ and $\ell(\gamma) = \ell(\overline{\gamma})$. Also, the representing affine Weyl group elements in the expressions $\lambda = w \cdot \lambda^-$ and $\gamma = y \cdot \gamma^-$, λ, γ in the alcove $\underline{\mathcal{C}}$ (see the beginning of this section) are the same as for $\overline{\lambda} = w \cdot \tau^-$ and $\overline{\gamma} = y \cdot \tau^-$. So, if $i = \overline{\lambda}$ and $\eta = \overline{\gamma}$ are related in the (up arrow) order, the same relation holds between $w \cdot \lambda^-$ and $y \cdot \lambda^-$, as well as between $w \cdot \tau^-$ and $u \cdot \gamma^-$. since Λ is an ideal, we can—and do—choose λ and γ to belong to the same orbit in this case.

Since $\overline{\gamma}$ belongs to a wall, there is only one $\mu \neq \gamma$ on $W_p \cdot \gamma$ with $\overline{\mu} = \overline{\gamma}$, and μ satisfies $\ell(\mu) = \ell(\gamma) + 1$. Thus, $\mathrm{Ext}^1_{\mathcal{C}}(L(\lambda), \nabla(\mu)) = 0$ by (4.4) for Λ, if $\mu \in \Lambda$. So the inequalities in the corollary to Proposition 5 are equalities if λ, γ belong to the same W_p orbit, and Λ is replaced by its intersection with that orbit. So we have

$$\mathrm{Ext}^1_{\mathcal{C}}(L(\lambda), \nabla(\gamma)) \cong \mathrm{Ext}^1_{\overline{\mathcal{C}}}(L(\overline{\lambda}), L(\overline{\gamma}))$$

in that case. Note this holds also when λ, γ are not able to be chosen in the same W_p orbit, since $\overline{\lambda}, \overline{\gamma}$ are not related, if such a choice is not possible.

The simple Kazhdan-Lusztig property ($\star$) in §1 holds for all $\lambda \in \Lambda$, by the Hypothesis and [CPS AKL]. Thus, $\mathrm{Ext}^1_{\mathcal{C}}(L(\lambda), \nabla(\gamma)) \cong \mathrm{Ext}^1_{\mathcal{C}}(L(\lambda), L(\gamma))$, if $\lambda > \gamma$. since the dimension of the latter group is, at most, that of $\mathrm{Ext}^1_{\overline{\mathcal{C}}}(L(\overline{\lambda}), L(\overline{\gamma}))$ here, by Proposition 6, we obtain the conclusions of Propositions 7′ and 8 in this case $\lambda > \gamma$. We also get the first surjectivity in Proposition 7, and the second surjectivity holds because its target is zero.

If $\lambda < \gamma$, dual arguments apply. It remains only to consider the case where λ, γ are unrelated. Here each group $\mathrm{Ext}^1_{\mathcal{C}}(L(\lambda), \nabla(\gamma))$, $\mathrm{Ext}^1_{\mathcal{C}}(\Delta(\lambda), L(\gamma))$, and $\mathrm{Ext}^1_{\mathcal{C}}(L(\lambda), L(\gamma))$, is zero. The same is true if λ, γ are interchanged. Thus, $\mathrm{Ext}^1_{\overline{\mathcal{C}}}(L(\overline{\lambda}), \nabla(\overline{\gamma})) = 0$, by applying the displayed isomorphism, and duality for G-modules. It follows easily that $\mathrm{Ext}^1_{\overline{\mathcal{C}}}(L(\overline{\lambda}), L(\overline{\gamma})) = 0$. This gives the vanishing required in Proposition 7′, since λ, γ are unrelated whenever $\overline{\lambda}, \overline{\gamma}$ are unrelated. We also obtain the isomorphism in Proposition 8, now, in all cases. The isomorphisms in Proposition 7′ also hold, now, since they both hold with both sides zero, when λ, γ are unrelated. This completes the proof of Propositions 7, 7′, and 8 in all cases. Q.E.D.

Remark 4. Each of the results just proved would be derivable from Conjecture 1, if it were true (and we would also have that each group $\mathrm{Ext}^1_{\overline{\mathcal{C}}}(\Delta(\tau), L(\eta))$, $\mathrm{Ext}^1_{\overline{\mathcal{C}}}(L(\tau), \nabla(\eta))$, $\mathrm{Ext}^1_{\overline{\mathcal{C}}}(L(\tau), L(\eta))$ is zero, when $\ell(\tau) \equiv \ell(\eta) \bmod 2$). See [CPS AKL] and [CPS Penn St]. Moreover, the latter paper shows that Proposition 7 or Proposition 7′ (or Proposition 8, given our Hypothesis on Λ) is sufficient to prove 1, if sufficient "Hecke operators" can be found. While naive adjoint constructions to not give Hecke operators in the singular weight case, it seems plausible that some variation might succeed. We will not attempt more detailed speculation here, but instead indicate a nice consequence of the conjecture. The left-hand sides of each part (1), (2), (3) below all have dimensions that can be calculated from Kazhdan-Lusztig polynomials of [CPS AKL]. Also, note that part (1) has already been proved in a special case, in Proposition 8, without the use of Conjecture 1.

Theorem 3. *Suppose Conjecture 1 is true. (We still assume the Hypothesis.) Suppose $\lambda, \gamma \in \Lambda$, with $\gamma \in W_p \cdot \lambda$, and $T(L(\lambda)) \neq 0$, $T(L(\gamma)) \neq 0$. Then*

(1) $\mathrm{Ext}^1_{\mathcal{C}}(L(\lambda), L(\gamma)) \cong \mathrm{Ext}^1_{\overline{\mathcal{C}}}(L(\overline{\lambda}), L(\overline{\gamma}))$.

(2) The maps

$$\mathrm{Ext}^n_{\mathcal{C}}(L(\lambda), L(\gamma)) \to \mathrm{Ext}^1_{\overline{\mathcal{C}}}(L(\overline{\lambda}), L(\overline{\gamma}))$$

are surjective, for each $n \geq 0$.

(3) The maps

$$\begin{aligned} \mathrm{Ext}^n_{\mathcal{C}}(\Delta(\lambda), L(\gamma)) &\to \mathrm{Ext}^n_{\overline{\mathcal{C}}}(\Delta(\overline{\lambda}), L(\overline{\gamma})), \\ \text{and } \mathrm{Ext}^n_{\mathcal{C}}(L(\lambda), \nabla(\gamma)) &\to \mathrm{Ext}^n_{\overline{\mathcal{C}}}(L(\overline{\lambda}), \nabla(\overline{\gamma})), \end{aligned}$$

are surjective, for each $n \geq 0$.

Proof. We have $\dim_k \mathrm{Ext}^1_{\mathcal{C}}(L(\lambda), L(\gamma)) \leq \dim_k \mathrm{Ext}^1_{\overline{\mathcal{C}}}(L(\overline{\lambda}), L(\overline{\gamma}))$ by Proposition 6 (and the proof even shows the natural map from the first Ext group to the second is injective, though we need only the inequality). So, to prove (1), its is sufficient to prove (2), then apply the case $n = 1$. To prove (2), it is sufficient to prove (3), by the argument of [PS, Feit volume]. The latter is formulated for a quotient functor, more precisely, for the projection functor associated with a Levy subgroup, and designed with a catalyzing result of Hemmer [H] in mind. Nevertheless, the arguments apply here. The results they establish for Levy subgroup projections, together with the common framework of the previous section for T, inspired the above theorem.

It remains to prove (3). We first need a formality regarding adjoints. Obviously, we may assume Λ is contained in a single W_p orbit $W_p \cdot \lambda^-$. Recall that $\overline{\Lambda}$ is contained in $W_p \cdot \tau^-$. As discussed above (4.2), the functor $T^{\tau^-}_{\lambda^-}$ has a (left and right) adjoint $T^{\lambda^-}_{\tau^-}$ at the G-module level. (We continue to only work with finite-dimensional modules.) If M, N are G-modules, it is a formality that, for any $f \in \mathrm{Hom}_G(M, N)$, the adjoint $\mathrm{adj}(T^{\tau^-}_{\lambda^-} f) : T^{\tau^-}_{\lambda^-} T^{\lambda^-}_{\tau^-} M \to N$ may be expressed as the composite of f and $\mathrm{adj}(T^{\tau^-}_{\lambda^-} id_M) : T^{\lambda^-}_{\tau^-} T^{\tau^-}_{\lambda^-} M \to M$. Here $id_M : M \to M$ is the identity map. Consequently, the composite

$$\mathrm{Hom}_G(M, N) \to \mathrm{Hom}_G(T^{\tau^-}_{\lambda^-} M, T^{\tau^-}_{\lambda^-} N) \cong \mathrm{Hom}_G(T^{\lambda^-}_{\tau^-} T^{\tau^-}_{\lambda^-} M, N)$$

agrees with the map

$$\mathrm{Hom}_G(\mathrm{adj}(T^{\tau^-}_{\lambda^-} id_M), N): \ \mathrm{Hom}_G(M,N) \to \mathrm{Hom}_G(T^{\lambda^-}_{\tau^-} T^{\tau^-}_{\lambda^-} M, N).$$

By taking N injective in a suitably large highest weight category, we can replace the Hom with Ext^n and obtain the same agreement.

Apply this for $M = \Delta(\lambda)$, $N = L(\gamma)$, and use (4.2), together with the assumed validity of Conjecture 1. The map

$$\mathrm{Ext}^n_G(T^{\lambda^-}_{\tau^-}\Delta(\overline{\lambda}), L(\gamma)) \to \mathrm{Ext}^n_G(\Delta(\lambda s), L(\gamma))$$

is zero for each n, since $\ell(\lambda s) = \ell(\lambda) + 1 = \ell(\overline{\lambda}) + 1$ and $\ell(\overline{\gamma}) = \ell(\gamma)$. (Note that $\mathrm{Ext}^n_G(T^{\lambda^-}_{\tau^-}\Delta(\overline{\lambda}), L(\gamma)) \cong \mathrm{Ext}^n_G(\Delta(\overline{\lambda}), L(\gamma))$.) Thus, the adjunction map $\mathrm{Ext}^n_G(\Delta(\lambda), L(\gamma)) \to \mathrm{Ext}^n_G(T^{\lambda^-}_{\tau^-}\Delta(\overline{\lambda}), L(\gamma))$ is surjective. (Note that $\Delta(\overline{\lambda}) \cong T^{\tau^-}_{\lambda^-}$.) Consequently, by the agreement of Ext^n maps noted above, the map $\mathrm{Ext}^n_G(\Delta(\lambda), L(\gamma)) \to \mathrm{Ext}^n_G(\Delta(\overline{\lambda}), L(\overline{\gamma}))$ in (3) is surjective. (Note that is induced by $T^{\tau^-}_{\lambda^-}$.)

Dual arguments, which we leave to the reader, show also that the map $\mathrm{Ext}^n_G(L(\lambda), \nabla(\gamma)) \to \mathrm{Ext}^n_G(\overline{\lambda}), \nabla(\overline{\gamma})$ in (3) is surjective. This completes the proof of the theorem. Q.E.D.

Remark 5. One cannot deduce a similar result for smaller facets than walls just from the validity of Conjecture 1 for them, should it be true. The reason is that $T^{\lambda^-}_{\tau^-}\Delta(\overline{\lambda})$ is much larger when τ^- belongs to a smaller facet. However, this difficulty might be partly overcome by a suitable bootstrap approach, taking λ^- also in a smaller facet (though the outward translations are still too large to be used naively).

Remark 6. All of the results and discussions (and questions and conjectures) carry over when "G modules" are replaced by "type 1 integrable U_ζ" modules, where U_ζ is the quantum enveloping algebra in char. 0 at a p^{th} root ζ of unit, p a positive integer, associated to the same root system as G. Here the Lusztig character formula is known (apparently) for $p > h$, the Coxeter number. See Tanisaki [T], and also [ABG].

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