LINEAR AND NONLINEAR GROUP ACTIONS, AND THE NEWTON INSTITUTE PROGRAM

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One of the simplest and most useful notions in mathematics is that of a group action: if G is a group and X is a nonempty set, then an *action of G on X* (or a *G-set* structure on X) consists of a multiplication operation $G \times X \to X$, with the image of a pair (g, x) written as, say, gx, with the following axioms satisfied:

(1) $1x = x$ for all $x \in X$ (here $1 \in G$ is the identity element of G);
(2) $(gh)x = g(hx)$ for all $g, h \in G$ and all $x \in X$.

Equivalently, a group action determines, and is determined by, a homomorphism $g \mapsto [x \mapsto gx]$ of G into the group of all bijective maps of X onto itself (the 'symmetric' group of all *permutations* of X). If X has the additional structure of a linear space over a field, we might impose an additional axiom that the mappings $[x \longmapsto gx]$ all be linear. In this case, we say the action of G is *linear*, or that it defines a *linear representation of G,* and we call X a *module* for G over k. In the linear or nonlinear case, if G and X have some topological structure, we often require that the map $G \times X \longrightarrow X$ be continuous, or in the algebraic geometry case, a morphism of algebraic varieties or schemes.

A main theme of the Newton Institute program[1] on the representation theory of algebraic and related finite groups, of which this conference volume is a part, may then be stated as follows: even in the simplest case where G is a finite group, and X is a finite set with no linear space or algebraic variety structure, there are, nevertheless, deep connections of these nonlinear actions with linear actions, and of the discrete theory of finite groups and their actions with the theory of algebraic groups and other groups and algebras arising in continuous Lie theory. I hope to explain these connections

[1]The author would like to thank the NSF for its support and the Newton Institute for its hospitality.

in the next few pages. The discussion will involve us in the nondefining and defining characteristic linear representation theory of finite groups of Lie type, and the theory of maximal subgroups, all major research topics in the Newton Institute program. I will also give a detailed discussion of the Lusztig conjecture, and recent progress on it, another central topic of the program.

To begin our picture, let us first consider the general action of a finite group G on a finite set X. To understand all such actions, it is obviously enough to understand the transitive case, because we can decompose X in terms of orbits. We are lucky here–if we were dealing with continuous actions, we would have to worry quite hard about how the orbits were stuck together.

Next, every student of group theory knows that a transitive action of a finite or discrete group G is isomorphic to one arising from the action of G on the left cosets G/H of one of its subgroups H. That is, $X \simeq G/H$ in the language of G-sets. The subgroup H is uniquely determined in G up to conjugacy. Given any such pair G and H, we can always interpolate any subgroup M of G containing H, and obtain some kind of reduction to the study of the pairs G, M and M, H. This reduction is not perfect, because we still have further work to do in understanding precisely the G-conjugacy classes of subgroups H from this data, even when we know the corresponding information for each pair. Nevertheless, it is a good start, so it is very natural to regard pairs G, M with M maximal in G as the basic building blocks for understanding transitive finite group actions. These are in turn the ingredients of all finite group actions, as noted above.

Next, the paper [5] gives an understanding of such pairs G, M (with G finite and M maximal in G) in terms of certain basic constructions. Often these involve much smaller groups than G, and/or more accessible information. For example, one might require the exact structure of the outer automorphism group of a simple group H in order to understand all diagonal embeddings of H in $H \times H$, each yielding a maximal subgroup of the latter. There are also more sophisticated examples,[2] but there are only two cases where no 'reduction' to a smaller or more tractable problem is possible. Let me loosely state this as a theorem:

Theorem 0.1 (Aschbacher and Scott [5]) *The determination up to conjugacy of all pairs G, M where G is a finite group and M is a maximal subgroup of G, reduces, modulo smaller or easier problems, to the following cases:*

[2]The reader might try to find a maximal copy of the alternating group on six letters inside the semidirect product of this group with a direct product of six (permuted) copies of the alternating group on five letters. There is one, easily understood with a little nonabelian 1-cohomology theory.

*(1) G is almost simple (*meaning that G is a group sandwiched between a simple finite group and its automorphism group*);*

*(2) $G = H.V$, a semidirect product of a quasisimple finite group H and one of its irreducible modules V over a field of p elements, and M is a complement to V. (*Recall that a quasisimple group is one which is perfect and simple modulo its center.*) The conjugacy classes in G of such maximal subgroups M are parameterized by the first cohomology group $H^1(H,V)$.*

Some permutation group theorists, focusing on the transitive case, try to define away the cohomology problem in case (2) by noting that all pairs G, M which arise there are at least conjugate by an automorphism of the group G. However, if we want to parameterize all intransitive actions, we cannot allow such equivalences, which need not be preserved under disjoint unions of G-sets. In any case, we must know all irreducible modules for H before knowing all pairs G, M with M maximal in G, and, to properly parameterize such pairs, we need to know the 1-cohomology of these irreducible modules.

More surprising perhaps is that case (1) is also intimately related to the problem of finding and understanding irreducible linear representations of quasisimple finite groups. Roughly speaking, the nonobvious maximal subgroups of a finite classical group all arise from other groups embedded irreducibly on the underlying natural module. A somewhat more precise statement, but still a paraphrase, is

Theorem 0.2 (Aschbacher [4]) *Let G be a finite classical group associated to a vector space V, and M a maximal subgroup of G. Then one of the following holds:*

(1) M belongs to a natural small list of 'suspects'; or

(2) M is the normalizer in G of a quasisimple subgroup H which acts irreducibly on V.

The exceptional groups of Lie type were the subject of many results at the conference, and much progress has been made, for example, by D.M. Testerman [98] and by J.-P. Serre [90], as explained by the latter author in several lectures in the NATO conference underlying this volume. Other excellent sources are Liebeck [67] and Liebeck's article [68] in these conference proceedings. (I would like to thank Jan Saxl for these references.) The prototype theorem of the above type was given by O'Nan and Scott in [85] for alternating groups (see [5] for corrections), and results for classical groups along the lines of the above theorem were suggested. In the alternating groups case, the simple groups in part (2) act primitively in the natural permutation representation (that is, with maximal point stabilizer), giving a kind of recursion for the maximal subgroups problem. Generally, all 'suspects' in the classical and alternating groups cases have now been convicted (shown to be maximal subgroups). See, for instance, the work of Liebeck, Praeger and Saxl [69], and the surveys of Saxl [84] and Seitz [89]. For a modern restatement of Theorem 1, see Liebeck, Praeger and Saxl [70], and also Kovács [65]. The paper [85], based on my talk at the 1979 Santa Cruz conference, perhaps marks the beginning of the modern day maximal subgroups program in finite group theory. For a revisiting, see Seitz [88]. The present paper is also a revisiting, since most topics we discuss here (including nondefining characteristic) were already mentioned in some early form in [85].

1. Linear Representations in the Defining Characteristic

The two theorems above leave linear representation theory the problem of finding all irreducible representations of all finite quasisimple groups in all characteristics. Some maximal subgroup theorists have perhaps wished that instead one could use maximal subgroup theory to find irreducible representations. While there are some results in that direction [97], realistically, the irreducible representations must be found to advance maximal subgroup theory.

For quasisimple finite groups $G(q)$ of Lie type over a finite field $\mathbb{F}_q$, there are two cases: representations in characteristic p dividing q, called the defining or describing characteristic case, and characteristic p not dividing q, called the nondefining or nondescribing characteristic case. We will take p to be the defining characteristic first. Though the above applications require, in any characteristic p, irreducible representations over all finite fields of characteristic p, it is an easy process to pass to these from representations over an algebraically closed field k of characteristic p, such as $k = \bar{\mathbb{F}}_q$. Sticking to that case, we have the following fundamental result of Steinberg, dating from 1963:

Theorem 1.1 (Steinberg [92, 93], or, alternatively, [94]) *All irreducible representations of $G(q)$ over k may be obtained by restriction from an irreducible representation of the ambient algebraic group $G{=}G(k)$.*

The irreducible representations are meant in the sense of algebraic groups. That is, if V is the underlying vector space, then the action $G \times V \to V$ is required to be a morphism of algebraic varieties. Steinberg's theorem is even more substantive than we have stated here, since he describes precisely the irreducible representations for G which are needed, in terms of the Chevalley highest weight parameterization. In the case of an untwisted group $G(q)$, it is just the irreducible modules whose high weights have fundamental weight coefficients at most $q - 1$. Even those with (nonnegative and integral) coefficients at most $p-1$ (the *restricted* weights) are enough, when their twists through powers of the Frobenius automorphism (p) are used in conjunction with Steinberg's tensor product theorem. In particular, Steinberg's work completely throws the defining characteristic problem into the world of algebraic groups.

One should not think, however, that this solves the problem. Though there is a parameterization of the irreducible modules for G by their highest weight, cf. [54], much as in the classical theory of semisimple Lie algebra representations over $\mathbb{C}$, there is no 'Weyl character formula' as there is in the latter theory, cf. [52]. We do not even know the degrees (dimensions for V) of the irreducible representations.

There is, however, a conjectured character formula due to Lusztig, provided p is at least as large as the Coxeter number h (which is n if G is the special linear group $SL_{n,}$ and in general has a root system definition). Let me state it for those familiar with basic root and weight terminology, say from Humphreys [52], after some background from Jantzen [54]. I have included much detail here, in keeping with a goal of the conference to clarify the conjecture and progress on it. The next paragraph, which is a kind of quick mini-course in algebraic group representations, sets the stage, and is intended to give the reader a better feeling for the place the conjecture occupies in the general theory. I also have in mind those in finite group representations who might be interested in possible future analogs of the Morita equivalences and 'isometries' involved. (The reader wishing to see only a correct statement of the conjecture may wish to skip this paragraph.)

First, the "linkage principle" says that irreducible modules $L(\lambda)$ indexed by weights λ in distinct orbits $W_p.\lambda$ under a certain "dot action" (see below) of the affine Weyl group W_p belong to distinct blocks. Here the affine Weyl group is the semidirect product of the Weyl group and p-multiples of translations by roots in their underlying Euclidean space. The 'block' terminology is used in the same way as in finite group theory. Thus, two indecomposable modules are in the same block precisely when they are the ends of a chain of indecomposable modules, with adjacent terms having common composition factors. Next, if $p \geq h$, the irreducible modules in the principal block are indexed by the dominant weights in $W_p.0$. (The 1-dimensional trivial module is $L(0)$; the condition $p \geq h$ insures the weight 0 is inside the 'lowest p-alcove' in the Euclidean space. This convex open region is defined by the hyperplanes orthogonal to roots and passing through the negative sum $-\rho$ of all fundamental weights, and through the hyperplane $(x+\rho, \check{\alpha}_0) = p$, where $\check{\alpha}_0$ is the dual $2\alpha_0/(\alpha_0, \alpha_0)$ of the maximum short root α_0. The "dot action" can now be made explicit, using ρ, as

$$w.x = w(x+\rho) - \rho.)$$

If we think of a character formula as expressing irreducible modules in a block in terms of the Weyl modules in that block (which are reductions of known irreducible characteristic 0 modules), then any character formula for all irreducible modules in the principal block now gives one for all other blocks. This is Jantzen's translation principle: there is a functor (called 'translation') from modules with composition factors irreducible modules $L(w.0)$, with $w.0$ dominant, to modules with composition factor $L(w.\tau)$, for any integral weight τ in the closure of the lowest p-alcove. If τ is in the interior, the functor is even invertible (defines an equivalence of categories). In any case, it takes an irreducible module $L(w.0)$ to either 0 or the corresponding irreducible module $L(w.\tau)$, the latter occurring precisely when w has minimum length among all $v \in W_p$ in any expressions $v.\tau = w.\tau$. The translation functor always takes the Weyl module $\Delta(w.0)$ to $\Delta(w.\tau)$. All dominant weights μ lie in some orbit $W_p.\tau$ for some τ in the closure of the lowest p-alcove.

We need just one more bit of terminology: The *Jantzen region* $\mathbf{J}{=}\mathbf{J}_p$ consists of all dominant integral weights μ satisfying $(\mu{+}\rho, \check{\alpha}) \leq p(p{-}h{+}2)$. Its significance is that, if we write $\mu = \mu_0{+}p\mu_1$ with μ_0, μ_1 dominant integral and μ_1 restricted, and $L(\mu) = L(\mu_0) \otimes L(\mu_1)^{(p)}$ using the Steinberg tensor product theorem, then $L(\mu_1) \cong \Delta(\mu_1)$. The reason is that μ_1 is forced to be so small that it is in the closure of the lowest p-alcove, thus minimal among the dominant weights in $W_p.\mu_1$, while it is maximal among weights appearing in composition factors of $\Delta(\mu_1)$. Thus, the character of $L(\mu)$ is effectively determined by the character of the corresponding 'restricted' irreducible module $L(\mu_0)$, if $\mu \in \mathbf{J}$. We can now state Lusztig's conjecture.

Conjecture 1.2 (Lusztig [71]) *If $p \geq h$, we have*

$$\operatorname{ch} L(w.0) = \sum_{y.0 \text{ dominant}} (-1)^{\ell(w)-\ell(y)} P_{yw_0, ww_0}(1) \operatorname{ch} \Delta(y.0)$$

whenever $w \in W_p$ and $w.0 \in \mathbf{J}$.

Here the expressions $P_{w_0 y, w_0 w}(1)$ are values at 1 of Kazhdan-Lusztig polynomials [59]. The function ℓ on the Coxeter group W_p gives the minimum length in terms of fundamental reflections. The element w_0 is the long word in the ordinary Weyl group $W \subseteq W_p$, and y ranges over all elements of the affine Weyl group W_p with $y.0$ dominant, as indicated. (It is enough to take elements $y \leq w$ in the Bruhat-Chevalley order.) The 'character' operator ch may be interpreted as passage to the Grothendieck group of G-modules, or passage to a list of all weight space dimensions. Using the remarks above, we may also, equivalently, state the conjecture for any dominant weight $\lambda \in \mathbf{J}$ of the form $w.\tau$ with τ in the closure of the lowest p-alcove, and w chosen of minimal length in representing λ as $w.\tau$. That is,

$$\mathrm{ch}\, L(w.\tau) = \sum_{y.0 \text{ dominant}} (-1)^{\ell(w)-\ell(y)} P_{yw_0, ww_0}(1)\, \mathrm{ch}\, \Delta(y.\tau).$$

When $p \geq 2h-3$, the Jantzen region $\mathbf{J}$ includes all restricted dominant weights, and, together with Steinberg's tensor product theorem, the above formula gives all characters of all irreducible G-modules. Actually, for some time after the conjecture appeared, it was widely assumed that the Jantzen region included all restricted weights just under the assumption of the conjecture that $p \geq h$. Kato [58] wrote down a formula equivalent to the above, conjectured it was true when $p \geq h$ for all restricted weights, and then said that his conjecture would follow if the Lusztig conjecture were true. However, the arithmetic doesn't work out that way. Nevertheless, it *is* quite reasonable to make such a conjecture, and researchers in the area have retained it, calling it *Kato's conjecture* or *Kato's extension of the Lusztig conjecture*. If it is true, then the above formula for $\lambda = w.\tau$ restricted, together with Steinberg's tensor product theorem, would give character formulas for all irreducible G modules, when $p \geq h$. To emphasize, this would give, by restriction, the characters of all irreducible $G(q)$-modules over k.

For variations in the indexing, see [18], and for an introductory discussion of the homological significance of the polynomials, see [86]. It is now understood that the general form of the character formula is to be expected in a wide variety of contexts involving finite-dimensional algebras, with homologically defined polynomials appropriate to the algebra.

For p very large (depending on the root system associated to G, explicit size unknown), the Lusztig conjecture is known to be true. This is not an easy theorem. It depends on four papers [61, 62, 63, 64] of Kazhdan and Lusztig (five, counting [60]), two of Kashiwara and Tanisaki [56, 57] (see also Kumar [66] and Tanisaki [96]) and the following theorem of Andersen, Jantzen and Soergel:

Theorem 1.3 (Andersen, Jantzen and Soergel [1]) *Let G be a semisimple algebraic group over an algebraically closed field k of characteristic p, with root system Φ. Let $\mathcal{U}_\mathcal{Z}$ denote the associated quantum enveloping algebra (in the sense of [73, 2]) over $\mathcal{Z} = \mathbb{Z}[v]_{(v-1)}$ with v a pth root of unity (the specialization of the indeterminate v in the notation of [2]).* (Thus, the algebra $\mathcal{U}_k = (\mathcal{U}_Z)_k$ is very close to the hyperalgebra (divided power enveloping algebra) of G, and all finite-dimensional G-modules are $\mathcal{U}_k$-modules.) *Then, for $p \gg 0$, depending on Φ, each finite-dimensional irreducible G-module $L = L(\lambda)$, with λ restricted, is isomorphic to L_k for some $\mathcal{Z}$-free $\mathcal{U}_\mathcal{Z}$-module $\tilde{L}$.*

The point is that character formulas for the finite-dimensional irreducible modules for $\mathbb{C} \otimes \mathcal{U}_{\mathbb{Z}}$ are given by the work cited of Kazhdan and Lusztig (relating quantum enveloping algebras and affine Lie algebras) and Kashiwara and Tanisaki (relating affine Lie algebras and perverse sheaves, where 'character' formulas were known, thanks to Kazhdan and Lusztig [60]). The result of all this is that the conjectured Lusztig character formula holds for the restricted irreducible L, and the restricted case is sufficient for all the others.

All the ingredients here, including the above result of Andersen, Jantzen and Soergel, those of Kazhdan and Lusztig, and those of Kashiwara and Tanisaki, had been conjectured by Lusztig in an elaborate program (see [76], and also [72, 74, 75, 78]). He and Kazhdan provided in their papers an essential link from quantum enveloping algebras to affine Lie algebras. It should be said that Andersen, Jantzen and Soergel work just as hard, and do introduce new ideas on their part, such as a genuine application of higher-dimensional deformation theory. Their paper stands alone as a fine result ("independence of p"), though the 'very large prime' condition is unfortunate. Similarly, Kashiwara and Tanisaki proved standing conjectures for affine Lie algebras, and had to overcome unexpected complications in the non-simply laced case. An earlier attempt (now revised) by Casian [15], [16] was incomplete. Altogether, these papers tell us that the Lusztig conjecture is "true", in the sense that some underlying principle has been vindicated, and the conjecture is not wildly false! For other approaches to proving the Lusztig conjecture, along with some reductions, see my surveys of the work of Cline, Parshall and Scott, e.g. [19], in [86, 87].

In terms of concrete answers, we only know the Lusztig conjecture is true for G of rank 2 or less, or type A_3, cf. [54]. The conjecture is even open for the case A_4 (the group $SL(5, k)$) for the prime $p = \operatorname{char} k = 5$. As I reported at the Newton Institute's NATO conference, I have been working with some University of Virginia undergraduates, Mike Konikoff and Chris McDowell, under the NSF REU program (Research Experience for Undergraduates) to settle this case by computer calculations. This case is also the first opportunity to distinguish between the Lusztig and Kato conjectures. I hope the answers will be available by the time this volume goes to press. Our approach would also be practical for $p = 7$, but the case $p = 11$ is probably beyond reach, since our methods involve determining maximal or minimal submodules in 'baby Verma modules' for the associated Lie algebra. These modules have dimension p^N, where N, the number of positive roots, is 10 in this case. Thus, the dimension p^N of the underlying vector space would be over 25 billion for $p = 11$, while it is 'only' about 10 million for $p = 5$, and is just 1024 for $p = 2$. It is also true, generally, that computer calculations tend to increase in difficulty with the size of the field, and all these calculations are carried out in the field of p elements. With the combination of these influences, the $p = 2$ case is much more tractable, and, using methods of Nanhua Xi similar to ours, Jia Chen Je has worked out by hand the characters of all irreducible modules for $SL(5, k)$ and even $SL(6, k)$ when $\operatorname{char} k = 2$, cf. [99]. The case of $SL(5, k)$ here, for $\operatorname{char} k = 2$, had previously been treated by Dowd and Sin by computer, along with the other semisimple groups of rank at most 4 in this characteristic [35]. This computational phenomenon, of easier calculations for smaller primes, is one reason why it would be nice to have a result like the original Lusztig conjecture, or, better, the Kato conjecture, that would tell us we do not have to compute for too many primes.

For type A there is yet another conjecture, due to Gordon James, which, together with the work of Kazhdan and Lusztig and of Kashiwara and Tanisaki above, would give character formulas for some irreducible modules for all primes, gradually getting more and more character formulas as the prime gets larger. James' conjecture has the advantage also of applying in the nondefining case, which we take up now.

2. Linear Representations in Nondefining Characteristics

The theory here exists in a state as satisfactory as the defining characteristic presently only for type A, where it is largely due to Dipper and James [31, 53], though there is much more to the history. There was considerable help, at least conceptually, from the block determinations by Fong and Srinivasan [45], which was a major step. Some earlier work had been done by Olssen [80], and by Curtis's student R. Boyce [8], the latter partly inspired by unpublished results of Curtis and myself from 1974. As recorded in my Santa Cruz talk [85], Alperin once expressed to me the rather prophetic view that a nondefining characteristic theory for a finite general linear group $GL(r, q)$ should parallel the representation theory of the symmetric group, because of the similarities in Sylow groups. I recorded his remarks because they confirmed my own prejudices, and they were the best 'authoritative' support for them I could find at the time. The future existence of a viable nondefining characteristic theory was a prerequisite for the maximal subgroups program I was proposing in [85]. Fong later mentioned my Santa Cruz talk and its Alperin reference in his talk at Arcata, though Srinivasan emphasized at the 'nondefining' conference here a problem of Feit as inspiring her work with Fong, cf. [44].

Aside from the work of Dipper and James, another treatment of the original work of Fong and Srinivasan was given by Broué [9]. It should be mentioned that his program [10] for obtaining local derived equivalences for blocks with abelian defect groups (achieved at the character isometry level for all finite groups of Lie type, cf. [13]) is at least relevant to the problem of finding Brauer characters. Also, there have been many interesting by-products of efforts to prove the conjectured derived equivalences, some discussed at this conference, including a theory of cyclotomic Hecke algebras [11] and generic blocks [12]. Nevertheless, the program would have to go quite far to get the information we seek for irreducible modules in positive characteristic. Ostensibly, this would require explicitly exhibiting derived equivalences, though possibly some more efficient approach might be found, and would require generalizing the program to the nonabelian case. Of course, many mathematicians working with blocks are more concerned with arithmetic questions and understanding ordinary character values, rather than the decomposition number and irreducible module questions that concern us.

I will mention some other approaches to blocks and decomposition numbers for more general finite groups of Lie type at the end of this section, but for now, let us return to $GL(r,q)$ and the theory of Dipper and James.

It is best to begin by describing the q-Schur algebra. A version of it had been invented in physics, by Jimbo [55], leading to the development of quantum groups. However, the development due to Dipper and James appears to be entirely independent, and certainly Jimbo had no idea of applications to finite group representations. The q-Schur algebra is so natural from this perspective that it is fair to say that, if the physicists had not invented quantum groups, the finite group theorists surely would have.

To define the q-Schur algebra, we first need the generic Hecke algebra (also called the Iwahori-Hecke algebra). In our case, there is one of these for each positive integer r. It is an algebra $\mathcal{H} = H(r)$ over $\mathbb{Z}[q]$, which is free over $\mathbb{Z}[q]$, and, when q is specialized to any prime power, becomes the endomorphism algebra of the familiar permutation module $M = \mathrm{Ind}_{B(q)}^{G(q)}\mathbb{Z}$. Here $B(q)$ denotes the Borel subgroup (invertible upper trianglular $r \times r$ matrices) of $G(q) = GL(r,q)$. It is also true that $\mathcal{H}$ specializes to the group algebra over $\mathbb{Z}$ of the symmetric group $S = S_r$ when q is specialized to 1. (Actually, every Coxeter group has a natural associated generic Hecke algebra, sometimes with multiple parameters.) There are natural q-analogs $x_\lambda\mathcal{H}$ of the standard transitive permutation modules $\mathbb{Z}S/S_\lambda$ for S associated to any partition $\lambda = (\lambda_1, \ldots, \lambda_m)$ of r. Here x_λ is the sum of standard basis elements τ_w of $\mathcal{H}$ specializing to elements w of the subgroups

$$S_\lambda = S_{\lambda_1} \times \cdots \times S_{\lambda_m}.$$

The q-Schur algebra may now be defined as the endomorphism algebra

$$S_q(n,r) = End_{\mathcal{H}}T, \text{ where } T = \bigoplus_{|\lambda|\leq n} x_\lambda\mathcal{H}.$$

Here the notation $|\lambda| \leq n$ means that the partition λ has at most n parts. The notation T stands for 'tensor space': when the symmetric group acts on the r-fold tensor power $V^{\otimes r}$ of an n-dimensional vector space V, the action may be decomposed into a direct sum of transitive permutation modules, associated to partitions with at most n parts. There may be a number of isomorphic copies of any one of these transitive modules, whereas in T there is only one (q-analog). However, if we modified the number of copies of each $x_\lambda \mathcal{H}$ in T, we would only replace its endomorphism ring by a Morita equivalent version. Thus, the q-Schur algebra is, up to Morita equivalence, the endomorphism algebra of a q-analog of the true tensor space $V^{\otimes r}$. In the nondefining characteristic representation theory for the general linear group, we will most often encounter here the case $n = r$ (as contrasted to the defining characteristic case, where, generally, n is fixed and r varies).

To motivate the next construction, recall that, for any commutative ring R, there is a *Steinberg module* St_R for $GL(r, q)$ defined as homology of the largest degree term in the Solomon-Tits complex [91]. If R is a domain of characteristic 0 in which the prime power q is invertible, there is another definition, used by Dipper and James [31, 25]. Let $M_R = \mathrm{Ind}_{B(q)}^{G(q)} R$ as above, let $y = y_{(r)} \in \mathcal{H}_\mathcal{R}$ be the alternating sum

$$\sum_{w \in S} (-1)^{-\ell(w)} \tau_w,$$

and let $\sqrt{y M_R}$ denote the smallest pure submodule of M_R containing yM_R. (Here we are using 'pure submodule' in the sense of having a torsion-free quotient.) Then $St_R \cong \sqrt{y M_R}$. The advantage of this description is that it is an expression that makes sense for any R-torsion-free $\mathcal{H}_\mathcal{R}$-module M_R. Moreover, S could be any finite Coxeter group, such as a direct product of symmetric groups. This gives an immediate generalization $St_R^\lambda = \sqrt{y_\lambda M_R}$, by replacing $S = S_{(r)}$ in the definition of y with the subgroup S_λ associated to a partition λ. This module is just the Harish-Chandra induction (meaning inflation followed by the usual tensor induction, on a parabolic subgroup) of a tensor product of Steinberg modules associated to smaller general linear groups. It is also meaningful, when considering a direct product of Coxeter groups, to form their Hecke algebra with different powers of the parameter q, and this is useful in thinking about the general statement below.

Before stating the main result, it is at least conceptually useful to recall from [45] that the blocks of $GL(r, q)$ may be divided into families associated to conjugacy classes of semisimple p'-elements s: a block is in the family associated to s if it contains an ordinary character with semisimple parameter s. The characters with semisimple parameter 1 for $GL(r, q)$

are those associated to constituents of $\mathrm{Ind}_{B(q)}^{G(q)}\mathbb{C}$. They are called *unipotent characters*. A block containing one of them is called a *unipotent block*.

Now let k be an algebraically closed field of characteristic p not dividing q, and regard k as the quotient of a complete DVR $\mathcal{O}$ with quotient field K of characteristic 0 (to name just one familiar situation where the theory may be formulated). It is also convenient to take K to be a splitting field, meaning that all irreducible modules of G over K remain irreducible over extension fields.

The following theorem is a reformulation of results in Dipper and James [31]; see also Dipper [25], with further details in Dipper and Du [29, 30], Takeuchi [95], and Cline, Parshall and Scott [22, 23].

Theorem 2.1 (Dipper and James [31]) *The nonisomorphic irreducible modules over k of $GL(r,q)$ which lie in some unipotent block may be parameterized as $D(\lambda) = D(1,\lambda)$, with λ ranging over the partitions of r. Moreover, the character of D^λ may be expressed in terms of ordinary unipotent characters, with $\mathbb{Z}$-coefficients determined by a corresponding expression for the q-Schur algebras $S_q(r,r)_k$ and $S_q(r,r)_K$. Here we require in $\mathcal{O} \subseteq \mathcal{K}$ that the indeterminate q has been specialized to the prime power q. Base change from $S_q(r,r)_\mathcal{O}$ by k yields $S_q(r,r)_k$. The indecomposable $GL(r,q)$-components of $\bigoplus_\lambda \sqrt{y_\lambda} M_\mathcal{O}$ have simple heads, which are the modules $D(\lambda)$. Given such a component, view it is the image of an idempotent projection in $\mathrm{End}_{GL(r,q)}(\bigoplus_\lambda \sqrt{y_\lambda} M_\mathcal{O})$. Then the irreducible module of $S_q(r,r)_k$ corresponding to its head D^λ is the head of the projective indecomposable $S_q(r,r)_\mathcal{O} = \mathrm{End}_{S(r,r)_\mathcal{O}}(S_q(r,r)_\mathcal{O})^{op}$-module which is the image of a corresponding idempotent projection under a natural isomorphism*

$$S_q(r,r)_\mathcal{O} \cong End_{GL(r,q)}\left(\bigoplus_\lambda \sqrt{y_\lambda} M_\mathcal{O}\right).$$

Also, the correspondence between irreducible unipotent representations of $GL(n,r)$ over K and irreducible representations of $S_q(r,r)_K$ may be obtained from the K-base change of the above isomorphism.

Moreover, if s is any semisimple p'-element in $GL(r,q)$, then there is a parameterization $D(s,\boldsymbol{\lambda})$ of the irreducible modules over k of $GL(r,q)$ belonging to a block associated to s, with $\boldsymbol{\lambda}$ ranging over multipartitions (finite tuples of partitions) *associated to conjugacy classes of unipotent elements in the centralizer of s. Analogs of all the above formulas and properties hold for a suitable $GL(r,q)$-module $M_\mathcal{O}$ with $S_q(r,r)$ replaced by a tensor product $\bigotimes_i S_{q^{a_i}}(r_i,r_i)$ of q^{a_i}-Schur algebras (the usual parameter q is "specialized" to q^{a_i} in such an algebra). Here $\sum_i a_i r_i = r$, and s requires precisely r_i blocks of size a_i to write its rational canonical form over $\mathbb{F}_q$.*

An extended version of the theorem in terms of a Morita equivalence with a quotient $\mathcal{O}GL(r,q)/J(q)$ of the group algebra has been given in Cline, Parshall and Scott [23], and used there to derive cohomology results (especially 1-cohomology) in the spirit of the defining characteristic work of Cline, Parshall, Scott and van der Kallen [24]. Such a Morita equivalence had been given for unipotent blocks by Takeuchi [95] over k.

The q-Schur algebras $S_q(n,r)$ are homomorphic images of quantum enveloping algebras of type A, at least after adjoining a square root of q; see Parshall and Wang [81], Dipper and Donkin [28], Beilinson, Lusztig and MacPherson [7], Du [42], Du and Scott [39], Lusztig [77], and Donkin [34]. The above theorem thus provides in some ways an analog of Steinberg's result, Theorem 1 above, in the defining characteristic, reducing the irreducible module problem to a similar problem in Lie theory. One may next ask if there is an analog of Lusztig's conjecture. It is indeed possible to state an almost perfectly analogous conjecture. This comes to us from a completely independent conjecture of Gordon James in type A, stated for q-Schur algebras, and dealing with both defining and nondefining characteristics. We will concentrate on its implications in the nondefining case first. Here is a precise statement of the conjecture:

Conjecture 2.2 (James [53]) *Let $p > 0$ be a prime, and ζ a primitive e^{th} root of unity in a number field K with $pe > r$. If p divides e, assume $p = e$.* (Thus e is always the smallest nonnegative integer satisfying $1 + \zeta + \cdots + \zeta^{e-1} = 0$, in K or any characteristic p base change.[3]) *Let $\mathcal{O}$ be the localization of the integers in K at any prime ideal $\mathcal{P}$ containing p, and k the finite residue field $\mathcal{O}/\mathcal{P}$. Let $S_q(r)_{\mathcal{O}} = S_q(r,r)_{\mathcal{O}}$ denote the q-Schur algebra over $\mathcal{O}$ with q specialized to ζ. Then all $S_q(r)_{\mathcal{O}}$-lattices in irreducible $S_q(r)_K$-modules reduce modulo $\mathcal{P}$ to give irreducible $S_q(r)_k$-modules.*

The resulting irreducible $S_q(r)_k$-modules would be absolutely irreducible, and give all irreducible $S_q(r)_k$-modules $L_k(\lambda)$. Character formulas for the corresponding irreducible modules $L_K(\lambda)$ could be obtained from the character formulas for quantum enveloping algebras at an $\ell = e^{th}$ root of unity, available through the work previously mentioned of Kazhdan and Lusztig, and Kashiwara and Tanisaki. (This was not available to James at the time of his conjecture, but he speculated in [53] that the corresponding q-Schur algebra problem should be accessible.) These formulas here would be in terms of the characters of Weyl modules $\Delta_k(\lambda)$. These modules exist over any field, and are the irreducible modules of $S_q(r)_K$ over the field K of the previous theorem, where the parameter q was specialized to the prime power q. So, the formulas that would be obtained in this way not only

[3] James' statement was more informal and omitted the base change requirement on e. However, he has confirmed verbally that it was part of his intention.

are very much like those appearing in the defining characteristic Lusztig conjecture, but also are just what is needed to apply the previous theorem. For example, if James' conjecture is true for p, r, and e as above, and $pe > r$, we have the following formula in the Grothendieck group of $S_q(r)_k$-modules:

$$\mathrm{ch}\ L_k(\lambda) = \sum_{\mu} (-1)^{\ell(w_{e,\lambda})) - \ell(w_{e,\mu})} P_{w_{e,\mu}w_0, w_{e,\lambda}w_0}(1)\, \mathrm{ch}\, \Delta_k(\mu).$$

The elements $w_{e,\lambda}$ for partitions λ of r are elements of the affine Weyl group $W_e = W_{r,e}$, the semidirect product of the symmetric group S_r with e-multiples of $\mathbb{Z}$-linear combinations of type A roots. To define these elements, view the $(r-1)$-dimensional affine space on which W_e acts naturally as having a $\mathbb{Z}$-basis consisting of 'weights'. The latter may be regarded as algebraic group homomorphisms $\mathbb{C}^{\times(r-1)} \longrightarrow \mathbb{C}^{\times}$ with $\mathbb{C}^{\times(r-1)}$ viewed as diagonal $r \times r$ matrices of determinant 1, to capture the action of S_r. The 'roots' in this interpretation are, of course, the homomorphisms obtained from root subgroups of $GL(r, \mathbb{C})$. Each partition λ defines such an algebraic group homomorphism $\bar{\lambda}$: $diag(t_1, t_2, \ldots, t_r) \longmapsto \prod_i t_i^{\lambda_i}$. The element $w_{e,\lambda}$ is then the inverse of the shortest element conjugating $\bar{\lambda}$ to the closure of the lowest e-alcove in the 'dot' action we discussed in the previous section. The function ℓ is the usual length function, and the P's are Kazhdan-Lusztig polynomials.

Applying the previous theorem, this would give

$$\mathrm{ch}\ D(\lambda) = \sum_{\mu} (-1)^{\ell(w_{e,\lambda})) - \ell(w_{e,\mu})} P_{w_{e,\mu}w_0, w_{e,\lambda}w_0}(1) \chi_{1,\lambda}.$$

Here we are using the Fong-Srinivasan notation $\chi_{1,\lambda}$ for unipotent characters, and have used the same notation for their restrictions to p'-elements (regarding, say, characters in the above display as functions on p'-elements). These characters (and all ordinary characters) of $GL(r, q))$ are known from work of J.A. Green [48], though Fong and Srinivasan use the language of Deligne-Lusztig theory, cf. [14]. If one starts with a prime p not dividing the prime power q, and then picks the smallest positive integer e such that $1 + q + \cdots + q^{e-1} \equiv 0 \pmod{p}$, then James' conjecture applies. More precisely, for such p and e, with $pe > r$, the validity of James' conjecture implies the above formula. Similar, but slightly more involved, statements can be made for nonunipotent blocks:

$$\mathrm{ch}\, D(s, \boldsymbol{\lambda}) = \sum_{\mu} (-1)^{\ell(w_{\boldsymbol{e},\boldsymbol{\lambda}}) - \ell(w_{\boldsymbol{e},\boldsymbol{\mu}})} P_{w_{\boldsymbol{e},\boldsymbol{\mu}}w_0, w_{\boldsymbol{e},\boldsymbol{\lambda}}w_0}(1) \chi_{s,\boldsymbol{\lambda}}.$$

Here, if a_i, r_i are defined for the semisimple p'-element s as above, then positive integers e_i are defined as the smallest positive integers e_i with

$1+q^{a_i}+\cdots+q^{a_i(e_i-1)}=0 \pmod p$, and $\boldsymbol{e}$ is defined as the tuple $(e_1, e_2, \ldots)$. The elements $w_{\boldsymbol{e},\boldsymbol{\lambda}}$ are defined component-wise. The $P(1)$'s are values at 1 of Kazhdan-Lusztig polynomials for the product $W_{\boldsymbol{r},\boldsymbol{e}} = W_{r_1,e_1} \times W_{r_2,e_2} \times \cdots$, and are products of $P(1)$'s for the factors. This formula for $\operatorname{ch} D(s,\boldsymbol{\lambda})$ should hold, in the presence of the James conjecture, whenever $pe_i > r_i$ for all i.

Thus, in the type A nondefining characteristic modular representation theory, there are conjectured formulas very close in spirit and in much detail to the original Lusztig conjecture in the defining characteristic. These formulas are even a little better in one respect: they give formulas for some irreducible modules even when p is small. One may now ask, in the spirit of the work of Andersen, Jantzen and Soergel, what happens when p is very large? Here we keep r fixed and let q be arbitrary. We could fix e, but there are only finitely many real possibilities, since the James conjecture does hold in a very easy way when $e > r$. The answer, now, is similar to the defining characteristic answer, and much easier: Gruber and Hiss, using an observation of Geck, remark in [51] that the James conjecture is true for p very large. They obtain this result so easily, that they do not even bother to mention it in their introduction! For further details, and a discussion of related homological phenomena, see Cline, Parshall and Scott [22, 23].

Observation (Geck, Gruber and Hiss [51]) For a given r, the James conjecture is true for $p \gg 0$, depending on r.

As in the defining characteristic case, the required size of p is unknown. Nevertheless, we may say that, with the exception of a finite number of primes, the irreducible p-modular characters $\operatorname{ch} D(s,\boldsymbol{\lambda})$ are given by the above formula for a tuple $\boldsymbol{e}$ defined in terms of s and p as described above. The prime power q participates in the determination of $\boldsymbol{e}$, but, otherwise, the coefficients in the formula are independent of q, and the formula completely describes $\operatorname{ch} D(s,\boldsymbol{\lambda})$ in terms of the known ordinary characters.

Finally, for small ranks, the nondefining characteristic theory for type A is in much better shape that in the defining characteristic case. In [53] James has determined all the irreducible modular characters, in terms of ordinary characters, for primes p not dividing q, for all $GL(r,q)$ with $r \leq 10$.

Thus, all in all, the theory is at least as satisfactory in the nondefining characteristic for type A, as it is for the defining characteristic, and in some ways better. The one glaring assumption here is the restriction to type A. Work is now being done in a similar direction for other types, by Geck, Hiss, Gruber and Malle, cf. [46, 47, 51]. In the third of these papers, in type B, Gruber and Hiss have been able to adapt the Dipper-James program in the so-called 'linear prime' case, using results of [32]. These primes are odd, and the prime power q cannot be of even order modulo p. Unfortunately, this disqualifies a lot of primes. One reason for this assumption is to have some working analog of the q-Schur algebra. Such an algebra has been provided by Du and Scott, now, in [40], called the q-Schur^2 algebra, and there is work in progress [41] in type D. These algebras use what might be called the 'Murphy basis' philosophy [79, 33]. A different approach to the required Hecke endomorphism algebras, for all types, using Kazhdan-Lusztig bases, has been proposed by Du, Parshall and Scott [36]. For related work in type A, see [37] and [38]. The algebras suggested would have long 'stratifications' in the sense of Cline, Parshall and Scott [20], but would not quite be quasihereditary, in general. Nor is it likely they would have a cellular basis, in the sense of Graham and Lehrer [50]. Both the q-Schur algebras and q-Schur^2 algebras are quasihereditary, and the Hecke algebra Murphy bases lead to cellular bases for them. The Kazhdan-Lusztig bases do this in type A, but the basis they provide for the Hecke algebra is not cellular in type B or other types. An innovation of Du, Parshall and Scott [36] and Du and Scott [40] is to consider endomorphism algebras of larger modules than the usual direct sum of q-analogs of permutation modules associated to parabolic subgroups.

3. Relations Between the Defining and Nondefining Characteristic, and to Symmetric Groups. Quantum and Finite Dimensional Algebras

Beyond the conjectures of James and Lusztig, one would want the characters of irreducible modules for the smaller primes, in both defining and nondefining characteristic. Some previous discussion of analogies between the defining and nondefining characteristic problems has been given in Dipper [26, 27], but our focus here is more on actual relationships.

Surprisingly, for any prime p whatsoever, the complete solution of the nondefining characteristic problem in type A breaks naturally into two subproblems, one of which is the defining characteristic problem! In addition, the complete solution of the defining characteristic problem would also give the characters of irreducible modules of the symmetric groups. The latter result is due to James; an account may be found in Green's book [49]. A remarkable converse has recently been proved by Erdmann [43].

To explain the connection between the nondefining and defining characteristic problems, we first consider the better-known case where the prime p divides $q-1$. As Dipper and James have observed, character formulas for $GL(r,q)$ unipotent blocks in these cases would give character formulas for all q-Schur algebras in characteristic p with the parameter q specialized to 1—that is, for all Schur algebras in characteristic p. It is well-known [49] that this gives all irreducible representations for the algebraic groups $GL(n,k)$ over algebraically closed fields k of characteristic p.

The general case is not so well known, though it is in part discussed by James [53]. However, our remarks here bring that discussion up to date, making use of a tensor product theorem [74, 81, 29] described below. The first version in characteristic 0 was proved by Lusztig [74]. This result was rediscovered for type A by Du and Scott with an entirely different argument (unpublished, except for its treatment in [29]), which inspired the second and third papers. The third paper, by Dipper and Du, requires restrictions on neither the roots of unity involved nor the characteristic. The paper [81] by Parshall and Wang did require some restrictions, but, together with [29], inspired a similar result for all types of root systems in Andersen and Wen [3]. We will use the language of these latter authors, keeping in mind that their restrictions are not required for our type A case, We will also make use of the recent results of Kazhdan and Lusztig, and Kashiwara and Tanisaki. Let e be defined for p and q as in the James conjecture. That is, e is the smallest positive integer satisfying $1 + q + \cdots + q^{e-1} \equiv 0 \pmod{p}$. In other words e is the order of q modulo p, unless p divides $q-1$, in which case e is p. To determine character formulas for the irreducible modular representation of all $GL(n,q)$'s in characteristic p, with q fixed, is equivalent to determining character formulas for all the irreducible modules for the q-Schur algebras $S_q(n,r)$ for all r. These algebras are homomorphic images of quantum enveloping algebras (at least, after an extension by $q^{1/2}$), and their irreducible modules are irreducible modules for the latter. We may also assume the indeterminate q has been (first) specialized to a complex primitive e^{th} of unity ζ. (Upon further reduction 'modulo p' we may assume that ζ agrees with the prime power q.) Then there is a tensor product theorem which holds over both $k = \mathbb{Q}(\zeta)$ and $k = \mathbb{F}_p$. Using the algebraic and quantum groups SL_n-notation for dominant weights μ as nonnegative integer linear combinations of fundamental weights, we have, for quantum SL_n irreducible modules $L_{q,k}(\mu)$,

$$L_{q,k}(\mu) = L_{q,k}(\mu^0) \otimes L_k(\mu^1)^{Fr_e}$$

where μ has been written $\mu = \mu^0 + e\mu^1$ for an e-restricted weight (meaning its coefficients lie in the interval $[0, e-1]$), and Fr_e is the 'Frobenius' homomorphism taking the specialization of the quantum group at $q = \zeta$ to the specialization at $q = 1$. All q-Schur algebra or quantum GL_n-irreducible modules, usually parameterized by a partition λ, are expressible as the product of (an extension to quantum GL_n of) $L_{q,k}(\mu)$ and some m^{th} power of the determinant. (To go from μ to λ, add the last $n-i$ coefficients of λ to m to get the i^{th} part of λ. Alternatively, see the statement in [29] which is entirely in terms of partitions, but not phrased directly as a tensor product. Also, their indexing, with some justification, starts with -1 instead of the superscript 0 above.) By the previously cited work of Kazhdan and Lusztig, and Kashiwara and Tanisaki, the characters of the irreducible modules displayed above are all known in the $Q(\zeta)$-case. In each case, the module $L_k(\mu^1)$ is the irreducible module for the usual Kostant $\mathbb{Z}$-form, base-changed to k, of the universal enveloping algebra of the Lie algebra sl_n of $n \times n$ matrices. The relationship of the two versions of $L_k(\mu_1)$ for the two versions of k is completely equivalent, and essentially identical, to the main problem of the defining characteristic theory (in type A) of finding the composition factors of Weyl modules in characteristic p for the algebraic group SL_n. The dominant weight μ^1 is arbitrary, just as μ is arbitrary. When $\mu = e\mu^1$, we have $\mu^0 = 0$, and $L_{q,k}(\mu^0)$ is the 1-dimensional trivial module. Thus, the defining characteristic problem for the algebraic group SL_n in any characteristic p is part of the nondefining characteristic p problem for the finite general linear groups $GL(r,q)$, with r here ranging over all values $r \geq n$, and e fixed as above. The nondefining characteristic problem, however, has a second part, of decomposing the (now known) modules $L_{q,Q(\zeta)}(\mu^0)$ into composition factors after base-change (of a suitable lattice) to $\mathbb{F}_p$.

James' conjecture proposes a simple answer for this in case μ^0 is sufficiently small: if the partition λ associated to μ (or μ^0) has r as its sum of parts, and $pe > r$, then the base-change to $\mathbb{F}_p$ of $L_{q,k}(\mu)$ (or $L_{q,k}(\mu^0)$) is irreducible. Though James did not have available the work of Kazhdan and Lusztig, and Kashiwara and Tanisaki, he in some sense anticipated them, suggesting in [53] that the problem of determining the characters of all $L_{q,Q(\zeta)}(\mu)$ should be accessible with known methods. Then, he continued, the problem would come down to the more difficult issue of determining composition factors of the "reductions modulo p" of the modules $L_{q,Q(\zeta)}(\mu)$.

James' conjecture also has something to say about the defining characteristic. Consider the case where p divides $q-1$, so that $e = p$. Also, as remarked above, the q-Schur algebra over $\mathbb{F}_p$ is just the Schur algebra, whose irreducible modules give those of the algebraic group GL_n or SL_n. Here James' conjecture asserts that, if $p^2 > r$, where r is the sum of parts of the partition associated to μ, then the base-change to $\mathbb{F}_p$ of $L_{q,k}(\mu)$ is irreducible. This part of James' conjecture can be viewed as a kind of version of the Lusztig conjecture in the defining characteristic. Because of the work of Kazhdan and Lusztig, and Kashiwara and Tanisaki, this part proposes the same character formula as the Lusztig conjecture, though the exact range of weights for which such a formula is proposed to hold is different. Although in many cases the restrictions imposed by James' conjecture are stronger than in the Lusztig conjecture (cf. [87]), James' conjecture is far better for primes smaller than n for GL_n, where the Lusztig conjecture proposes no answer at all. In this case James' conjecture almost always proposes useful character formulas for some weights, no matter how small the prime.

James' conjecture is true for very large primes, size dependent on r, as noted above. There is also a result of Du and Scott [39] which establishes it for any $L_{q,Q(\zeta)}(\mu)$ for which the associated Weyl module ($q = 1$) has multiplicity-free 'reduction modulo p'. Some crude analogs of James' conjecture for other types were proved under a similar hypothesis. Unfortunately, except for type A, absolutely no connection is yet known between quantum enveloping algebras and the nondefining characteristic representation theory of finite groups of Lie type. One part of the type A theory which does presently have some analogs is the endomorphism algebra theory. That is, the finite dimensional q-Schur algebras in type A, in their enveloping algebra role, do appear to have reasonable analogs in other types. See, for instance, the work cited above of Du and Scott, and Du, Parshall and Scott, and the survey [22]. These papers involve quasihereditary algebras and stratified algebras [17, 20], both notions being algebraic concepts based in the geometry and derived category properties of perverse sheaves, cf. [82, 18, 83]. Whatever heroic transformations it takes to get there, all the deeper theorems on Kazhdan-Lusztig- (and Lusztig-) type character formulas reduce eventually to the case of perverse sheaves, where the properties one wants can at least be formulated (cf. [85, 86, 83, 21]) in terms of finite-dimensional algebras like these. Conceivably, one could understand these finite-dimensional algebras sufficiently well and never require quantum enveloping algebras, or the perverse sheaves themselves. It is also appealing to believe that some more direct algebraic theory—sufficiently deep, of course, and perhaps involving some form of quivers and relations—might eventually replace and improve upon our existing highly indirect understanding of all the Kazhdan-Lusztig character formulas.

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