ENDOMORPHISM ALGEBRAS AND REPRESENTATION THEORY

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Endomorphism algebras figure prominently in group representation theory. For example, if G is a finite group of Lie type, the representation theory of the endomorphism algebra $\mathrm{End}_G(\mathbb{C}|_B^G)$—sometimes known as the Hecke algebra over $\mathbb{C}$ of G—plays a central role in unraveling the complex unipotent characters on G [2], [7]. Another example arises in the modular representation theory of the finite general linear group $G = GL_n(q)$ over an algebraically closed field k of characteristic p not dividing q. In this so-called non-describing characteristic representation theory, the Hecke algebras H over k associated with symmetric groups provide a link between the representation theory of G and that of quantum groups. In fact, if T denotes the direct sum of the various "transitive" q-permutation modules for H, then the endomorphism algebras $\mathrm{End}_H(T)$ are Morita equivalent to q-Schur algebras over k. In work by Dipper and James (see [10], [11], [8]) the decomposition numbers for kG are proved to be completely determined by the decomposition numbers for certain of these "quantized Schur algebras". In turn, the representation theory of these latter algebras relates closely to that of the quantum linear group $GL_{n,q}(k)$ over k.

Several years ago, the authors began a general homological investigation of endomorphism algebras [4]. Motivation came from several sources, including the theory of Schur algebras, but more particularly work of Dlab, Heath, and Marko [12] on quasi-hereditary endomorphism algebras as well as Soergel's work [32]. The latter realizes the principal block $\mathcal{O}_{\mathrm{triv}}$ for the category $\mathcal{O}$ of a complex semisimple Lie algebra $\mathfrak{g}$ as the module category for a certain endomorphism algebra $A = \mathrm{End}_R(T)$ in which $R = H^\bullet(G/B, \mathbb{C})$ is the cohomology algebra of the associated flag manifold of $\mathfrak{g}$. Section 1 of the present paper provides a guide for some of this work by beginning with the idea of a stratified algebra A. Stratified algebras are natural generalizations of quasi-hereditary algebras. When $A = \mathrm{End}_R(T)$ is an endomorphism algebra, a stratification on A is roughly equivalent to having a "Specht module theory" for the algebra R, much in the same spirit as the classical theory of Specht modules for symmetric groups. The precise theorem is quite complicated, but simplifies remarkably when the data A, T, R lifts to similar data $\widetilde{A}, \widetilde{T}, \widetilde{R}$ over a discrete valuation ring $\mathcal{O}$ such that $\widetilde{R}_K$ is semisimple over the quotient field K of $\mathcal{O}$. (The complexity at the field level is perhaps suggested by the corresponding complication of the theory of Specht modules for the symmetric groups $\mathfrak{S}_n$ over fields of characteristic 2.)

Section 2 applies this theory in the case $A = \mathrm{End}_R(T)$ and R is a Hecke algebra H over k associated to the general finite reductive group G. In this setting, there is another isomorphism

$$A \cong \mathrm{End}_{kG}\left(\bigoplus_J \mathrm{ind}_{P_J}^G k\right),$$

suggesting a close relationship between the non-describing representation theory of G and that of the algebra A. These algebras do appear in [21] and [23], which are aimed at generalizing the Dipper-James theory to other types. In this section, we present an overview of the recent joint work [15], [16] of the second two authors and Jie Du dealing with the structure of the algebra A from the point of view of stratified algebras. Some of this research was presented in talks at the Newton Institute. A key point centers on a very strong homological condition for Hecke algebras of finite Coxeter groups; in turn, this homologicial property depends on the Kazhdan-Lusztig theory of cells for Coxeter groups.

Section 3 returns to the case of $G = GL_n(q)$ to discuss very recent results in [5]. Making heavy use of [10] as well as work of Fong-Srinivasan [18] on the blocks for kG, we obtain an interesting Morita equivalence between a quotient algebra kG/J_k and a second algebra which is a direct sum of tensor products of q-Schur algebras. (A similar result had been obtained by Takeuchi [33] in the special case of unipotent blocks.) Moreover, J_k is contained in the radical of kG, so that kG and kG/J_k have the same irreducible modules. Through this Morita equivalence, we can seriously study the cohomology groups $H^\bullet(GL_n(q), L)$ at a general irreducible module L.

Because of the connection between the representation theory of q-Schur algebras and quantum groups, the cohomology groups above can be "generically" determined in terms of the cohomology of affine Lie algebras in characteristic 0, where we expect that explicit answers can be obtained eventually. This is explained in §4.

The authors wish to thank the National Science Foundation for their support of this research.

1. Stratified Algebras

Let A be a finite dimensional algebra over a field k. (Usually, k is algebraically closed.)

Given an ideal $J \triangleleft A$, and left A/J-modules M, N, we can regard M, N as A-modules through the quotient map $A \to A/J$. Moreover, there is a natural morphism

$$i_* = i_*(M,N) : \mathrm{Ext}^\bullet_{A/J}(M,N) \longrightarrow \mathrm{Ext}^\bullet_A(M,N) \qquad (1.1)$$

of Ext-groups. When $i_*(M,N)$ is an isomorphism for all M, N, we say that J is a (left) *stratifying ideal* of A. A sufficient condition that J is a stratifying ideal is that the following two conditions hold:–

(1) J is an idempotent ideal (i.e., $J = J^2$ or $J = AeA$ for an idempotent $e \in A$), and
(2) J is projective as a left ideal.

In case (1) and (2) hold, we call J a (left) *standard stratifying ideal* of A. (More generally, consult [4], §3, for necessary and sufficient conditions for J to be stratifying.) A *standard stratification* of A (of length n) is a sequence

$$0 = J_0 \subsetneqq J_1 \subsetneqq \cdots \subsetneqq J_n = A \qquad (1.2)$$

of ideals such that for $0 < i \leq t$, the ideal J_i/J_{i-1} is a standard stratifying ideal in A/J_{i-1}. If each J_i/J_{i-1} is merely a stratifying ideal, then (1.2) is a *stratification* of A. (Observe that (1.2) is a stratification of A if and only if each J_i is a stratifying ideal of A. However, a similar statement does not hold for a standard stratification.)

Example 1.3 Suppose that A is a self-injective algebra (e. g., A may be the group algebra kG of a finite group). Let J be a two-sided ideal of A which is projective as a left A-module. Then J is also injective as a left A-module, so that $A = J \bigoplus I$ for some left ideal I of A. Since $J = JA = J^2 \bigoplus JI$, we conclude that $JI = 0$. Hence, $\mathrm{Hom}_A(J, I) = 0$, so the heads of the indecomposable summands of J (as a left A-module) do not appear as a composition factor of I. The hypothesis on A implies that any irreducible module of A appears in both the socle and the head of the left A-module ${}_AA$. It follows that the head and socle of J have the same irreducible modules (but perhaps occurring with different multiplicities).

Therefore, $\mathrm{Hom}_A(I,J)=0$ and so $IJ=0$. We conclude that I is a two-sided ideal of A and so J is a direct sum of blocks of A.

It follows that any standard stratification of A corresponds to a decomposition of A into disjoint collections of blocks of A. Hence, the notion of a standard stratification for self-injective algebras leads to nothing new. However, as we see in the next example, this is not at all the case when we consider endomorphism algebras associated to such algebras.

Example 1.4 To consider a very different example, let k be a field of characteristic 2 and consider the cyclic group G of order 2. View $T=kG\bigoplus k$ as a right kG-module and form the endomorphism algebra $A=\mathrm{End}_{kG}(T)$. We can identify A with an algebra of 2×2 matrices $\begin{pmatrix} kG & k \\ k & k \end{pmatrix}$ which, in calculating the $(1,2)$- and $(2,1)$-positions k is regarded as a kG-module via the augmentation $kG\to k$. If $e=\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, then the idempotent ideal $J=AeA=\begin{pmatrix} k & k \\ k & k \end{pmatrix}$ is a projective left A-module. Hence, J is a stratifying ideal which is *not* a direct summand of A (and hence is not a block). The category A-mod of finitely generated left A-modules has two irreducible modules, labeled a,b which have uniserial PIMs with structure

$$P(a)=\boxed{\begin{matrix} a \\ b \\ a \end{matrix}}\ \text{and}\ P(b)=\boxed{\begin{matrix} b \\ a \end{matrix}}. \tag{1.4.1}$$

We will return to stratifications of endomorphism algebras in (2.11)–(2.15).

Example 1.5 Recall that A is a quasi-hereditary algebra provided that it has a standard stratification (1.2) such that, writing

$$J_i/J_{i-1}=(A/J_{i-1})\bar{e}(A/J_{i-1})$$

for an idempotent $\bar{e}\in A/J_{i-1}$, the algebra $\bar{e}(A/J_{i-1})\bar{e}$ is semisimple. For example, the algebra A in (1.4) satisfies this condition with the standard stratification $0\subsetneqq J\subsetneqq A$ of length 2. In general, if A is quasi-hereditary it has a standard stratification of length equal to the number of simple A-modules (the maximal length possible for any stratification of an algebra A).

Let Λ be a finite set with a preorder (transitive and reflexive relation) $\leq$. For $\lambda,\mu\in\Lambda$, write $\lambda\sim\mu$ for the equivalence relation: $\lambda\leq\mu$ and $\mu\leq\lambda$. The preorder $\leq$ defines a natural poset structure on the set $\bar{\Lambda}$ of equivalence classes (or cells) defined by $\sim$: if $\lambda\mapsto\bar{\lambda}$ assigns to λ the equivalence class containing it, then $\bar{\lambda}\leq\bar{\mu}$ if and only if $\lambda\leq\mu$. There is an alternative module-theoretic way to say that A has a (standard) stratification.

Theorem 1.6 *([4]) The algebra A has a stratification (1.2) of length n if and only if there exists a finite set Λ with a preorder $\leq$ such that:–*

(1) $|\bar{\Lambda}| = n$;

(2) There is a collection $\{\Delta(\lambda)\}_{\lambda\in\Lambda}$ of objects in A-mod such that any irreducible A-module is a homomorphic image of some $\Delta(\lambda)$;

(3) For $\lambda \in \Lambda$, there exists a projective object $P(\lambda)$ such that, for all $\lambda,\mu \in \Lambda$: (i) $\mathrm{Hom}_A(P(\mu),\Delta(\lambda)) \neq 0$ implies that $\mu \leq \lambda$; and (ii) $P(\lambda)$ has a filtration $P(\lambda) = F_m \supseteq F_{m-1} \supseteq \cdots \supseteq F_1 \supseteq F_0 = 0$ with top section $F_m/F_{m-1} \cong \Delta(\lambda)$ and lower sections $F_i/F_{i-1} \cong \Delta(\mu)$ $(i < m)$ for some $\mu \in \Lambda$ satisfying $\bar{\mu} \geq \bar{\lambda}$.

The stratification can be taken to be standard if and only if the data can be chosen so that in (3(ii)) the inequalities can be taken to be strict. In this case, it can be assumed (after possibily changing Λ and the $\Delta(\lambda)$) that Λ indexes the set of irreducible A-modules.

We will call the modules $\Delta(\lambda)$ in (1.6) the *standard modules* in A-mod.

Of course, (1.6) is inspired by the corresponding result for quasi-hereditary algebras A and highest weight categories. Thus, if Λ is a poset indexing the irreducible A-modules, then A-mod is a highest weight category if there exists a collection $\{\Delta(\lambda)\}_{\lambda\in\Lambda}$ of A-modules such that: (1) For $\lambda \in \Lambda$, $\Delta(\lambda)$ has irreducible head $L(\lambda)$ and all other composition factors $L(\mu)$ satisfy $\mu < \lambda$; and (2) the projective cover $P(\lambda)$ of $L(\lambda)$ has a filtration with top section $\Delta(\lambda)$ and lower sections $\Delta(\mu)$ for $\mu > \lambda$. Clearly, in this case, the hypotheses of (1.6) are satisfied. By [3], A-mod is a highest weight category if and only if A is a quasi-hereditary algebra.

We remark that, for any standardly stratified algebra A, the set Λ can be taken to index the irreducible A-modules, and standard modules $\Delta(\lambda)$ can be chosen so that all the highest weight conditions above are satisfied but the one on the composition factors of $\Delta(\lambda)$. (Instead of $\mu < \lambda$, one has $\mu \leq \lambda$.) This chosen set of standard modules is unique, and is the set of non-isomorphic direct summands of those in (1.6).

Now suppose that A, R are finite dimensional algebras over k and T is a finite dimensional (A,R)-bimodule T. Define contravariant functors (denoted by the same symbol):–

$$\begin{cases} (-)^\diamond = \mathrm{Hom}_A(-,T): & A\text{-mod} \longrightarrow \text{mod-}R \\ (-)^\diamond = \mathrm{Hom}_R(-,T): & \text{mod-}R \longrightarrow A\text{-mod} \end{cases} \tag{1.7}$$

between the category A–mod of f.g. left A-modules and the category mod–R of f.g. right R-modules. A main result in [4] presents necessary and sufficient conditions (in terms of the structure of T as an R-module) in order that A have a standard stratification. These conditions are quite complicated, so we will be content to describe only briefly some of their features.

Assume that $\leq$ is a fixed preorder on Λ and that $T = \bigoplus_{\lambda\in\Lambda} Y_\lambda^{\oplus m_\lambda}$ is a decomposition of T into a direct sum of indecomposable, distinct summands Y_λ. For each λ, assume given a fixed "Specht" submodule $S_\lambda \subseteq Y_\lambda$ as well as a filtration $F_\lambda : 0 = F_\lambda^0 \subseteq F_\lambda^1 \subseteq \cdots \subseteq F_\lambda^{n_\lambda} = Y_\lambda$ of Y_λ with bottom section $F_\lambda^1 = S_\lambda$ and higher sections $F_\lambda^i/F_\lambda^{i-1} \cong S_\mu$ for some $\mu \in \Lambda$ with $\bar\mu > \bar\lambda$. For elementary reasons, $P(\lambda) = Y_\lambda^\diamond$ (in the notation of (1.7)) is a projective indecomposable A-module. If $L(\lambda) = \mathrm{head}(P(\lambda))$, then $\{L(\lambda)\}_{\lambda\in\Lambda}$ is a set of representatives of the distinct irreducible A-modules. In order to verify the hypotheses of (1.6), one might first try setting $\Delta(\lambda) = S_\lambda^\diamond$. Unfortunately, this simple approach does not always work, as shown by the following three examples, where we also illustrate a successful, more sophisticated approach.

Example 1.8 Let $R = k\mathfrak{S}_r$ be the group algebra for the symmetric group $W = \mathfrak{S}_r$ of degree r. For a positive integer n, let $\Lambda^+(n,r)$ be the set of partitions λ of r into at most n parts. The set $\Lambda^+(n,r)$ comes equipped with the dominance order $\trianglelefteq$: if $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$ and $\mu = (\mu_1 \geq \mu_2 \geq \cdots)$ belong to $\Lambda^+(n,r)$, then $\lambda \trianglelefteq \mu$ if and only if $\lambda_1 \leq \mu_1$, $\lambda_1 + \lambda_2 \leq \mu_1 + \mu_2$, etc. For $\lambda \in \Lambda^+(n,r)$, let W_λ be an associated Young subgroup (i. e., the stabilizer in W of a tableau of shape λ with distinct entries chosen from $\{1, 2, \cdots, n\}$) and let $T_\lambda = \mathrm{ind}_{W_\lambda}^W k$ be the corresponding permutation module. Put $T = \bigoplus_{\lambda\in\Lambda^+(n,r)} T_\lambda^{\oplus n_\lambda}$ for some choice of positive integers n_λ, and let $A = \mathrm{End}_R(T)$. Of course, the choice of the n_λ is purely a matter of convenience: changing their values leads to a Morita equivalent algebra. (For example, let V be a vector space over k of dimension n. Regard $V^{\otimes r}$ as a right $\mathfrak{S}_r$-module by place permutation. For some choice of the n_λ, $V^{\otimes r} \cong \bigoplus_{\lambda\in\Lambda^+(n,r)} T_\lambda^{\oplus n_\lambda}$. Classically, $\mathrm{End}_R(V^{\otimes r})$ is the Schur algebra $S(n,r)$ [22].) Suppose first that k does not have characteristic 2. For $\lambda \in \Lambda^+(n,r)$, let λ' be the dual partition and let $\mathrm{sgn} : \mathfrak{S}_r \to k$ be the sign representation. Classically, $\dim \mathrm{Hom}_{\mathfrak{S}_r}(T_{\lambda'} \otimes \mathrm{sgn}, T_\lambda) = 1$, determining unique indecomposable summands $Y_\lambda^\natural$ and Y_λ of $T_{\lambda'} \otimes \mathrm{sgn}$ and T_λ, respectively, with $\mathrm{Hom}_R(Y_\lambda^\natural, Y_\lambda) \neq 0$. If $\phi_\lambda : Y_\lambda^\natural \to Y_\lambda$ is a basis vector for the space of morphisms $Y_\lambda^\natural \to Y_\lambda$, set $S_\lambda = \mathrm{Im}(\phi_\lambda)$. Then Y_λ (resp., S_λ) is the Young (resp., Specht) module associated to λ. The Y_λ, $\lambda \in \Lambda^+(n,r)$, are precisely the indecomposable summands of T above. We put $\Delta(\lambda) = S_\lambda^\diamond$.

If k does have characteristic 2, however, the Specht and Young modules are defined first for the group algebra $\mathcal{O}\mathfrak{S}_r$ over a DVR $\mathcal{O}$ with residue field k. Then their analogues over the field k are obtained by base change. In this case, the S_λ need not be distinct for distinct λ. For example, $S_{(1^r)} \cong S_{(r)}$ in characteristic 2. Now define $\Delta(\lambda)$ to be the set of all morphisms $S_\lambda \to T$ which lift to a morphism $Y_\lambda \to T$. Clearly, $\Delta(\lambda)$ is an A-submodule of $S_\lambda^\diamond$.

In both cases, the collection $\{\Delta(\lambda)\}_{\lambda\in\Lambda^+(n,r)}$ of A-modules *does* satisfy the hypotheses of (1.6), so that A has a stratification of length $|\Lambda^+(n,r)|$; in fact, A-mod is a highest weight category with standard modules the $\Delta(\lambda)$. We always have $\Delta(\lambda)^\diamond = S_\lambda$, and if char $k \neq 2$, $\Delta(\lambda) = \mathrm{Hom}_R(S_\lambda, T) = S_\lambda^\diamond$ (i. e., the stratifying system $\{\Delta(\lambda)\}$ is "Δ-reflexive"). In all cases, the standard modules $\Delta(\lambda)$ are distinct for distinct λ.

For more details, consult [4], (1.6), (3.8.3), (4.4.15), and (5.2), which presents a complete development of the above discussion, including a reorganization of most of the modular representation theory of symmetric groups dealing with Specht modules, etc.

Example 1.9 Let R be a self-injective commutative local ring with radical quotient $R/\mathrm{rad}(R) = k$. Let Λ be a poset of cardinality equal to dim R. Suppose for $\lambda \in \Lambda$, there is given a local ideal $Y_\lambda \subseteq R$ such that $Y_\lambda \subseteq Y_\mu$ if and only if $\lambda \geq \mu$. Put $T = \bigoplus_{\lambda\in\Lambda} Y_\lambda$ and $A = \mathrm{End}_R(T)$. By [12], A is quasi-hereditary if and only if $\mathrm{rad}(Y_\lambda) = \sum_{\mu>\lambda} Y_\mu$ for all $\lambda \in \Lambda$. When this condition holds, put $S_\lambda = \mathrm{soc}\,(Y_\lambda)$, and let $\Delta(\lambda)$ be the A-module consisting of all morphisms $f : S_\lambda \to T$ which lift to a morphism $Y_\lambda \to T$. Then $\{\Delta(\lambda)\}_\lambda$ satisfies the hypotheses of (1.6), as shown in [5]. Observe that here all the "Specht modules" S_λ are isomorphic!

Example 1.10 Let $\mathfrak{g}$ be a complex semisimple Lie algebra. Consider the principal block $\mathcal{O}_{\mathrm{triv}}$ in the category $\mathcal{O}$ associated to $\mathfrak{g}$. Then $\mathcal{O}_{\mathrm{triv}}$ is a highest weight category with poset the Weyl group $\Lambda = W$ (using the Chevalley-Bruhat ordering) and standard modules $V(w\cdot 0)$, the Verma module of high weight $w\cdot 0 = w\rho - \rho$, where ρ denotes the half-sum of the positive roots. (When $\mathfrak{g} \cong \mathfrak{sl}_2(\mathbb{C})$, the PIMs in $\mathcal{O}_{\mathrm{triv}}$ are described by the diagrams (1.4.1).) Let $T' = P(-2\rho)$ denote the projective cover of $V(w_0)$, where w_0 is the long word of W. It known that T' is also an injective object in $\mathcal{O}$ and that any $V(w\cdot 0) \subseteq T'$. By [32], $R = \mathrm{End}_{\mathfrak{g}}(T')^{\mathrm{op}} \cong H^\bullet(G/B, \mathbb{C})$, the cohomology algebra of the flag manifold G/B associated to $\mathfrak{g}$. In particular, R is a commutative, Frobenius algebra.

Let A be a finite dimensional algebra so that A-mod is equivalent to $\mathcal{O}_{\mathrm{triv}}$. Let T, $\Delta(w)$, $w \in W$, be the A-modules which correspond under this equivalence to T', $V(w)$, $w \in W$, respectively. Then $R \cong \mathrm{End}_A(T)$ and it can be shown using [32] that the double centralizer property $A \cong \mathrm{End}_R(T)$ also holds.

For $w \in W$, let $\bar{X}_w$ be the associated Schubert variety in the flag manifold G/B. The intersection cohomology $Y_w = \mathbb{H}^\bullet(\bar{X}_w, \mathbb{C})$ (regarded as an R-module) play the role of the Young modules—they are the indecomposable components of the R-module T. Let S_w be the socle of Y_w. For $w \in W$, $\Delta(w)$ identifies with the A-module of all R-morphisms $S_w \to T$

which lift to Y_w. See [4], (5.4), for more details. Again all "Specht" modules are isomorphic.

We have purposely avoided describing the conditions on the data consisting of the Y_λ, the Specht modules S_λ, and the filtration F_λ of Y_λ which are necessary and sufficient in order that the hypotheses of (1.6) hold with $\Delta(\lambda)$ defined just as in the above three examples. (See [4], (3.1.1), (3.1.3), and (3.3).) Although it sometimes is necessary to work at the field-theoretic level, we have learned (painfully) that it is usually easier to "lift" the problem to the setting of orders in semisimple algebras. We will end this section by discussing *integral conditions* guaranteeing a stratification. The next section shows how this theory works for Hecke endomorphism algebras.

First, some more notation: Let $\mathcal{Z}$ be a commutative, Noetherian domain with fraction field K. Let $\widetilde{R}$ be a $\mathcal{Z}$-algebra which is finitely generated and projective as a $\mathcal{Z}$-module, let $\widetilde{T}$ be a finitely generated (right) $\widetilde{R}$-module, projective over $\mathcal{Z}$, and put $\widetilde{A} = \mathrm{End}_{\widetilde{R}}(\widetilde{T})$. Now suppose that $\leq$ is a preorder on a finite set Λ. Let $\widetilde{T} = \bigoplus_{\lambda\in\Lambda} \widetilde{T}^{\oplus n_\lambda}$ be a direct sum decomposition into $\widetilde{R}$-submodules. (We make no assumptions that the $\widetilde{T}_\lambda$ be indecomposable, or that the decomposition be unique up to isomorphism.) For $\lambda \in \Lambda$, form the left $\widetilde{A}$-module $\widetilde{\Delta}(\lambda) = \mathrm{Hom}_{\widetilde{R}}(\widetilde{S}_\lambda, \widetilde{T})$.

Theorem 1.11 *([15], (1.2.10), (1.2.5), and (1.2.12)) In addition to the above assumptions, assume that $\mathcal{Z}$ is a regular ring of Krull dimension ≤ 2. For $\lambda \in \Lambda$, suppose given an $\widetilde{R}$-submodule $\widetilde{S}_\lambda$ of $\widetilde{Y}_\lambda$ and a filtration $\widetilde{F}_\lambda : 0 = \widetilde{F}^0_\lambda \subseteq \widetilde{F}^1_\lambda \subseteq \cdots \subseteq \widetilde{F}^{t(\lambda)}_\lambda$ of Y_λ. Assume that the following conditions (1)–(3) hold:–*

(1) For $\lambda \in \Lambda$, there is a fixed sequence $\nu_{\lambda,0}, \nu_{\lambda,1}, \cdots, \nu_{\lambda,t(\lambda)-1}$ in Λ such that $\nu_{\lambda,0} = \lambda$ and, for $i > 0$, $\nu_{\lambda,i} \geq \lambda$. For $0 \leq i < t(\lambda)$, there is given a fixed isomorphism $\mathrm{Gr}^i \widetilde{F}_\lambda \cong \widetilde{S}_{\nu_{\lambda,i}}$.

(2) For $\lambda, \mu \in \Lambda$, $\mathrm{Hom}_{\widetilde{R}}(\widetilde{S}_\mu, \widetilde{T}_\lambda) \neq 0 \Rightarrow \lambda \leq \mu$.

(3) For all $\lambda \in \Lambda$, we have $\mathrm{Ext}^1_{\widetilde{R}}(\widetilde{T}_\lambda/\widetilde{F}^i_\lambda, \widetilde{T}) = 0$ for all i.

Then for any field k which is a $\mathcal{Z}$-algebra, the $\widetilde{A}_k$-modules $\{\widetilde{\Delta}(\lambda)_k\}_{\lambda\in\Lambda}$ satisfy the hypotheses of (1.6), so that $\widetilde{A}_k$ has a stratification of length $|\bar{\Lambda}|$. Furthermore, if, in condition (1), the inequalities $\bar{\nu}_{\lambda,i} > \bar{\lambda}$ hold, then the stratification is standard.

The homological requirement (3) is very natural and surprisingly easy to check—see (2.9) below. By contrast, its analogue in the field case is often false, even when it holds integrally, cf. (2.13) below.

2. Hecke Endomorphism Algebras

In this section, $\mathcal{Z} = \mathbb{Z}[q, q^{-1}]$ is the ring of Laurent polynomials in an indeterminate q. (Sometimes q also denotes the specialization of the variable q to a prime power ρ^d in the field k or in a DVR $\mathcal{O}$ having k as residue field.) Let (W, S) be a finite Coxeter system associated to a reductive group $\mathbb{G}$, defined over a finite field $\mathbb{F}_{\rho^d}$, ρ a prime. For $s \in S$, let $\rho^{c_s} = [\mathbb{B}(\rho^d) : \mathbb{B}(\rho^d) \bigcap {}^s\mathbb{B}(\rho^d)]$, where $\mathbb{B}$ is a Borel subgroup defined over $\mathbb{F}_{\rho^d}$. If $s, t \in S$ are W-conjugate then $c_s = c_t$. The Hecke algebra $\widetilde{H}$ of $\mathbb{G}$ over $\mathcal{Z}$ has $\mathcal{Z}$-basis $\{\tau_w\}_{w\in W}$ satisfying the multiplicative relations ($w \in W, s \in S$):–

$$\tau_s\tau_w = \begin{cases} \tau_{sw} & \text{if } sw > w, \\ q^{c_s}\tau_{sw} + (q^{c_s} - 1)\tau_w & \text{otherwise.} \end{cases} \tag{2.1}$$

If $w \in W$ has reduced expression $w = s_{i_1} \cdots s_{i_t}$, the expression $q_w = q^{c_{s_{i_1}}} \cdots q^{c_{s_{i_t}}}$ is well-defined.

For $\lambda \subseteq S$ and $W_\lambda = \langle s \mid s \in \lambda\rangle$, define

$$x_\lambda = \sum_{w\in W_\lambda} \tau_w \text{ and } y_\lambda = \sum_{w\in W_\lambda} (-1)^{\ell(w)} q_w^{-1}\tau_w. \tag{2.2}$$

In the same spirit as (1.8), let

$$\widetilde{T} = \bigoplus_{\lambda\subseteq S} \widetilde{T}_\lambda^{\oplus n_\lambda} \text{ and } \widetilde{A} = \mathrm{End}_{\widetilde{H}}(\widetilde{T}), \tag{2.3}$$

where each $n_\lambda > 0$ and $\widetilde{T}_\lambda$ is the right "q-permutation module" $x_\lambda\widetilde{H}$. (Similarly, we can define the "twisted q-permutation module" $\widetilde{T}_\lambda^{\#} = y_\lambda\widetilde{H}$.)

Example 2.4 Let $\widetilde{V}$ be a free $\mathcal{Z}$-module of rank n. Let $\widetilde{H}$ be the Hecke algebra over $\mathcal{Z}$ corresponding to the Coxeter group $W = \mathfrak{S}_r$ (with all $c_s = 1$). Fix a basis $\{v_1, \cdots, v_n\}$ for $\widetilde{V}$. For a sequence $I = (i_1, \cdots, i_r)$ of integers i_j, $1 \le i_j \le n$, write $v_I = v_{i_1} \otimes \cdots \otimes v_{i_r}$. The structure of $\widetilde{V}^{\otimes r}$ as a right $\widetilde{H}$-module was first investigated by Jimbo [25]. Here, we will use the right action of $\widetilde{H}$ on $\widetilde{V}^{\otimes r}$ defined in Dipper-Donkin [9], (3.1.5), by setting, for $s = (j, j+1) \in \mathfrak{S}_r$:–

$$v_I\tau_s = \begin{cases} qv_{Is} & \text{if } i_j \le i_{j+1} \\ v_{Is} + (q-1)v_I & \text{otherwise .} \end{cases} \tag{2.4.1}$$

(Here we are using the natural right action of $\mathfrak{S}_r$ on the set of sequences I.) For suitable choice of n_λ in (2.3), $\widetilde{V}^{\otimes r} \cong \widetilde{T}$. Thus, the algebra $\widetilde{A}$ in (2.3) is Morita equivalent to the *q-Schur algebra* $\widetilde{S}_q(n, r) \stackrel{\text{def}}{=} \mathrm{End}_{\widetilde{H}}(\widetilde{V}^{\otimes r})$.

Now consider a homomorphism $\mathcal{Z} \to k$ (and continue to let q denote the image in k of the variable q) and form the q-Schur algebra $S_q(n,r) = \widetilde{S}_q(n,r)_k$ over k. Let $GL_{n,q}(k)$ be the (Manin) quantum general linear group over k [30]. As proved in [30], the category $S_q(n,r)$-mod of finite dimensional $S_q(n,r)$-modules is isomorphic to a full subcategorgy of the category of finite dimensional rational modules for $GL_{n,q}(k)$—namely, the full subcategory of rational modules which are *homogeneous of degree* r. In particular, the irreducible $S_q(n,r)$-modules are indexed by the set $\Lambda^+(n,r)$: for $\lambda \in \Lambda^+(n,r)$, let $L_q^k(\lambda)$ be the $S_q(n,r)$-module corresponding to the irreducible $GL_{n,q}(k)$-module of highest weight λ (which will necessarily be homogeneous of degree r). It is also known that $S_q(n,r)$ is a quasi-hereditary algebra. In fact, this result can be deduced very conceptually from the methods of this section and is discussed more fully in (2.15) below.

To apply (1.11), we require candidate "Specht modules", together with an appropriate filtration on each $\widetilde{T}_\lambda$. For this, we let $\{C'_w\}_{w\in W}$ be the C'-basis for the Hecke algebra $\widetilde{H}_0 = \widetilde{H} \otimes \mathbb{Z}[q^{1/2}, q^{-1/2}]$. This basis is defined in [27] in the case in which all $c_s = 1$ (i. e., $\mathbb{G}$ is split), and in general in [29]. Putting

$$C_w^+ = q^{\ell(w)/2} C'_w, \ w \in W, \tag{2.5}$$

defines a $\mathcal{Z}$-basis $\{C_w^+\}_{w\in W}$ for $\widetilde{H}$ satisfying

$$\tau_s C_w^+ = \begin{cases} q_s C_w^+, & \text{if } sw < w; \\ -C_w^+ + C_{sw}^+ + \sum_z \widetilde{M}_{z,w}^s C_z^+, & \text{otherwise,} \end{cases} \tag{2.6a}$$

for $s \in S, w \in W$. Here $\widetilde{M}_{z,w}^s \in \mathcal{Z}$. Also, we have:–

$$C_w^+ \tau_s = \begin{cases} q_s C_w^+, & \text{if } ws < w; \\ -C_w^+ + C_{ws}^+ + \sum_z \widetilde{M}_{z^{-1},w^{-1}}^s C_z^+, & \text{otherwise.} \end{cases} \tag{2.6b}$$

Let $\leq_L$, $\leq_R$, and $\leq_{LR}$ the Kazhdan-Lusztig preorders on W. The corresponding cells (i. e., equivalence classes) are, respectively, the Kazhdan-Lusztig left, right, and two-sided cells of W. (The preorders as well as the cells depend on the choice of the integers c_s. When all $c_s = 1$, the $\widetilde{M}_{z,w}^s$ have a simple description as leading coefficients of Kazhdan-Lusztig polynomials.)

If $w \in W$, let $\mathcal{R}(w) = \{s \in S \mid ws < w\}$ be the *right-set* of w. It is known that if x, y belong to the same left cell ω, then $\mathcal{R}(x) = \mathcal{R}(y)$. Thus, for a left-cell ω, $\mathcal{R}(\omega)$ is defined. The set $\{C_y^+\}_{\lambda\subseteq\mathcal{R}(y)}$ is a basis for the left ideal $\widetilde{H}x_\lambda$ for any $\lambda \subseteq S$. Then (2.6a) implies that $\leq_L$ induces a filtration on $\widetilde{H}x_\lambda$: Let Ω be the set of left cells in W, and put $\Omega_\lambda = \{\omega \in \Omega \mid \lambda \subseteq \mathcal{R}(\omega)\}$.

Fix a listing $\omega_{\lambda,1},\cdots,\omega_{\lambda,n_\lambda}$ of Ω_λ so that if $x\in\omega_{\lambda,i}, y\in\omega_{\lambda,j}$ and $x\leq_L y$, then $i\leq j$. (Exercise: $\omega_{\lambda,n_\lambda}$ is the left cell containing the long word in W_λ.) The *left cell filtration* $\widetilde{E}^\lambda : \widetilde{H}x_\lambda = \widetilde{E}^\lambda_0 \supseteq \cdots \supseteq \widetilde{E}^\lambda_{n_\lambda}$ of $\widetilde{H}x_\lambda$ is defined by putting:–

$$\begin{cases} \widetilde{E}^\lambda_i &= \operatorname{span}\left(C^+_y \,|\, y\in\omega_{\lambda,j},\ j\leq n_\lambda - i\right),\ \ 0\leq i< n_\lambda;\ \ \text{and} \\ \widetilde{E}^\lambda_{n_\lambda} &= 0. \end{cases} \tag{2.7}$$

For a $\mathcal{Z}$-module $\widetilde{M}$, let $\widetilde{H}^* = \operatorname{Hom}_{\mathcal{Z}}(\widetilde{M},\mathcal{Z})$, so that $(\widetilde{H}x_\lambda)^*\cong x_\lambda\widetilde{H}$. Thus, taking $\mathcal{Z}$-duals in (2.7) defines a filtration $\widetilde{F}_\lambda : 0=\widetilde{F}^0_\lambda\subseteq\widetilde{F}^1_\lambda\subseteq\cdots\subseteq\widetilde{F}^{n_\lambda}_\lambda = \widetilde{T}_\lambda$ on $\widetilde{T}_\lambda$, *viz.*,, we set $\widetilde{F}^i_\lambda = (\widetilde{H}'x_\lambda/\widetilde{E}^\lambda_i)^*$. Let $\widetilde{S}_\lambda$ be the bottom section $\widetilde{F}^1_\lambda$ of this filtration. The other sections of $\widetilde{F}_\lambda$ have the form $\widetilde{S}_\mu$ for μ satisfying $\omega_{\mu,n_\mu}\leq_L\omega_{\lambda,n_\lambda}$.

There is also a coarser filtration $\widetilde{F}_{\lambda,\mathcal{R}}$ of $\widetilde{T}_\lambda$. The *right-set* preorder $\leq^{\mathcal{R}}$ on W is obtained by putting $x\leq^{\mathcal{R}} y \iff \mathcal{R}(x)\supseteq\mathcal{R}(y)$. Now put

$$\begin{cases} \widetilde{E}^{\lambda,\mathcal{R}}_i &= \operatorname{span}\left(C^+_y \,|\, \mathcal{R}(y)=\lambda_{i_j}, j\leq m_\lambda - i\right) \text{ if } 0\leq i<m_\lambda, \\ \widetilde{E}^{\lambda,\mathcal{R}}_{m_\lambda} &= 0. \end{cases} \tag{2.8}$$

Let $\widetilde{F}^i_{\lambda,\mathcal{R}} = (\widetilde{H}x_\lambda/\widetilde{E}^{\lambda,\mathcal{R}}_i)^*$. We call $\widetilde{F}_{\lambda,\mathcal{R}}$ the *dual right-set* filtration of $\widetilde{T}_\lambda$. For $\lambda\subseteq S$, let $\widetilde{S}^{\mathcal{R}}_\lambda = \widetilde{F}^1_{\lambda,\mathcal{R}}$. The $\widetilde{S}^{\mathcal{R}}_\lambda$ are the *dual right-set* modules. The other sections of $\widetilde{F}_{\lambda,\mathcal{R}}$ have the form $\widetilde{S}^{\mathcal{R}}_\mu$ for $\mu\supsetneqq\lambda$.

Theorem 2.9 *With the above notation, we have for $\lambda\subseteq S$ and all i:–*

$$\begin{cases} \operatorname{Ext}^1_{\widetilde{H}}(x_\lambda\widetilde{H}/\widetilde{F}^i_{\lambda,\mathcal{R}}, x_\mu\widetilde{H}) &= 0, \\ \operatorname{Ext}^1_{\widetilde{H}}(x_\lambda\widetilde{H}/\widetilde{F}^i_\lambda, x_\mu\widetilde{H}) &= 0. \end{cases}$$

This is proved in [15], (2.3.9), making use of the following observation: suppose that $\widetilde{R}$ is a $\mathcal{Z}$-algebra (finite as a $\mathcal{Z}$-module) with $\widetilde{R}_K$ semisimple (where K is the fraction field of $\mathcal{Z}$). Let $\widetilde{M},\widetilde{T}$ be finitely generated $\widetilde{R}$-modules, $\widetilde{T}$ being $\mathcal{Z}$-torsion free. If, for every $d\in\mathcal{Z}$, the natural map

$$\operatorname{Hom}_{\widetilde{A}}(\widetilde{M},\widetilde{T}) \longrightarrow \operatorname{Hom}_{\widetilde{A}}(\widetilde{M}/d\widetilde{M},\widetilde{T}/d\widetilde{T})$$

is surjective. Then $\operatorname{Ext}^1_{\widetilde{A}}(\widetilde{M},\widetilde{T})=0$. See [15], (1.2.13), for the proof. Using this fact, (2.9) follows from properties of the C^+-basis.

Because of this result, condition (1.11(3)) holds using either the filtrations $\widetilde{F}_\lambda$ or $\widetilde{F}_{\lambda,\mathcal{R}}$ of $\widetilde{T}_\lambda$. Let Λ be the power-set of S. Define the preorder $\leq$ on Λ to be the smallest preorder containing the set

$$\left\{(\lambda,\mu)\in\Lambda\times\Lambda \,|\, \lambda\subseteq\mu \text{ or } \operatorname{Hom}_{\widetilde{H}}(\widetilde{S}^{\mathcal{R}}_\lambda,\widetilde{S}^{\mathcal{R}}_\mu)\neq 0\right\}. \tag{2.10}$$

If $|S| > 1$, it is proved in [15], (2.4.3), that $|\bar{\Lambda}| \geq 3$. Now we can state the main result in [15]:

Theorem 2.11 *Assume that $|S| > 1$. Let k be a field which is a $\mathcal{Z}$-algebra. Then:–*

(a) $\widetilde{A}_k \cong \mathrm{End}_{\widetilde{H}_k}(\widetilde{T}_k)$.

(b) For $\lambda \in \Lambda$, set $\widetilde{\Delta}(\lambda) = \mathrm{Hom}_{\widetilde{H}}(\widetilde{S}^{\mathcal{R}}_{\lambda}, \widetilde{T})$. Then the $\widetilde{A}_k$-modules $\{\widetilde{\Delta}(\lambda)_k\}_\lambda$ satisfy the hypotheses of (1.6) for a stratification, using the preorder (2.10) on the power set Λ of S. Hence, $\widetilde{A}_k$ has a stratification of length ≥ 3.

Example 2.12 Suppose that under the structure map $\mathcal{Z} \to k$ in (2.11), the variable $q \mapsto 1$. Then $\widetilde{H}_k \cong kW$, the group algebra over k of the Coxeter group. Also, $\widetilde{T}_{\lambda k} = T_\lambda$ is the permutation module $\mathrm{ind}_{W_\lambda}^W k$ defined by the parabolic subgroup W_λ of W. Then (2.11) shows that $A = \mathrm{End}_W(T)$ has a stratification of length at least 3, proving the conjecture [4], (6.3.1).

Example 2.13 Assume that (W, S) has rank 2. It is proved in [15], (2.4.11), as a corollary of the above theorem, that the stratification of $A = \widetilde{A}_k$ is standard. For example, suppose that $G = Sp_4(q)$, where q is a prime power satisfying $q \equiv -1 \bmod p$. Then $H = \widetilde{H} \otimes k$ has two distinct irreducible modules, labeled ϕ_1, ϕ_2, with ϕ_2 projective. The space $\mathrm{Ext}^1_H(\phi_1, \phi_1)$ has dimension 2; let ξ, ζ be a basis. If P denotes the projective cover of ϕ_1, then $T = \bigoplus x_\lambda H$ is isomorphic to $P \oplus \phi_2^{\oplus 4} \oplus \xi \oplus \zeta \oplus \phi_1$. Put $Y_{-1} = P$, $Y_0 = \xi$, $Y_{0'} = \zeta$, $Y_{0''} = \phi_2$ and $Y_1\phi_1$. Thus, we take $\Lambda = \{-1, 0, 0', 0'', 1\}$ with the preorder defined by $\overline{-1} < \bar{0} = \overline{0'} = \overline{0''} < \bar{1}$. Letting $S_\lambda = \mathrm{soc}\,(Y_\lambda)$, and taking $\Delta(\lambda)$ to be the A-submodule of $\mathrm{Hom}_H(S_\lambda, T)$ of morphisms that extend to Y_λ, the set $\{\Delta(\lambda)\}_\lambda$ satisfies the hypotheses of (1.6). The verification of this (over k, without using $\mathcal{Z}$) takes some work! Thus, A has a standard stratification of length 3. Observe that $\mathrm{Ext}^1_H(\phi_1, \phi_1) \neq 0$, so that condition (1.11(3)) fails at the field level—though it holds at the integral level.

For more details of this example, see [15], (3.2). All told, there are 26 different possible cases to consider for the finite groups of Lie type of rank 2. Though eventually we found a uniform argument, our first proof that A has a standard stratification of length 3 was case by case!

Returning to the general case of a Hecke algebra of Lie type over $\mathcal{Z}$, consider the $\widetilde{A}$-modules $\widetilde{S}_\omega$, $\omega \in \Omega$, as defined above. Generalizing the conjectures in [4], §6, we have the following conjecture. (A somewhat more precise version appears in [15], (2.5.2), where all rank 2 cases, twisted and untwisted, are checked. The type A case is checked in [16] with $\tilde{X} = 0$.)

Conjecture 2.14 ([15], (2.5.2)) There exists a right $\widetilde{H}$-module $\widetilde{X}$ which has a filtration with sections $\widetilde{S}_\omega$, $\omega \in \Omega$, such that if $\widetilde{T}^+ = \widetilde{T} \bigoplus \widetilde{X}$, and if k is any field which is a $\mathcal{Z}$-algebra, then the algebra $\widetilde{A}_k^+ = \mathrm{End}_{\widetilde{H}}(\widetilde{T}^+)_k$ has a standard stratification of length equal to the number of two-sided Kazhdan-Lusztig cells in the Coxeter group W.

We wrap this section up with a brief description of some of the further results for the q-Schur algebras which can be obtained using the methods above.

Example 2.15 Let $W = \mathfrak{S}_r$, etc. be as in (2.4). Let Ξ be the set of two-sided Kazhdan-Lusztig cells in W, and form the poset $(\Xi, \leq_{LR}^{\mathrm{op}})$ using the opposite of the poset structure defined by $\leq_{LR}$. It follows as in [27], (1.4), that if ω, ω' are two left cells contained in the same two sided cells, then $\widetilde{S}_\omega \cong \widetilde{S}_{\omega'}$ as right $\widetilde{H}$-modules. So, if $\xi \in \Xi$, put $\widetilde{S}_\xi = \widetilde{S}_\omega$ for any left cell ω contained in ξ. (It is easy to see that, if $\mathcal{O}$ is a discrete valuation ring which is a $\mathcal{Z}$-algebra, then the $\widetilde{S}_{\xi\mathcal{O}}$ are actually Specht modules for $\widetilde{H}_{\mathcal{O}}$ in the sense of [10]. See [16].) For $\xi \in \Xi$, let $\widetilde{\Delta}(\xi) = \mathrm{Hom}_{\widetilde{H}}(\widetilde{S}_\xi, \widetilde{T})$. It is proved in [16] that, for any field k which is a $\mathcal{Z}$-algebra, $\widetilde{S}(n,r)_k$-mod is a highest weight category with standard modules $\{\widetilde{\Delta}(\xi)_k\}_{\xi\in\Xi}$ and poset $\Xi(n,r)$ for an ideal $X(n,r)$ of $(\Xi, \leq_{LR}^{\mathrm{op}})$. The proof uses the homological property (2.9) and gives a very easy proof of the quasi-heredity of $\widetilde{S}(n,r)$ in the integral case. (It turns out that $(\Xi(n,r), \leq_{LR}^{\mathrm{op}}) \cong (\Lambda^+(n,r), \trianglelefteq)$. However, a proof of this fact presently requires the positivity of the Kazhdan-Lusztig polynomials.)

We also mention that [17] uses cell theory to obtain a new determination of tilting modules of q-Schur algebras (in the integral and well as the field case), extending and generalizing earlier work in the direction for Schur algebras in [4], (5.2), and in [13], [14] (the latter using completely different methods). Consider the case $n \geq r$ and put

$$\widetilde{X} = \bigoplus_{\lambda \subseteq S} \widetilde{V}^{\otimes r} y_\lambda,$$

where y_λ is as in (2.2). Then $\widetilde{X}_k \cong \bigoplus_\lambda \widetilde{V}_k^{\otimes r} y_\lambda$ is a full tilting module for $S_q(n,r) = \widetilde{S}_q(n,r)_k$. Also,

$$\mathrm{End}_{S_q(n,r)}(\widetilde{X}_k) \cong S_q(n,r),$$

so that $S_q(n,r)$ is isomorphic to its own Ringel dual (in case $n \geq r$). (This result has also been obtained in [14] by other methods.) The details are given in [17], §7, which also calculates the Ringel dual of $S_q(n,r)$ for $n < r$. As long as $q \neq -1$ in k, the partial tilting module $X(\lambda)$ of highest weight λ is isomorphic to $\mathrm{Hom}_{S_q(n,r)}(Y_\lambda^\natural, \widetilde{V}_k^{\otimes r})$, where $Y_\lambda^\natural$ is the "twisted" Young module for $\widetilde{H}_k$ defined by analogy with (1.8).

Conjecture 2.16 In the spirit of the discussion in (2.15) above, we conjecture that the algebras A defined in (1.10) for any complex semisimple Lie algebra are Morita equivalent to their Ringel duals.

3. Non-Describing Representation Theory and Cohomology of $GL_n(q)$

Fix a discrete valuation ring $\mathcal{O}$ with fraction field K of characteristic 0 and residue field k of characteristic $p > 0$. We let $G = GL_n(\rho^d)$, where the prime ρ is distinct from p. We assume that K is a splitting field for G. For simplicity, we often denote ρ^d by q.

Let $\mathcal{C}$ (resp., $\mathcal{C}_{ss}$, $\mathcal{C}_{ss,p'}$) be a set of representatives from the conjugacy classes (resp., semisimple conjugacy classes, semisimple p'-conjugacy classes) of G. (Recall that $x \in G$ is semisimple provided that it is semisimple in the usual sense in the algebraic group $GL_n(\bar{\mathbb{F}}_{\rho^d})$, where $\bar{\mathbb{F}}_{\rho^d}$ is the algebraic closure of $\mathbb{F}_{\rho^d}$.) Given $s \in \mathcal{C}_{ss}$, its centralizer $Z_G(x)$ has the form

$$Z_G(s) \cong \prod_{i=1}^{m(s)} GL_{n_i(s)}(q^{a_i(s)}), \quad \text{where } \sum a_i(s) n_i(s) = n. \tag{3.1}$$

Put $\mathbf{n}(s) = (n_1(s), \cdots, n_{m(s)}(s))$ and let $\Lambda^+(\mathbf{n}(s))$ denote the set of multipartitions $\lambda \vdash \mathbf{n}(s)$, i.e., $\lambda = (\lambda^{(1)}, \cdots, \lambda^{(m(s))})$, where $\lambda^{(1)} \vdash n_1(s), \cdots, \lambda^{(m(s))} \vdash n_{m(s)}(s)$.

The following result is proved in [5], (9.17). (As mentioned in the introduction, the special case for unipotent blocks already appears in [33].)

Theorem 3.2 *There exists an ideal $J \lhd \mathcal{O}G$ such that the algebras $\mathcal{O}G$ and $\mathcal{O}G/J$ have the same irreducible modules. Further, the there is a Morita equivalence*

$$\mathcal{O}G/J \underset{\text{Morita}}{\sim} \bigoplus_{s \in \mathcal{C}_{ss,p'}} \bigotimes_{i=1}^{m(s)} \widetilde{S}_{q^{a_i(s)}}(n_i(s), n_i(s))_{\mathcal{O}}. \tag{3.2.1}$$

To explain the notation for the algebras $\widetilde{S}_{q^a}(m,m)_{\mathcal{O}}$ above, let $\widetilde{S}_q(m,m)$ be the q-Schur algebra over $\mathcal{Z} = \mathbb{Z}[q, q^{-1}]$ defined in (2.3). Regard $\mathcal{O}$ as a $\mathcal{Z}$-algebra under which q maps to the image of ρ^a in $\mathcal{O}$, and set $\widetilde{S}_{q^a}(m,m)_{\mathcal{O}} = \widetilde{S}_q(m,m) \otimes_{\mathcal{Z}} \mathcal{O}$.

The Morita equivalence above induces a Morita equivalence

$$kG/J_k \underset{\text{Morita}}{\sim} \bigoplus_{s \in \mathcal{C}_{ss,p'}} \bigotimes_{i=1}^{m(s)} \widetilde{S}_{\rho^{da_i(s)}}(n_i(s), n_i(s))_k \tag{3.3}$$

for the indicated algebras over the field k. (As defined, $\mathcal{O}G/J$ is $\mathcal{O}$-torsion free, so that $J_k = J \otimes_{\mathcal{O}} k \subset kG$.) As explained in (2.4), for any field k which is a $\mathcal{Z}$-algebra, the irreducible modules $L_q^k(\lambda)$ for the algebra $\widetilde{S}_q(n,r)_k = S_q(n,r)$ are indexed by the poset $\Lambda^+(n,r)$. (In the Morita equivalence (3.3), we only consider q-Schur algebras $S_q(a,b)$ in which $a=b$.) Thus, (3.3) provides a natural indexing of the irreducible kG-modules: if $s \in \mathcal{C}_{ss,p'}$ and $\lambda = (\lambda^{(1)}, \cdots, \lambda^{m(s)}) \in \Lambda^+(\mathbf{n}(s))$, let $D(s,\lambda)$ be the irreducible kG-module corresponding to $\bigotimes L^k_{\rho^{d a_i(s)}}(\lambda^{(i)})$. In particular, the trivial module for kG is labeled $D(1,(1^n))$ and corresponds to the quantum determinant representation $\det_q$ of $S_q(n,n)$. More generally, the labeling $D(s,\lambda)$ coincides with that used in [10] for the irreducible modules.

We emphasize that the above result is proved using the methods of [10], together with those of [18]. Besides providing a conceptual framework for considering the non-describing representation theory of the finite general linear groups, we can apply our Morita equivalence (or that of Takeuchi [33]) to obtain new results on the cohomology groups $H^\bullet(G,L)$ for kG acting on an irreducible module L. We wish to conclude by briefly mentioning some of our results along these lines. The proofs will be given in [5], §§10, 11. Besides (3.3), the arguments make use of the Kazhdan-Lusztig cell theory methods of §2.

We can now state the following cohomology result.

Theorem 3.4 *Assume that p is relatively prime to q and to $q^i - 1$ for $i = 1, \cdots, m$. Then for any kG/J_k-module V (e. g., any $V = D(s,\lambda)$, $s \in \mathcal{C}_{ss,p'}$ and $\lambda \vdash \mathbf{n(s)}$), there is an isomorphism*

$$H^i(G,V) \cong \mathrm{Ext}^i_{S_q(n,n)}(\det_q, V), \ 0 \le i \le m+1$$

and an injection

$$\mathrm{Ext}^{m+2}_{S_q(n,n)}(\det_q, V) \hookrightarrow H^{m+2}(G,V).$$

In both these expressions, we have identified the kG/J_k-module V with the $S_q(n,n)$-module to which it corresponds under the Morita equivalence given in (3.3).

In the above result, if $s \neq 1$, the module $D(s,\lambda)$ does not lie in the principal block for kG, so that the cohomology groups $H^\bullet(G, D(s,\lambda))$ vanish identically. We do not know much about Ext^n-groups for non-unipotent blocks, except that (3.2) obviously gives some inequalities for Ext^1. (For $n > 1$, there are some well-known "change of rings" spectral sequences for quotient algebras.) We also emphasize that the ideal J_k in kG is generally *not* a stratifying ideal (since it is nilpotent); in fact, the algebra kG/J_k

has finite global dimension, while the group algebra kG does not have finite global dimension if p divides $|G|$. The stated arithmetic conditions are required in order to pass from the cohomology of kG/J_k to that of kG.

4. From Characteristic p to Characteristic 0: Generic Representation Theory

Generic representation theory refers to phenomena which stabilize for large values of the parameters. For example, consider the describing characteristic representation theory case. If $\mathbb{G}$ is a reductive algebraic group defined and split over $\mathbb{F}_p$ and if V is a finite dimensional rational $\mathbb{G}$-module, then for any non-negative integer n, dim $H^n(\mathbb{G}(p^d), V)$ achieves a stable value as $d \to \infty$ [6]. If λ is a dominant weight and $L(\lambda)$ is the irreducible $\mathbb{G}$-module of highest weight λ, then dim $H^n(\mathbb{G}(p^d), L(\lambda))$ stabilizes as $p \to \infty$ [19], [20]. Furthermore, in some cases, these generic values can even be explicitly calculated and explicit bounds on the size of p given for those calculations [19], [20]. More recently, the work of [1], [27] and [26] shows that the characters of the irreducible $\mathbb{G}$-modules $L(\lambda)$ can be calculated explicitly in terms of Kazhdan-Lusztig polynomials once the prime p is large enough (but nobody knows how large is "large enough").

In the non-describing representation theory, we can ask for similar generic results. For example, is there an explicit formula for the Brauer characters of the irreducible $kGL_n(q)$-modules $D(s, \lambda)$ in terms of ordinary characters in the same spirit as in the work [1], etc., cited above if p is large enough? In fact, the answer is that there is such a formula [23], (10.2), (see also [5], §8, and [31]). Similarly, one can expect explicit generic formulas for the cohomology calculations we have indicated in the previous section. The answer is not yet known, but the problem can be translated into one involving affine Lie algebra cohomology in characteristic 0. In this last section, we briefly indicate some results along this line, following part of the development in [5].

Let $\mathcal{O}$ be a commutative, Noetherian domain with fraction field K. Set $X = \operatorname{Spec} \mathcal{O}$. Let $\widetilde{A}$ be an $\mathcal{O}$-algebra. We are interested in comparing properties of the algebra $\widetilde{A}_K$ and its representation theory with analogous properties for the algebra $\widetilde{A}_{k(\mathfrak{p})}$ over the residue field $k(\mathfrak{p})$ at $\mathfrak{p} \in X$. For example, if $\widetilde{A}_K$ is a separable algebra over K, there exists a non-empty open subset $\Omega \subseteq X$ such that if $\mathfrak{p} \in \Omega$ then $\widetilde{A}_{k(\mathfrak{p})}$ is a separable $k(\mathfrak{p})$-algebra. However, if $\widetilde{A}_K$ is only semisimple, the residue algebras $\widetilde{A}_{k(\mathfrak{p})}$ may fail to be semisimple on any non-empty open subset Ω of X. (See [5], (1.6), (1.7).)

We have the following elementary result, comparing irreducible modules.

Proposition 4.1 *([5], (1.9)) Assume that $\widetilde{A}_K/\mathrm{rad}(\widetilde{A}_K)$ is a separable algebra over K. Let L_i^K, $i = 1, \cdots, n$, be the distinct irreducible $\widetilde{A}_K$-modules.*

There exists a nonzero $f \in \mathcal{O}$ such that each L_i^K has an $\widetilde{A}_f$-lattice L_i^f with the property that, for $\mathfrak{p} \in X_f$, the set $\{L^f_{1k(\mathfrak{p})}, \cdots, L^f_{nk(\mathfrak{p})}\}$ is a complete set of representatives for the isomorphism classes of irreducible $\widetilde{A}_{k(\mathfrak{p})}$-modules.

We also have the following cohomology comparison:–

Theorem 4.2 *Let $M, N \in \mathrm{Ob}(\widetilde{A}\text{–mod})$. If m is a non-negative integer, there is a nonempty open subset $\Omega_m \subset X$ such that for $\mathfrak{p} \in \Omega_m$, each extension field E of the residue field $k(\mathfrak{p})$, and $0 \leq n \leq m$, we have*

$$\dim\ \mathrm{Ext}^n_{\widetilde{A}_E}(M_E, N_E) = \dim\ \mathrm{Ext}^n_{\widetilde{A}_K}(M_K, N_K).$$

To apply these results to q-Schur algebras and the representation theory of finite general linear groups in non-describing characteristic, let $\zeta \in \mathbb{C}$ be a primitive ℓth root of unity (for some positive integer ℓ), set $K = \mathbb{Q}(\zeta)$ and form the Dedekind domain $\mathcal{O}$ of algebraic integers in the number field K. There is a homomorphism $\mathcal{Z} \to \mathcal{O}$ under which $q \mapsto \zeta$. Let $S_q(n,m) = \widetilde{S}_q(n,m)_k$, etc. be as in (2.15). Thus, $\tilde{S}_q(n,m)$-mod is a highest weight category with weight poset $\Lambda^+(n,r)$ and standard modules $\{\tilde{\Delta}(\lambda)_k\}_{\lambda\in\Lambda^+(n,r)}$. Then (4.1) and (4.2) apply to yield the following result:–

Theorem 4.3 *There exists a nonempty basic open subset $X_g \subseteq X =$ $\mathrm{Spec}\,\mathcal{O}$ such that if $\mathfrak{p} \in X_f$ and $k = k(\mathfrak{p})$, then reduction "modulo $\mathfrak{p}$" carries the irreducible modules $L_q^K(\lambda)$, $\lambda \in \Lambda^+(n,r)$, for the highest weight category $S_q(n,m)_K$-mod to the irreducible modules $L_q^k(\lambda)$ for the highest weight category $S_q(n,r)_k$-mod. Furthermore, $[\widetilde{\Delta}(\lambda)_K : L_q^K(\mu)] = [\widetilde{\Delta}(\lambda)_k : L_q^k(\mu)]$ and*

$$\dim\ \mathrm{Ext}^m_{S_q(n,r)_K}(L_q^K(\lambda), L_q^K(\mu)) = \dim\ \mathrm{Ext}^m_{S_q(n,r)_k}(L_q^k(\lambda), L_q^k(\mu)),$$

for all $\lambda, \mu \in \Lambda$, $m \in \mathbb{Z}^+$.

Coming back to the cohomology discussion at the end of §3, we can, exactly in the spirit of [6], conclude the following stability result:–

Theorem 4.4 *(H^1-stability) Let n be a fixed positive integer, p a sufficiently large prime (the size requirement depending only on n), and k an algebraically closed field of characteristic p. Let $q = \rho^d$ for any prime ρ distinct from p. Then $\dim\ H^1(GL_n(q), D(1,\lambda))$ depends only on λ and the order ℓ of q modulo p. Also, for any prime p and $1 \neq s \in \mathcal{C}_{ss,p'}$, we have $H^1(GL_n(q), D(s,\lambda)) = 0$.*

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