REDUCED STANDARD MODULES AND COHOMOLOGY

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Abstract. First cohomology groups of finite groups with nontrivial irreducible coefficients have been useful in several geometric and arithmetic contexts, including Wiles's famous paper [42]. Internal to group theory, 1-cohomology plays a role in the general theory of maximal subgroups of finite groups, as developed in Aschbacher-Scott [5]. One can easily pass to the case where the group acts faithfully, and the underlying module is absolutely irreducible. In this case, R. Guralnick [23] conjectured that there is a universal constant bounding all of the dimensions of these cohomology groups. This paper provides the first general positive results on this conjecture, proving that the generic 1-cohomology $H^1_{\rm gen}(G,L) := \lim\limits_{q\to\infty} H^1(G(q),L)$ (see [18]) of a finite group $G(q)$ of Lie type, with absolutely irreducible coefficients L, is bounded by a constant depending only on the root system. This result emerges here as a consequence of a general study, of interest in its own right, of the homological properties of certain rational modules $\Delta^{\rm red}(\lambda), \nabla_{\rm red}(\lambda)$, indexed by dominant weights λ, for a reductive group G. The modules $\Delta^{\rm red}(\lambda)$ and $\nabla_{\rm red}(\lambda)$ arise naturally from irreducible representations of the quantum enveloping algebra U_ζ (of the same type as G) at a pth root of unity, where $p > 0$ is the characteristic of the defining field for G. When the Lusztig character formula holds for irreducible G-modules having restricted high weight, the modules $\Delta^{\rm red}(\lambda)$ and $\nabla_{\rm red}(\lambda)$, for *all* regular dominant weights λ, have very strong homological properties. These homological properties then provide a way to determine the desired bounds on generic 1-cohomology. The investigation leads to numerous related results and conjectures, involving, for example, structural relations between the $\Delta^{\rm red}$-modules and classical Weyl modules, and calculations of $\mathrm{Ext}^n_G(\Delta^{\rm red}(\lambda), \nabla_{\rm red}(\mu))$ for all $n \geq 0$ and regular dominant weights λ, μ.

Contents

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INTRODUCTION

Let G be a semisimple, simply connected algebraic group defined over an algebraically closed field $k = \overline{\mathbb{F}}_p$. The group G has a well-known construction, first discovered by Chevalley [9], in terms of the corresponding complex semisimple Lie algebra $\mathfrak{g}_{\mathbb{C}}$. In addition, given a dominant weight λ for $\mathfrak{g}_{\mathbb{C}}$, one can construct two rational G-modules $\Delta(\lambda)$ and $\nabla(\lambda)$ by a process of "reduction modulo p". More precisely, $\Delta(\lambda)$ (resp., $\nabla(\lambda)$) is obtained from a minimal (resp., maximal) lattice in the complex irreducible $\mathfrak{g}_{\mathbb{C}}$-module of high weight λ. The modules $\Delta(\lambda)$ and $\nabla(\lambda)$ were broadly popularized by Steinberg [40] who also established some basic properties. In addition, these modules have remarkable homological properties, often dependent on the validity of Kempf's vanishing theorem [26]. A number of these properties were first discovered in [18]. For example, if λ, μ are dominant weights then $\mathrm{Ext}^n_G(\Delta(\lambda), \nabla(\mu)) = 0$ unless $n = 0$ and $\lambda = \mu$, in which case the Ext group is a Hom, and is equal to k. Thus, the various modules $\Delta(\lambda)$ and $\nabla(\mu)$ are almost homologically invisible to each other, and give the appearance of forming, collectively, some kind of pair of dual bases at the derived category level.

Now let U_ζ be the (Lusztig) quantum enveloping algebra of the same type as G, where ζ is a pth root of unity. The process described in the previous paragraph has been carried out by Lusztig [32] and then Lin [30], where now the role of $\mathfrak{g}_{\mathbb{C}}$ is played by U_ζ. In this way, we obtain rational G-modules $\Delta^{\mathrm{red}}(\lambda)$ and $\nabla_{\mathrm{red}}(\lambda)$ which are indexed by λ. In many ways, these modules are quite analogous to the standard modules $\Delta(\lambda)$ and costandard modules $\nabla(\lambda)$, and they may be viewed as *reduced* versions of them. For example, $\Delta(\lambda)$ and $\Delta^{\mathrm{red}}(\lambda)$ both have irreducible head $L(\lambda)$. The character of $\Delta(\lambda)$ is given by Weyl's character formula, while the character of $\Delta^{\mathrm{red}}(\lambda)$ is given by the (Kazhdan-)Lusztig character formula (subject to some restrictions on p). In this paper, we begin an investigation into the homological properties of the Δ^{red}– and ∇_{red}–modules, together with their deeper structural relationships to their predecessors.

Let $\mathcal{B}$ be a block (or a union of blocks) in the category G–mod of finite dimensional rational G-modules. The irreducible modules $L(\lambda)$ in $\mathcal{B}$ have high weights λ lying in a union $X^+_{\mathcal{B}}$ of orbits of appropriate subgroups $W_p^{(r)}$ of the affine Weyl group W_p attached to G. We associate to $\mathcal{B}$ two Grothendieck groups $K_0^L = K_0^L(\mathcal{B})$ and $K_0^R = K_0^R(\mathcal{B})$. Both K_0^L and K_0^R are free modules over the ring $\mathbb{Z}[t, t^{-1}]$ of Laurent polynomials, and each has a basis indexed by $X^+_{\mathcal{B}}$. These bases are paired to each other with respect to a natural sesquilinear pairing

$$\langle\, ,\, \rangle : K_0^L \times K_0^R \to \mathbb{Z}[t, t^{-1}]. \tag{0.0.1}$$

The groups K_0^L and K_0^R, first introduced in [13], can be viewed as deformations of the ordinary Grothendieck group K_0 of the bounded derived category $D^b(\mathcal{B})$ of the abelian category $\mathcal{B}$. However, K_0 is isomorphic to the ordinary Grothendieck group of $\mathcal{B}$, while K_0^L and K_0^R preserve homological degree information by taking into account parity conditions on modules. Given modules $M, N \in \mathcal{B}$, when M and N are "represented" in K_0^L and K_0^R, respectively—say by $[M]$ and $[N]$—then

$$\sum_i \dim \operatorname{Ext}^i_G(M, N)t^i = \langle [M], [N]\rangle. \tag{0.0.2}$$

Thus, if $[M], [N]$ can be explicitly written in terms of the paired bases alluded to above, (0.0.2) explicitly calculates $\operatorname{Ext}^\bullet_G(M, N)$ using the form (0.0.1).

More precisely, an object $M \in D^b(\mathcal{B})$ is represented in K_0^L when M or its shift $M[1]$ belongs to $\mathcal{E}^L = \mathcal{E}^L(\mathcal{B})$, and direct sums of such objects—those in $\mathcal{E}^L \oplus \mathcal{E}^L[1]$—are also represented in K_0^L. Here $\mathcal{E}^L$ is a full subcategory of $D^b(\mathcal{B})$ roughly consisting of those objects which have a "filtration" with "sections" of the form $\Delta(\lambda)[i]$, $\lambda \in X^+_{\mathcal{B}}$, $i \equiv l(\lambda) \bmod 2$, for a suitable "length" function $l : X^+_{\mathcal{B}} \to \mathbb{N}$. In this expression, $\Delta(\lambda)$ is the Weyl or standard module of high weight λ. A similar description applies to K_0^R, involving another full subcategory $\mathcal{E}^R = \mathcal{E}^R(\mathcal{B})$. The "length" function l could just as well be called a "parity" function, since we only use its values modulo 2. However, it often arises naturally in terms of lengths of Coxeter group elements associated with weights $\lambda \in X^+_{\mathcal{B}}$.

In a special situation, the question of when irreducible modules $L(\lambda)$ belong to $\widehat{\mathcal{E}^L} = \mathcal{E}^L \oplus \mathcal{E}^L[1]$ (and its counterpart $\widehat{\mathcal{E}^R}$) is closely related to the validity of certain character formulas. Specifically, suppose that $X^+_{\mathcal{B}}$ consists of regular weights. Then a well-known theorem states that, given a saturated subset $\Gamma \subset X^+_{\mathcal{B}}$, $L(\lambda) \in \widehat{\mathcal{E}^L}$ for all $\lambda \in \Gamma$ if and only if each $L(\lambda)$, $\lambda \in \Gamma$, has formal character given by Lusztig's famous character formula.

The algebra U_ζ above is defined over the cyclotomic field generated by ζ and it has a natural integral form $\widetilde{U}_\zeta$ over a discrete valuation ring with residue field $\mathbb{F}_p$. Let $\overline{U}_\zeta$ be the reduction modulo p of $\widetilde{U}_\zeta$. As proved by Lusztig, the distribution algebra $\mathrm{hy}(G)$ of G is a natural quotient of $k \otimes_{\mathbb{F}_p} \overline{U}_\zeta$. In this way, the irreducible (type 1 and integrable) U_ζ-modules $L_\zeta(\lambda)$, $\lambda \in X^+$, give rise by reduction modulo p to certain rational G-modules $\Delta^{\mathrm{red}}(\lambda)$ and $\nabla_{\mathrm{red}}(\lambda)$. Lusztig's modular conjecture states that for $p \geq h$ (the Coxeter number of G) and λ a regular dominant weight in the Jantzen region, $\Delta^{\mathrm{red}}(\lambda) \cong \nabla_{\mathrm{red}}(\lambda) \cong L(\lambda)$. An extension proposed by Kato posits the same result provided only that $p \geq h$ and λ is a restricted dominant weight.

A main result, proved in Theorem 6.7, establishes that if the Kato version of the Lusztig conjecture holds, then $\Delta^{\mathrm{red}}(\lambda) \in \widehat{\mathcal{E}^L}$ and $\nabla_{\mathrm{red}}(\lambda) \in \widehat{\mathcal{E}^R}$ for *all* regular dominant weights λ. As a consequence, we are able to compute $\mathrm{Ext}^\bullet_G(\Delta^{\mathrm{red}}(\lambda), \nabla_{\mathrm{red}}(\mu))$ for all regular weights λ and μ in terms of Kazhdan-Lusztig polynomials, cf. Theorem 5.4(b). This result generalizes to all regular dominant weights a theorem of [13] for regular dominant weights inside the Jantzen region, where the modules $\Delta^{\mathrm{red}}(\lambda), \nabla_{\mathrm{red}}(\mu)$ are irreducible. In one substantive case when $\lambda = p\mu$ for a dominant weight μ, we are even able to establish Theorem 5.4 without assuming the Lusztig conjecture. Section 6 contains additional results and conjectures concerning the modules $\Delta^{\mathrm{red}}(\lambda)$ and $\nabla_{\mathrm{red}}(\lambda)$. In particular, we conjecture that for $p > h$ and λ a dominant weight, the standard module $\Delta(\lambda)$ has a filtration with sections of the form $\Delta^{\mathrm{red}}(\mu)$ and, in Theorem 6.8, we prove a partial result in this direction.

In Section 7, we apply Theorem 5.4 and other arguments to show there is a bound on dim $H^1(G, L)$, depending only on the root system and valid for any irreducible module L and underlying characteristic p, cf. Theorem 7.3. (In this proof, the Lusztig conjecture is used, but only to handle very large primes, where the conjecture is a theorem, cf. [3].) The prize application of our theory is the analogue of this result, given in Theorem 7.4, for generic cohomology $H^1_{\mathrm{gen}}(G; L)$ (in the sense of [18]). This theorem provides the first general positive evidence for a conjecture given by R. Guralnick [24] that 1-cohomology groups for finite groups with irreducible coecients are universally bounded. As pointed out more recently in [25], this conjecture remains an open question. The current largest dimension for $H^1(G, L)$ is 3, but such cohomology groups are usually 1-dimensional. First cohomology groups of finite groups with irreducible coefficient modules are important for the general study of maximal subgroups of finite groups [5]. We believe our results here on generic cohomology represent a significant step in understanding their asymptotic behavior, irrespective of the final verdict on Guralnick's conjecture. Also, the result appears to depend heavily on the use of quantum groups and on Theorem 6.7, possibly requiring the full machinery of this paper.

Finally, Section 8 contains examples suggesting the assumption $p > h$ occurring in §6 can be relaxed to $p \geq h$.

Some of the results in this paper were reported in [35].

1. Preliminaries

This section reviews some basic notions from the representation theory of a semisimple group and its relation to that of the associated quantum enveloping algebra. We then introduce the modules $\Delta^{\mathrm{red}}(\lambda)$ and $\nabla_{\mathrm{red}}(\lambda)$ which play a central role in this paper.

1.1. **General notation.** Fix the algebraically closed field $k = \overline{\mathbb{F}}_p$ of positive characteristic p. If H is a group scheme over k, let H-Mod (resp., H-mod) be the category of rational H-modules (resp., finite dimensional rational H-modules). By definition, a rational H-module is a comodule for the coordinate algebra $k[H]$ of H; see [26, Ch. 2]. Unless otherwise noted, we work with left modules.

Throughout this paper, G will denote a simply connected, semisimple algebraic group over k. We assume that G is defined and split over $\mathbb{F}_p$. In particular, there is a commutative Hopf algebra A_0 over $\mathbb{F}_p$ such that $k[G] \cong k \otimes_{\mathbb{F}_p} A_0$ as k-Hopf algebras. The Hopf algebra morphism $F^* : k[G] \to k[G]$ defined by putting $F^*(s \otimes a) = s \otimes a^p$, $s \in k$, $a \in A_0$, is the comorphism of the Frobenius morphism $F : G \to G$. If $V \in G$–Mod and r is a positive integer, write $V^{(r)}$ for the rational G-module obtained from V by making $g \in G$ act on V by $F^r(g)$. If $V \cong W^{(r)}$ for some rational G-module W, we write $W = V^{(-r)}$.

For a positive integer r, set G_r equal to the scheme-theoretic kernel of F^r. If H is a subgroup scheme of G, then $G_rH = (F^r)^{-1}(H)$ denotes the pull-back through F^r of H. In this paper, we consider only the case $r = 1$ and the group schemes G_1, G_1T and G_1B, where T is a fixed maximal $\mathbb{F}_p$-split torus and $B \supset T$ is a fixed Borel subgroup.

The Lie algebra $\mathfrak{g}$ of G is a restricted Lie algebra; $u(\mathfrak{g})$ denotes its restricted enveloping algebra. Then G_1-mod (resp., G_1T–mod, G_1B–mod) is equivalent to $u(\mathfrak{g})$–mod (resp., $u(\mathfrak{g}).T$, $u(\mathfrak{g}).B$-mod). Here $u(\mathfrak{g})$–mod denotes the category of finite dimensional $u(\mathfrak{g})$-modules, while $u(\mathfrak{g}).T$–mod and $u(\mathfrak{g}).B$–mod are the categories of finite dimensional $u(\mathfrak{g})$-modules with a compatible (rational) action of T and B, respectively.

Let $X = X(T)$ be the character group and $X^\vee = \mathrm{Hom}(\mathbb{G}_m, T)$ the cocharacter group of T. There is a natural pairing $(\, , \,) : X \times X^\vee \to \mathbb{Z} \cong \mathrm{End}(\mathbb{G}_m)$. Let $\Phi \subset X(T)$ be the root system of G with respect to T. If $\alpha \in \Phi$, let $\alpha^\vee \in X^\vee$ be the associated coroot. Let $Q := \mathbb{Z}\Phi \subset X(T)$ be the root lattice.

Let Φ^+ be a fixed set of positive roots, and let $\Pi = \{\alpha_1, \cdots, \alpha_n\} \subseteq \Phi^+$ be the simple roots in Φ. We assume that if $B^+ \supseteq T$ is the positive Borel subgroup defined by Φ^+, then B (the original Borel subgroup) is its opposite Borel subgroup (and so corresponds to the set $\Phi^- := -\Phi^+$ of negative roots). We regard X as a poset by putting $\lambda \leq \mu$, for $\lambda, \mu \in X$, if and only if $\mu - \lambda \in \mathbb{N}\Phi^+$. A second partial order $\uparrow$ on X is defined in terms of the affine Weyl group discussed below.

Let $X^+ \subset X$ be the set of dominant weights on T, i. e., $\lambda \in X$ belongs to X^+ if and only if $(\lambda, \alpha^\vee) \in \mathbb{Z}^+$ for all $\alpha \in \Pi$. Denote the fundamental dominant weights by $\varpi_1, \cdots, \varpi_n$; thus, $(\varpi_i, \alpha_j^\vee) = \delta_{i,j}$. We list Π and the fundamental dominant weights as in [8, Appendix]. We let X_1^+ be the set of restricted dominant weights, i. e., $\lambda \in X_1^+$ if and only if $0 \leq (\lambda, \alpha^\vee) < p$ for all $\alpha \in \Pi$.

1.2. **The affine and extended affine Weyl groups.** Let $\mathbb{E} = \mathbb{R} \otimes_{\mathbb{Z}} X$ be endowed with a positive definite, symmetric bilinear form $(\, , \,)$, invariant under the Weyl group W of Φ. We identify $X^\vee$ as a subgroup of $\mathbb{E}$, so that $\alpha^\vee = \frac{2}{(\alpha,\alpha)}\alpha$ and the pairing $X \times X^\vee \to \mathbb{Z}$ is compatible with the inner product.

The affine Weyl group $W_p = p\mathbb{Z}\Phi \rtimes W$ is the group of transformations on $\mathbb{E}$ generated by W and the normal subgroup consisting of translations by elements in $p\mathbb{Z}\Phi$. If $\alpha \in \Phi$ and $r \in \mathbb{Z}$, define $s_{\alpha,r} : \mathbb{E} \to \mathbb{E}$ by $s_{\alpha,r}(x) = x - ((x, \alpha^\vee) - rp)\alpha$. Then $s_{\alpha,r} \in W_p$. If $\alpha_0 \in \Phi$ denotes the maximal short root, then (W_p, S_p) is a Coxeter system, putting $S_p = \{s_{\alpha_1}, \cdots, s_{\alpha_n}, s_{\alpha_0,-1}\}$.

In this paper, we will usually use the "dot" action of W_p on $\mathbb{E}$, given by setting $w \cdot x = w(x+\rho) - \rho$, where $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$ is the Weyl weight. Let $C^+ \subset \mathbb{E}$ be the positive fundamental alcove; it consists of all $x \in \mathbb{E}$ satisfying the inequalities $0 < (x+\rho, \alpha_i^\vee)$, $i = 1, \cdots, n$, and $(x+\rho, \alpha_0^\vee) < p$. Put $C^- = w_0 \cdot C^+$, where w_0 is the maximal word in W. The closures $\overline{C^+}$ and $\overline{C^-}$ are fundamental domains for the action of W_p on $\mathbb{E}$. The subsets $w \cdot C^+ \subset \mathbb{E}$, $w \in W_p$, are the alcoves for W_p. If $C = w \cdot C^+$ is an alcove, put $C_\mathbb{Z} = C \cap X$ and $\overline{C}_\mathbb{Z} = \overline{C} \cap X$.

Using W_p we can define the $\uparrow$ partial ordering on X as follows: for $\lambda, \mu \in X$, $\lambda \uparrow \mu$ if and only if there is a sequence $\lambda = \lambda_0 \leq \lambda_1 \cdots \leq \lambda_t = \mu$ in which, for $0 \leq i < t$, $\lambda_{i+1} = s_{\beta_i, n_i} \cdot \lambda_i$ for some $\beta_i \in \Phi$ and some $n_i \in \mathbb{Z}$. (This partial ordering, taken from [26, 6.4], differs somewhat from that defined in [13, p. 527], although the two agree on $X^+ - \rho$.)

The walls of the simplex $\overline{C^-}$ are labeled by the simple reflections $s_{\alpha_0,-1}, s_{\alpha_1}, \cdots, s_{\alpha_n}$. Thus, the walls of any alcove C are labeled by the same set of simple reflections, since there is a unique $w \in W_p$ satisfying $w \cdot C = C^-$. Given $s \in S_p$, we can speak of the s-wall of an alcove C. Thus, the dot action of W_p carries an s-wall into an s-wall. If $\lambda \in C$, let Cs be the alcove which is obtained from C by reflection through the s-wall of C. Similarly, given $\lambda \in C$ and $s \in S_p$, let $\lambda s \in Cs$ be the image of λ through reflection through the s-wall of C.

The extended affine Weyl group $\widetilde{W_p}$ is defined to be $\widetilde{W_p} = pX \rtimes W$. Clearly, $W_p \trianglelefteq \widetilde{W_p}$ and $\widetilde{W_p}/W_p \cong X/Q$ is a finite abelian group. Also, $\widetilde{W_p}$ acts faithfully on $\mathbb{E}$, preserving the alcoves. Define N to be the stabilizer in $\widetilde{W_p}$ of C^+. Then $\widetilde{W_p} = N \rtimes W_p$ and $N \cong X/Q$.

Let $\phi : \widetilde{W_p} \to W$ be the surjective homomorphism with kernel X. For $\alpha \in \Phi$, $r \in \mathbb{Z}$, and $w \in \widetilde{W_p}$,

$$ws_{\alpha,r}w^{-1} = s_{\phi(w)\alpha, r+(\lambda,\phi(w)(\alpha^\vee))}.$$

Thus, $\widetilde{W_p}$ acts on the set of reflections $s_{\alpha,r}$ in $\widetilde{W_p}$, and N is also isomorphic to the stabilizer in $\widetilde{W_p}$ of S_p.

Let $l : W_p \to \mathbb{N}$ be the length function for the Coxeter system (W_p, S_p). We extend l to a length function $l : \widetilde{W_p} \to \mathbb{N}$ by setting $l(nw) = l(w)$, for $nw \in \widetilde{W_p}$, $n \in N$, $w \in W_p$. By the previous paragraph, $l(wn) = l(w)$.

The length function defines a function $l : X \to \mathbb{N}$ as follows. Given $\lambda \in X$, choose $w \in W_p$ of minimal length such that $w \cdot \lambda \in \overline{C^-}$. Then we set $l(\lambda) := l(w)$.

A weight $\lambda \in X$ is *regular* provided that $\lambda \in C$ for some alcove C. Let X_{reg} be the set of regular weights, so that $X_{\text{reg}} \neq \emptyset$ if and only if $p \geq h$. Otherwise, λ is called singular. We let X^+_{reg} be the set of all regular dominant weights.

1.3. **Representation theory.** If H is a group scheme over k, let H–Mod (resp., H-mod) be the abelian category of rational (resp., finite dimensional rational) H-modules. Given a closed subgroup scheme H of a group scheme K, let $\text{res}^K_H : K\text{–Mod} \to H\text{–Mod}$, be the (exact) restriction functor. It has a left exact right adjoint $\mathrm{ind}_H^K : H\text{–Mod} \to K\text{–Mod}$ (the induction functor).

We first consider the representation theory of the semisimple group G. For each $\lambda \in X^+$, let $L(\lambda) \in G$–mod be the irreducible representation of high weight λ. Write $\nabla(\lambda) = \mathrm{ind}_B^G \lambda$, where λ here denotes the one-dimensional rational B-module defined by extending λ to a linear character on B through the quotient map $B \twoheadrightarrow B/U \cong T$. If $w_0 \in W$ is the long word, define $\star : X \to X$ by $\tau \mapsto \tau^\star := -w_0(\tau)$. Set $\Delta(\lambda) = \nabla(\lambda^\star)^*$, where $(-)^*$ means linear dual. The module $\nabla(\lambda)$ (resp., $\Delta(\lambda)$) has socle (resp., head) isomorphic to $L(\lambda)$. We call $\Delta(\lambda)$ (resp., $\nabla(\lambda)$) the standard (resp., costandard) G-module of high weight λ. The $\Delta(\lambda)$ (resp., $\nabla(\lambda)$), $\lambda \in X^+$, form the standard (resp., costandard) modules in the highest weight categories G–mod and G–Mod.

The integral group ring $\mathbb{Z}X$ of X has $\mathbb{Z}$-basis e^ξ, $\xi \in X$. If $V \in G$–mod, put $\mathrm{ch}\ V \in \mathbb{Z}X$ equal to the formal character of V. If $\chi = \sum n_\xi e^\xi \in \mathbb{Z}X$, set $\chi^{(1)} = \sum n_\xi e^{p\xi}$. Thus, $\mathrm{ch}\ V^{(1)} = (\mathrm{ch}\ V)^{(1)}$. For $\lambda \in X^+$, $\mathrm{ch}\ \Delta(\lambda) = \mathrm{ch}\ \nabla(\lambda) = \chi(\lambda)$, where $\chi(\lambda)$ is given by the Weyl character formula

$$\chi(\lambda) = \sum_{w \in W} (-1)^{l(w)} e^{w\cdot\lambda} / \sum_{w \in W} (-1)^{l(w)} e^{w\cdot 0}. \tag{1.0.3}$$

Given a finite dimensional rational G-module M, define its Poincaré polynomial to be

$$p_{\mu,M}(t) = \sum_{n=0}^{\infty} \dim\ \mathrm{Ext}^n_G(M, \nabla(\mu)) t^n. \tag{1.0.4}$$

For later use, we recall the following well-known cohomological fact. For (a), see [13, Lemma 2.2] and for part (b), see [13, Lemma 3.2]

Lemma 1.1. *(a) Let* $\lambda, \mu \in X^+$. *Then*

$$\dim\ \mathrm{Ext}^n_G(\Delta(\lambda), \nabla(\mu)) = \delta_{n,0}\delta_{\lambda,\mu}.$$

(b) For any finite dimensional rational G*-module* M*,*

$$\mathrm{ch}\ M = \sum_{\mu \in X^+} p_{\mu,M}(-1)\, \mathrm{ch}\ \nabla(\mu). \tag{1.1.1}$$

If $V \in B$–mod, let $\mathcal{L}_V$ be the induced G-homogenous vector bundle on G/B with fibre V at B/B. Write $H^\bullet(G/B, V)$ for $H^\bullet(G/B, \mathcal{L}_V)$. Thus, for $\lambda \in X^+$, if k_λ denotes the 1-dimensional B-module defined by λ, then $\nabla(\lambda) \cong H^0(G/B, k_\lambda)$ and $\Delta(\lambda) \cong H^N(G/B, k_{w_0\cdot\lambda})$, where $N = \dim\ G/B$.

Given a finite dimensional rational B-module V, we denote its *Euler characteristic* to be

$$\chi(V) := \sum_{n=0}^{\infty} (-1)^n\, \mathrm{ch}\ H^n(G/B, V) \in \mathbb{Z}X. \tag{1.1.2}$$

Thus, if $V = k_\lambda$, for $\lambda \in X^+$, (1.1.2) is consistent with (1.0.3).

We will require some basic facts about the representation theory of the group schemes G_1B and B_1T. The set X indexes the irreducible objects in both G_1B–mod and G_1T–mod. Thus, if $\lambda \in X$, let $\widehat{L}_1(\lambda)$ be the irreducible G_1B-module of high weight λ. Its restriction to G_1T, which we usually continue to denote by $\widehat{L}_1(\lambda)$, remains irreducible. Given $\lambda \in X$, write $\lambda = \lambda_0 + p\lambda_1$, where $\lambda_0 \in X_1^+$ and $\lambda \in X$. Then in G_1B–mod or G_1T–mod, $\widehat{L}_1(\lambda) \cong p\lambda_1 \otimes \widehat{L}_1(\lambda_0)$, where $p\lambda_1$ denotes the one-dimensional module defined by the character $p\lambda_1$ on G_1B. In addition, $\widehat{L}_1(\lambda_0) \cong \operatorname{res}^G_{G_1B} L(\lambda_0)$.

Lemma 1.2. *For any rational G_1B-module V, we have*

$$R^\bullet \operatorname{ind}_{G_1B}^G V^{(1)} \cong (R^\bullet \operatorname{ind}_B^G V)^{(1)}.$$

Proof. In degree 0,

$$\begin{aligned}\operatorname{ind}_{G_1B}^G V^{(1)} &\cong (k[G] \otimes V^{(1)})^{G_1B} \\ &\cong (k[G]^{G_1} \otimes V^{(1)})^B \\ &\cong (k[G/G_1] \otimes V)^{B/B_1} \\ &\cong (\operatorname{ind}_B^G V)^{(1)},\end{aligned}$$

functorially in V. Let $I \in G_1B$-mod be injective. Then I is also an injective object in B-mod, so that the required isomorphism holds in all degrees by dimension shifting. □

Given $\lambda \in X$, put $\widehat{Z}_1(\lambda) := u(\mathfrak{g}) \otimes_{u(\mathfrak{b}^+)} k_\lambda$, where k_λ denotes the 1-dimensional rational B^+-module defined by the weight λ on B^+. Then $\widehat{Z}_1(\lambda) \in G_1B$–mod has dimension p^N, where $N = |\Phi^+|$, and irreducible head $\widehat{L}(\lambda)$.

1.4. **Character formulas.** Let $\lambda \in X^+$ and write $\lambda = w \cdot \lambda^-$, where $\lambda^- \in \overline{C^-_{\mathbb{Z}}}$ and w has minimal length among all elements $w' \in W_p$ which satisfy $w' \cdot \lambda^- = \lambda$. Because the isotropy subgroup of λ^- in W_p has the form W_J for some $J \subset S_p$, w is uniquely determined as a distinguished left coset representative of W_J in W. For $y, w \in W_p$, let $P_{y,w} \in \mathbb{Z}[t, t^{-1}]$ be the associated Kazhdan-Lusztig polynomial.[1] Define[2]

$$\chi_{\mathrm{KL}}(\lambda) = \sum_{y \in W_p, y\cdot\lambda^- \in X^+} (-1)^{l(w)-l(y)} P_{y,w}(-1)\chi(y \cdot \lambda^-). \tag{1.2.1}$$

The following result is proved by Kato for $p \geq h$, but the argument works for all p. (In Kato's argument, replace the weight λ in the interior of an alcove by a weight in its closure.)

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[1] Recall that $P_{y,w}$ is a polynomial in $q := t^2$ of degree $\leq (l(w) - l(y) - 1))/2$ (in q). We prefer below and throughout this paper to regard $P_{y,w}$ as a polynomial in t, albeit one which is a polynomial also in t^2.

[2] Let F be the unique facet containing λ. Then, using [26, 6.11], F lies in the upper closure of a unique alcove C. If C' is a second alcove satisfying $F \subseteq \overline{C'}$, then $C \uparrow C'$. In particular, if $w \in W_p$ satisfies $w \cdot C^- = C$, then w is the shortest element in W_p satisfying $w \cdot \lambda^- = \lambda$. In the expression below, given $\mu \in X^+$, there may well exist several $y \leq w$ such that $y \cdot \lambda^- = \mu$.

Lemma 1.3. *(Kato [27])Let $\lambda \in X^+$ have expansion $\lambda = \lambda_0 + p\lambda_1$ where $\lambda_0 \in X_1^+$ and $\lambda_1 \in X^+$. Then*

$$\chi_{\rm KL}(\lambda) = \chi_{\rm KL}(\lambda_0)\chi(\lambda_1)^{(1)}.$$

Following [34], we say that $\lambda \in X^+$ satisfies the Lusztig character formula (LCF) provided that ch $L(\lambda) = \chi_{\rm KL}(\lambda)$. Also, we say that λ satisfies the *homological* LCF (hLCF) provided that

$$t^{l(w)-l(y)}\overline{P}_{y,w} = p_{\mu,L(w\cdot\lambda^-)} = \sum_{n=0}^{\infty} \dim \operatorname{Ext}^n_G(L(w\cdot\lambda^-), \nabla(y\cdot\lambda^-))t^n. \tag{1.3.1}$$

In this expression, $\overline{P}_{y,w}$ is obtained from $P_{y,w}$ by replacing t by t^{-1} throughout, and the second equality is just the definition (1.0.4) of the Poincaré polynomial $p_{y.\lambda^-,L(w.\lambda^-)}$. If λ satisfies the LCF (resp., hLCF), we will often say that $L(\lambda)$ satisfies the LCF (resp., hLCF) *condition.*

Although the definitions in the previous paragraph have been made for arbitrary p, we will in this paper be concerned with the special case when $p \geq h$ (so that $X^+_{\rm reg} \neq \emptyset$).

1.5. **Quantum enveloping algebras.** From now on, the prime p is odd, and, if G has a component of type G_2, then $p > 3$. Let $C = (c_{i,j})$ be the Cartan matrix of G: thus, $c_{i,j} = (\alpha_j, \alpha_i^\vee)$, for $1 \leq i, j \leq n$.

Inside the function field $\mathbb{Q}(v)$, let $\mathcal{A} = \mathbb{Z}[v]_{\mathfrak{m}}$, where $\mathfrak{m} = (v-1, p)$. Let U' be the quantum enveloping algebra over $\mathbb{Q}(v)$ corresponding to the matrix C. It has generators $E_1, \cdots, E_n$, $F_1, \cdots, F_n$, $K_1^{\pm 1}, \cdots, K_n^{\pm 1}$ satisfying the familiar relations; cf. [4, §0], whose notation we generally follow. In particular, let U be the $\mathcal{A}$-subalgebra of U' generated by the divided powers $E_i^{(N)}$, $F_i^{(N)}$, $N \geq 1$, together with the elements $K_i^{\pm 1}$, $1 \leq i \leq n$.

Now let $\phi_p = 1 + v + \cdots + v^{p-1} \in \mathfrak{m}$ be the pth cyclotomic polynomial, and let $\mathcal{O} = \mathcal{A}/(\phi_p)$. Thus, $\mathcal{O}$ is a discrete valuation ring with maximal ideal $\mathfrak{n} = (p)$, quotient field K and residue field $\mathbb{F}_p$. Then $\zeta := v + (\phi_p) \in \mathcal{O}$ is a primitive pth root of unity. We regard $\mathcal{O}$ as an $\mathcal{A}$-module via $v \mapsto \zeta$. We put $\widetilde{U}_\zeta = \mathcal{O} \otimes U$ and $U_\zeta = K \otimes U$. Thus, $\widetilde{U}_\zeta$ is an integral $\mathcal{O}$-form of U_ζ. Put $\overline{U}_\zeta = \widetilde{U}_\zeta/\pi\widetilde{U}_\zeta$, and let I be the ideal in $\overline{U}_\zeta$ generated by the images of the elements $K_i - 1$, $1 \leq i \leq n$. Then, [32, (8.15)] states that

$$\overline{U}_\zeta/I \cong \mathrm{hy}_0(G), \tag{1.3.2}$$

the distribution algebra of G over $\mathbb{F}_p$. Put $\mathrm{hy}(G) = k \otimes_{\mathbb{F}_p} \mathrm{hy}_0(G)$, the distribution algebra of G over k.

Throughout this paper, the category of finite dimensional integrable, type 1 U_ζ-modules will be denoted by $\mathcal{C}_\zeta$. It is a highest weight category with irreducible (resp. standard, costandard) modules $L_\zeta(\lambda)$ (resp., $\Delta_\zeta(\lambda)$, $\nabla_\zeta(\lambda)$), $\lambda \in X^+$. For $\mu \in X^+$, $\operatorname{ch}\Delta_\zeta(\mu) = \operatorname{ch}\nabla_\zeta(\mu) = \chi(\mu)$.

Remark 1.4. As noted above, the irreducible, type 1, integral U_ζ modules are indexed by X^+. Subject to some possible restrictions, for any $\lambda \in X^+$, $\operatorname{ch} L_\zeta(\lambda) = \chi_{\mathrm{KL}}(\lambda)$. In type A_n and D_{2n}, there is no restriction on p (except our blanket assumption that p is odd). In type D_{2n+1}, we require $p > 3$. In other types, we require $p > h$; see [41, §7] for a detailed discussion and further references. In this case, we say λ satisfies the LCF$_\zeta$ condition. *Throughout this paper, we will always assume that the LCF$_\zeta$ holds for all $\lambda \in X^+$.* In addition, for $\lambda = w \cdot \lambda^- \in X^+_{\mathrm{reg}}$ with $\lambda^- \in C^-_{\mathbb{Z}}$, we have

$$t^{l(w)-l(y)}\overline{P}_{y,w} = \sum_{n=0}^{\infty} \dim \operatorname{Ext}^n_{\mathcal{C}_\zeta}(L_\zeta(w\cdot\lambda^-), \nabla_\zeta(y\cdot\lambda^-))t^n. \tag{1.4.1}$$

In other words, the hLCF$_\zeta$ holds for λ. It is not known to the authors if the assumption that λ is regular in (1.4.1) is necessary. In addition, we have, given $x\cdot\lambda^-, y\cdot\lambda^- \in X^+$,

$$\sum_{n=0}^{\infty} \dim \operatorname{Ext}^n_{\mathcal{C}_\zeta}(L_\zeta(x\cdot(-2\rho)), L_\zeta(y\cdot(-2\rho))t^n = \sum_z t^{l(x)+l(y)-2l(z)}\overline{P}_{z,x}\overline{P}_{z,y}, \tag{1.4.2}$$

where the summation on the right is over all $z \in W_p$ for which $z\cdot(-2\rho) \in X^+$. See the argument in [13, Thm. 3.9.1]

An *admissible lattice* in $M \in \mathcal{C}_\zeta$ is a free $\mathcal{O}$-submodule $\widetilde{M}$ of M such that $\widetilde{M}_K = K\otimes_{\mathcal{O}}\widetilde{M} \cong M$ and such that $\widetilde{M}$ is $\widetilde{U}_\zeta$-stable. If M_μ is a nonzero weight space in M, then $\widetilde{M}\cap M_\mu$ contains a basis for M_μ [20, (3.3)]. Define $\widetilde{\mathcal{C}}_\zeta$ to be the category of integrable, type 1 $\widetilde{U}_\zeta$-modules which are finitely generated over $\mathcal{O}$; see [20] for a discussion. For $M, N \in \mathcal{C}_\zeta$, we have

$$\operatorname{Ext}^\bullet_{\widetilde{\mathcal{C}}_\zeta}(\widetilde{M}, \widetilde{N})_K \cong \operatorname{Ext}^\bullet_{\mathcal{C}_\zeta}(M, N) \tag{1.4.3}$$

whenever $\widetilde{M}, \widetilde{N}$ are admissible lattices for M, N, respectively. In addition, suppose that $\widetilde{M}$ is an admissible lattice as above, and put $\overline{M} := \widetilde{M}/\pi\widetilde{M}$. For any $V \in G$–mod,

$$\operatorname{Ext}^\bullet_{\widetilde{\mathcal{C}}_\zeta}(\widetilde{M}, V) \cong \operatorname{Ext}^\bullet_G(\overline{M}, V). \tag{1.4.4}$$

The proof of (1.4.4) is given in [20, (2.9) & Thm. 3.2]. The proof of the [20, (2.9)] (a reduction to standard issues with projective resolutions) also establishes (1.4.3). These observations will be used from §5 on.

In addition, for $\lambda \in X^+$, there exist admissible lattices $\widetilde{\Delta}_\zeta(\lambda)$ and $\widetilde{\nabla}_\zeta(\lambda)$ for $\Delta_\zeta(\lambda)$ and $\nabla_\zeta(\lambda)$, respectively, so that $\widetilde{\Delta}_\zeta(\lambda)/\pi\widetilde{\Delta}_\zeta(\lambda) \cong \Delta(\lambda)$ and $\widetilde{\nabla}_\zeta(\lambda)/\pi\widetilde{\nabla}_\zeta(\lambda) \cong \nabla(\lambda)$ [20, p. 159].

Given an irreducible U_ζ-module $L_\zeta(\lambda)$ of high weight λ, fix a high weight vector $v^+ \in L_\zeta(\lambda)$. Then there is a unique admissible lattice $\widetilde{L}^{\min}_\zeta(\lambda)$ (resp., $\widetilde{L}^{\max}_\zeta(\lambda)$) of $L_\zeta(\lambda)$ which is minimal (resp., maximal) with respect to all admissible lattices $\widetilde{L}$ such that $\widetilde{L}\cap L_\zeta(\lambda)_\lambda = \mathcal{O}v^+$. For example, we can take $\widetilde{L}^{\min}_\zeta(\lambda) = \widetilde{U}_\zeta\cdot v^+$. By abuse of notation, we call $\widetilde{L}^{\min}_\zeta(\lambda)$ (resp., $\widetilde{L}^{\max}_\zeta(\lambda)$) the minimal (resp., maximal) lattice of $L_\zeta(\lambda)$. Any two "minimal" (resp., "maximal") lattices are isomorphic as $\widetilde{U}_\zeta$-modules.

For $\lambda \in X^+$, put

$$\Delta^{\rm red}(\lambda) = \widetilde{L}^{\min}_\zeta(\lambda)/\pi\widetilde{L}^{\min}_\zeta(\lambda) \text{ and } \nabla_{\rm red}(\lambda) = \widetilde{L}^{\max}_\zeta(\lambda)/\pi\widetilde{L}^{\max}_\zeta(\lambda).$$

We call the rational G-modules $\Delta^{\rm red}(\lambda)$ (resp., $\nabla_{\rm red}(\lambda)$) the ζ-*reduced* standard (resp., costandard) module of highest weight λ. Under the assumption that the LCF holds for U_ζ for all $\lambda \in X^+$(see above), our remarks show that

$$\text{(1.4.5)} \qquad {\rm ch}\ \Delta^{\rm red}(\lambda) = {\rm ch}\ \nabla_{\rm red}(\lambda) = \chi_{\rm KL}(\lambda)$$

for all $\lambda \in X^+$.

In order to give another description of $\Delta^{\rm red}(\lambda)$, we require the following unpublished result of E. Cline.

Lemma 1.5. *(E. Cline) For a restricted weight $\lambda \in X_1^+$, there is a surjective morphism $\widehat{Z}_1(\lambda) \twoheadrightarrow \Delta(\lambda)$ in G_1B–mod.*

Proof. Let $v^+ \in \Delta(\lambda)$ be a high weight vector. There is a natural G_1B-module map $\widehat{Z}_1(\lambda) \to \Delta(\lambda)$ which sends $u \otimes 1 \in \widehat{Z}_1(\lambda)$ to $u \cdot v^+$. The image $E := u(\mathfrak{g}) \cdot v^+ = u(\mathfrak{b}) \cdot v$ of this map is a $u(\mathfrak{g})$-submodule of $\Delta(\lambda)$. Of course, $v^+ \in E$, and we must show that $\Delta(\lambda) = kG \cdot v^+ \subseteq E$.

Let $\alpha \in -\Pi$ be the negative of a simple root and fix an α-root vector $x_\alpha \in \mathfrak{g}$. For a positive integer n, let $x_\alpha^{(n)}$ be the corresponding divided power element in ${\rm hy}(G)$. Since $\lambda \in X_1^+$, for $n \geq p$, $\lambda - n\alpha$ is not a weight in $\Delta(\lambda)$, it follows that $x_\alpha^{(n)} \cdot v^+ = 0$. (For example, $\Delta(\lambda)$ is obtained by "reduction mod p" from the complex irreducible module of high weight λ, and $(\lambda, -\alpha^\vee) < p$.) Hence, $\exp(tx_\alpha) := \sum_{n\geq 0} t^n x_\alpha^{(n)}$ satisfies $\exp(tx_\alpha) \cdot v^+ = \sum_{n=0}^{p-1} t^n x_\alpha^{(n)} \cdot v^+ \in E$, for all $t \in k$. If $U_\alpha \subset B$ is the root subgroup determined by α, we obtain that $U_\alpha \cdot v^+ \subseteq E$. But B is generated by T and the U_α, $-\alpha \in \Pi$, so $kBB^+ \cdot v^+ \subset E$. Since BB^+ is dense in G, we get $kG \cdot v^+ \subseteq E$, as required. □

The following proposition is the $r = 1$ case of a result of Lin [30, Thm. 2.7]. It shows that the $\Delta^{\rm red}$- and $\nabla_{\rm red}$-construction behaves well with respect to tensor products. (The proof we give would also work for $r > 1$, and seems similar to Lin's proof which does not explicitly use Lemma 1.5, or its $r > 1$ analogue.)

Proposition 1.6. *Suppose $\lambda = \lambda_0 + p\lambda_1$ where $\lambda_0 \in X_1^+$ and $\lambda_1 \in X^+$. Then $\Delta^{\rm red}(\lambda) = \Delta^{\rm red}(\lambda_0) \otimes \Delta(\lambda_1)^{(1)}$ and $\nabla_{\rm red}(\lambda) = \nabla_{\rm red}(\lambda_0) \otimes \nabla(\lambda_1)^{(1)}$.*

Proof. It suffices to prove the first equality; the second then follows by duality.

Let $v_0^+ \in \Delta^{\rm red}(\lambda_0)$ and $v_1^+ \in \Delta(\lambda_1)^{(1)}$ be high weight vectors, so that $v_0^+ \otimes v_1^+$ is a high weight vector in $\Delta^{\rm red}(\lambda_0) \otimes \Delta(\lambda_1)^{(1)}$. The latter module is the reduction mod

p of the lattice $\widetilde{L}_q^{\min}(\lambda_0)\otimes\widetilde{L}_q^{\min}(p\lambda_1)$ in $L_q(\lambda)$, so has the same dimension as $\Delta^{\rm red}(\lambda)$ since $L_q(\lambda)\cong L_q(\lambda_0)\otimes L_q(p\lambda)$ by the quantum tensor product theorem. Also, we can arrange an inclusion $\widetilde{L}_q^{\min}(\lambda)\subseteq\widetilde{L}_q^{\min}(\lambda_0)\otimes\widetilde{L}_q^{\min}(p\lambda_1)$ with reduction mod p of the inclusion sending a high weight vector of $\Delta^{\rm red}(\lambda)$ to $v_0^+\otimes v_1^+$. Thus, it suffices to prove that $v_0^+\otimes v_1^+$ generates $\Delta^{\rm red}(\lambda_0)\otimes\Delta(\lambda_1)^{(1)}$.

Let $\mathfrak{r}=\operatorname{rad}(\Delta^{\rm red}(\lambda_0))$, so that $\Delta^{\rm red}(\lambda_0)/\mathfrak{r}\cong L(\lambda_0)$. Let $\mu=\mu_0+p\mu_1\in X^+$ with $\mu_0\in X_1^+$ and $\mu_1\in X^+$. Then

$$\operatorname{Hom}_G(L(\lambda_0)\otimes\Delta(\lambda_1)^{(1)},L(\mu))\cong\operatorname{Hom}_G(\Delta(\lambda_1),L(\mu_1))\otimes\operatorname{Hom}_{G_1}(L(\lambda_0),L(\mu_0)),$$

since $L(\mu)\cong L(\mu_0)\otimes L(\mu_1)^{(1)}$ by the tensor product theorem. It follows that that $L(\lambda_0)\otimes\Delta(\lambda_1)^{(1)}$ has a irreducible head $L(\lambda_0)\otimes L(\lambda_1)^{(1)}$. In particular, the image of $v_0^+\otimes v_1^+$ generates the quotient $\Delta^{\rm red}(\lambda_0)\otimes\Delta(\lambda_1)^{(1)}/\mathfrak{n}\otimes\Delta(\lambda_1)^{(1)}\cong L(\lambda_0)\otimes\Delta(\lambda_1)^{(1)}$. In other words, if E is the G-module generated by $v_0^+\otimes v_1^+$ in $\Delta^{\rm red}(\lambda_0)\otimes\Delta(\lambda_1)^{(1)}$, then $E+\mathfrak{r}\otimes\Delta(\lambda_1)^{(1)}=\Delta^{\rm red}(\lambda_0)\otimes\Delta(\lambda_1)^{(1)}$.

Next, consider $\Delta^{\rm red}(\lambda_0)$ as a module for $u(\mathfrak{g})-T$. By Lemma 1.5, $\Delta^{\rm red}(\lambda_0)$ is a homomorphic image of $\widehat{Z}_1(\lambda_0)$. Since $\widehat{Z}_1(\lambda_0)$ has an irreducible head, $\Delta^{\rm red}(\lambda_0)$ must also, as a $u(\mathfrak{g}).T$-module, have an irreducible head. and, consequently, $\mathfrak{r}\otimes\Delta(\lambda_1)^{(1)}$ is the $u(\mathfrak{g}).T$-radical of $\Delta^{\rm red}(\lambda_0)\otimes\Delta(\lambda_1)^{(1)}$. (Note that $u(\mathfrak{g})$ acts trivially on the second factor.) Thus, the above discussion and Nakayama's lemma for $u(\mathfrak{g}).T$-modules implies that $E=\Delta^{\rm red}(\lambda_0)\otimes\Delta(\lambda_1)^{(1)}$, proving the proposition. □

2. Grothendieck groups

We introduce and study the categories $\mathcal{E}^L$ and $\mathcal{E}^R$ associated to the semisimple group G, as well as their attached "enriched" Grothendieck groups. Our setup follows that introduced in [13]. Its chief advantage is that it allows the tracking of degree information in the derived category of rational G-modules, which the ordinary Grothendieck group of the derived category does not do. There are other ways to track degree information. For instance, the homological degree can be tracked using the Grothendieck group of a suitable (e. g., Koszul) $\mathbb{Z}$-graded highest weight category; see [15]. In later paper [36], we show how even a suitable $\mathbb{Z}/2$-grading is sufficient to track degree information, using again a suitably enriched Grothendieck group structure. However, no such graded structures will be required in this paper.

2.1. **The categories $\mathcal{E}^L$, $\mathcal{E}^R$.** In this section, we work with $\mathcal{D}=D^b(G\text{–mod})$, the bounded derived category of the abelian category G–mod. If we write $G\text{–mod}=\amalg\mathcal{B}$, $\mathcal{B}$ running over the set of blocks in G–mod, then $\mathcal{D}=\amalg D^b(\mathcal{B})$ in terms of the derived categories of the various blocks. In practice, we will generally consider those blocks associated to the regular dominant weights in a W_p-orbit.

Recall that [7, (1.3.9] defines an associative product $*$ on the set of isomorphism classes of objects in any triangulated category. The categories $\mathcal{E}^L$ and $\mathcal{E}^R$ are full subcategories of $\mathcal{D}$ obtained by applying the $*$-product to shifts $\Delta(\lambda)[m]$ (resp., $\nabla(\lambda)[m]$)

of standard (resp., costandard) modules, where it is required that $m \equiv l(\lambda) \bmod 2$. The categories $\mathcal{E}^L, \mathcal{E}^R$ were first introduced in [14]; for a recent treatment, see [26, Appendix C].

In more detail, regard G–mod as embedded as a full subcategory of $\mathcal{D}$: if $M \in G$–mod, then M is also regarded as a complex in $\mathcal{D}$ concentrated in degree 0. Define $\mathcal{E}^L_0$ to be the full subcategory of $\mathcal{D}$ consisting of all objects isomorphic to finite direct sums of various $\Delta(\lambda)[m]$, where $\lambda \in \Xi$ and $m \equiv l(\lambda) \bmod 2$. Here [1] is the shifting functor in $\mathcal{D}$, and, if $m > 0$, $[m] = \underbrace{[1] \circ \cdots \circ [1]}_{m}$ (with the standard convention if $m < 0$). If $\mathcal{E}^L_i$ is defined, $i = 0, 1, \cdots$, let $\mathcal{E}^L_{i+1}$ be the strict, full subcategory of $\mathcal{D}$ consisting of all objects X for which there exists a distinguished triangle $A \to X \to B \to$, with $A, B \in \mathcal{E}^L_i$. Then $\mathcal{E}^L_0 \subseteq \mathcal{E}^L_1 \subseteq \cdots$, and we set

$$\mathcal{E}^L = \mathcal{E}^L(G\text{–mod}) = \bigcup_{i=0}^{\infty} \mathcal{E}^L_i. \tag{2.0.1}$$

The category $\mathcal{E}^R$ is obtained by replacing the $\Delta(\lambda)$ throughout by $\nabla(\lambda)$. Also, if $\mathcal{B}$ is a block in G–mod, then we can construct $\mathcal{E}^L(\mathcal{B})$ and $\mathcal{E}^R(\mathcal{B})$ by restricting to the block $\mathcal{B}$. Thus, $\mathcal{E}^L$ (resp., $\mathcal{E}^R$) decomposes into the direct sum of the various $\mathcal{E}^L(\mathcal{B})$ (resp., $\mathcal{E}^R(\mathcal{B})$) as $\mathcal{B}$ runs over the blocks of G–mod.

In what follows, $\widehat{\mathcal{E}^L} = \mathcal{E}^L \oplus \mathcal{E}^L[1]$ and $\widehat{\mathcal{E}^R} = \mathcal{E}^R \oplus \mathcal{E}^R[1]$.

Lemma 2.1. *(a)* $\mathrm{Hom}_{\mathcal{D}}(\mathcal{E}^L[m], \mathcal{E}^R) = 0$ *for any odd integer* m.

(b) Given $\lambda \in X^+$*, the following are equivalent:*

1. $L(\lambda) \in \widehat{\mathcal{E}^L}$;
2. $L(\lambda)[-l(\lambda)] \in \mathcal{E}^L$;
3. $L(\lambda) \in \widehat{\mathcal{E}^R}$;
4. $L(\lambda)[-l(\lambda)] \in \mathcal{E}^R$.

Proof. (a) is clear from Lemma 1.1, so we prove (b):

(1) $\Longrightarrow$ (2): Suppose $L(\lambda) \in \widehat{\mathcal{E}^L}$, so that $L(\lambda) \in \mathcal{E}^L \oplus \mathcal{E}^L[1]$. Thus, $L(\lambda)[-l(\lambda)] \in \mathcal{E}^L$ or $L(\lambda)[-l(\lambda)] \in \mathcal{E}^L[1]$. If $L(\lambda)[-l(\lambda)] \in \mathcal{E}^L[1]$, then $\mathrm{Hom}_{\mathcal{D}}(L(\lambda)[-l(\lambda)], \mathcal{E}^R) = 0$. But

$$\mathrm{Hom}_{\mathcal{D}}(L(\lambda)[-l(\lambda)], \nabla(\lambda)[-l(\lambda)]) \neq 0,$$

contradiction.

(2) $\Longrightarrow$ (1): trivial

(3) $\Longleftrightarrow$ (4): similar to (1) $\Longleftrightarrow$ (2).

(1) $\Longleftrightarrow$ (3): The category G–mod has a duality D fixing irreducible objects and taking each $\Delta(\lambda)$ to $\nabla(\lambda)$, $\lambda \in X^+$; see [12]. It induces a duality D on $\mathcal{D}$ which interchanges $\widehat{\mathcal{E}^L}$ and $\widehat{\mathcal{E}^R}$. Thus, if (1) holds so does (3), and conversely. □

Lemma 2.2. *In the poset* $(X^+, \uparrow)$*, let* Γ *be a finite ideal of regular weights. The following statements are equivalent:*

(1) $L(\lambda) \in \widehat{\mathcal{E}^R}$ *for all* $\lambda \in \Gamma$*;*
(2) $L(\lambda) \in \widehat{\mathcal{E}^L}$ *for all* $\lambda \in \Gamma$*;*
(3) $\mathrm{Ext}^n_G(L(\lambda), \nabla(\mu)) \neq 0 \implies n \equiv l(\lambda) - l(\mu) \bmod 2$*, for all* $\lambda, \mu \in \Gamma$*;*
(4) $\mathrm{Ext}^n_G(\Delta(\lambda), L(\mu)) \neq 0 \implies n \equiv l(\lambda) - l(\mu) \bmod 2$*, for all* $\lambda, \mu \in \Gamma$*;*
(5) *the LCF holds for all* $\lambda \in \Gamma$*;*
(6) *the hLCF holds for all* $\lambda \in \Gamma$*.*

Proof. (1) $\iff$ (2) follows as in the proof of (1) $\iff$ (3) in Lemma 2.1. Next, (2) $\iff$ (3), using Lemma 2.1(b) and [13, Thm. 2.4]. Also, (3) $\iff$ (4) since $\mathrm{Ext}^\bullet_G(L(\lambda), \nabla(\mu)) \cong \mathrm{Ext}^\bullet_G(\Delta(\mu), L(\lambda))$. For (1) $\iff$ (5) $\implies$ (6), see [13, Thm. 5.3]. Finally, suppose that (6) holds. Since $P_{y,w}$ is a polynomial in $q = t^2$, (3) follows. □

Remark 2.3. The proof of the above lemma carries over easily to the quantum enveloping algebra case. In view of Remark 1.4, and Lemma 2.2(5), all six conditions holds for $L_\zeta(\lambda)$.

We also have the following related result, which follows easily from [13, Thm. 2.4].

Lemma 2.4. *Let* $M \in G - \mathrm{mod}$*. Then* $M \in \mathcal{E}^L$ *if and only if* $\mathrm{Ext}^n_G(M, \nabla(\lambda)) \neq 0$ *implies that* $n \equiv l(\lambda) \bmod 2$ *for all* $\lambda \in X^+$*. If the composition factors of* M *have the form* $L(w \cdot \lambda^-)$ *for some* $\lambda^- \in \overline{C^-_{\mathbb{Z}}}$*, then it is sufficient to consider only those* λ *of the form* $y \cdot \lambda^-$ *which satisfy* $\lambda \leq w \cdot \lambda^-$ *for some composition factor* $L(w \cdot \lambda^-)$ *in* M*.*

Section appendix: application to Levi subgroups. We recast some results from [17] and [34] using the categories $\mathcal{E}^L$ and $\mathcal{E}^R$.

Let $H \supseteq T$ be a Levi subgroup of G. Let $\Phi_H \subseteq \Phi$ be the root system of T in H. For $\omega \in X$, let $\Omega = \omega + \mathbb{Z}\Phi_H \subset X$. There is an exact truncation functor $\tau_\Omega : G\text{–mod} \to H\text{–mod}$ which assigns to a rational H-module V the module $\tau_\Omega V = \bigoplus_{\mu \in \Omega} V_\mu$, where V_μ is the μ-weight space of T in V. For $\lambda \in \Omega$, τ_Ω carries $\Delta(\lambda)$, $\nabla(\lambda)$ and $L(\lambda)$ to the corresponding objects $\Delta_H(\lambda)$, $\nabla_H(\lambda)$ and $L_H(\lambda)$, respectively. If $\lambda \notin \Omega$, then τ_Ω maps these modules to the zero module. (See [17] for more discussion.)

Let $\mathcal{D}_H$ be the bounded derived category of H–mod.

Theorem 2.5. *Let* $X \in \widehat{\mathcal{E}^L}$ *and* $Y \in \widehat{\mathcal{E}^R}$*. Then the map*

$$\mathrm{Hom}^\bullet_{\mathcal{D}}(X, Y) \to \mathrm{Hom}^\bullet_{\mathcal{D}_H}(\pi_\Omega X, \pi_\Omega Y) \tag{2.5.1}$$

induced by π_Ω *is surjective.*

Proof. Let $\mathcal{D}_H := D^b(H\text{–mod})$. We can replace X or Y by $X[1]$ of $Y[1]$ to assume that $X \in \mathcal{E}^L$ and $Y \in \mathcal{E}^R$ to begin with. Now assume that $X \in \mathcal{E}^L_s$ and $Y \in \mathcal{E}^R_t$ for $s, t \geq 0$. By Lemma 1.1

$$\dim \mathrm{Ext}^n_G(\Delta(\lambda), \nabla(\mu)) = \delta_{n,0}\delta_{\lambda,\mu} = \dim \mathrm{Ext}^n_H(\Delta_H(\lambda), \nabla_H(\mu)), \tag{2.5.2}$$

the result is clear if $s + t = 0$.

We prove (2.5.1) by induction on $s+t$. By (2.5.2), $\mathrm{Hom}^n_{\mathcal{D}}(U,V)=0$ if n odd and $U\in\mathcal{E}^L$, $V\in\mathcal{E}^R$. An analogous statement holds for H. Now suppose that $t\in\mathbb{N}^+$, so there is a distinguished triangle $Y'\to Y\to Y''\to$ with $Y',Y''\in\mathcal{E}^R_{t-1}$. To prove that $\mathrm{Hom}^n_{\mathcal{D}}(X,Y)$ maps surjectively to $\mathrm{Hom}^n_{\mathcal{D}_H}(\pi_\Omega X,\pi_\Omega Y)$, we can assume that n is even. Since $\mathrm{Hom}_{\mathcal{D}}(X,-)$ and $\mathrm{Hom}_{\mathcal{D}_H}(\pi_\Omega X,-)$ are cohomological functors, there is a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}^n(X,Y') \arrow[r] \arrow[d] & \mathrm{Hom}^n(X,Y) \arrow[r] \arrow[d] & \mathrm{Hom}^n(X,Y'') \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Hom}^n(\pi_\Omega X,\pi_\Omega Y') \arrow[r] & \mathrm{Hom}^n(\pi_\Omega X,\pi_\Omega Y) \arrow[r] & \mathrm{Hom}^n(\pi_\Omega X,\pi_\Omega Y'') \arrow[r] & 0
\end{tikzcd}$$

with exact rows. By induction the left and right vertical maps are surjective, so the middle vertical map is surjective by the snake lemma. A similar contravariant argument works if $s>0$. Thus, (2.5.1) is surjective, as required. □

The above result can be used to give another explanation of the following result [34].

Corollary 2.6. *Let λ,μ be dominant weights lying in Ω. Assume that $L(\lambda),L(\mu)\in\widehat{\mathcal{E}^L}$. Then the map $\mathrm{Ext}^\bullet_G(L(\lambda),L(\mu))\to\mathrm{Ext}^\bullet(L_H(\lambda),L_H(\mu))$ induced by π_Ω is surjective.*

2.2. **The enriched Grothendieck groups.** We continue to work with the category G–mod with $\Lambda=X^+$, regarded as a poset with respect to the $\leq$ partial ordering introduced before. For our applications, we could restrict to the sum of the regular blocks in G–mod (or even to an individual regular block, corresponding to $W_p\cdot\lambda^-\cap X^+$, where $\lambda^-\in C^-_{\mathbb{Z}}$ is some fixed weight). In any event, we use the length function $l:X^+\to\mathbb{Z}$ defined in §1.2.

The enriched Grothendieck group $K_0^L=K_0^L(G\text{–mod})$ is the quotient of the free abelian group on objects of $\widehat{\mathcal{E}^L}$ by the subgroup spanned by all relations $X+Z-Y$ where $X\to Y\to Z\to$ is a distinguished triangle with terms X,Y,Z either $\mathcal{E}^L$ or in $\mathcal{E}^L[1]$, or a direct sum of two such distinguished triangles. If $X\in\widehat{\mathcal{E}^L}$, let $[X]_L$ denote its image in K_0^L. The definition easily implies that K_0^L is a free module for the ring $\mathbb{Z}[t,t^{-1}]$ of Laurent polynomials in which $t[X]_L=X[-1]_L$. Then K_0^L has basis $\{[\Delta(\lambda)]_L\}_{\lambda\in X^+}$.

The discussion in this and the next paragraph is largely taken from [13]; see especially Proposition 2.3 there. The right enriched Grothendieck group $K_0^R=K_0^R(G\text{–mod})$ is defined similarly, using $\widehat{\mathcal{E}^R}$. It is again a free $\mathbb{Z}[t,t^{-1}]$-module with basis $\{[\nabla(\lambda)]_R\}_{\lambda\in X^+}$. Again, $t[X]_R:=[X[-1]]_R$ for $X\in\widehat{\mathcal{E}^R}$. There is a natural bilinear form

$$\langle\ ,\ \rangle : K_0^L\times K_0^R\to\mathbb{Z}[t,t^{-1}],$$

given by putting

$$\langle[M]_L,[N]_R\rangle=\sum_n \dim\mathrm{Hom}^n_{\mathcal{D}}(M,N)t^n$$

in $\mathbb{Z}[t,t^{-1}]$ for $M \in \widehat{\mathcal{E}^L}, N \in \widehat{\mathcal{E}^R}$. Since

$$\mathrm{Hom}^i_{\mathcal{D}}(M[-n],N) = \mathrm{Hom}^i_{\mathcal{D}}(M,N[n]) = \mathrm{Hom}^{n+i}_{\mathcal{D}}(M,N),$$

this form is sesquilinear with respect to the action of $\mathbb{Z}[t,t^{-1}]$. That is,

$$\langle f(t)[M]_L, [N]_R\rangle = \langle [M]_L, [N]_R f(t^{-1})\rangle = \langle [M]_L, [N]_R\rangle f(t^{-1})$$

for any $f(t) \in \mathbb{Z}[t,t^{-1}]$. By Lemma 1.1, the bases $\{[\Delta(\lambda)]_L\}_{\lambda\in X^+}$ and $\{[\nabla(\lambda)]_R\}_{\lambda\in X^+}$ are naturally "dual" to one another with respect to this pairing.

In particular, if $M \in \widehat{\mathcal{E}^L}$ and

$$[M]_L = \sum_\mu \overline{p}_{M,\mu}[\Delta(\mu)], \tag{2.6.1}$$

then

$$p_{M,\mu} = \langle [M]_L, [\nabla(\mu)]_R\rangle = \sum \dim \mathrm{Hom}^n_{\mathcal{D}}(M,\nabla(\mu))t^n. \tag{2.6.2}$$

The Grothendieck group $K_0 = K_0(G\text{–mod})$ of G–mod is canonically isomorphic to the Grothendieck group of the triangulated category $\mathcal{D}$ [22]; in practice, we identify the two. If $X \in \mathcal{D}$, let $[X]$ denote its image in K_0. Then K_0 is a free abelian group with basis $\{[\Delta(\lambda)] = [\nabla(\lambda)]\}_{\lambda\in X^+}$. Regarding K_0 as a $\mathbb{Z}[t,t^{-1}]$-module in which t acts as multiplication by -1, there module morphisms $\pi_L : K_0^L \to K_0$ and $\pi_R : K_0^R \to K_0$ satisfying

$$\pi_L \sum_\lambda p_\lambda[\Delta(\lambda)]_L = \sum_\lambda p_\lambda(-1)[\Delta(\lambda)] = \pi_R \sum_\lambda p_\lambda[\nabla(\lambda)]_R \tag{2.6.3}$$

for $\sum_\lambda p_\lambda[\Delta(\lambda)]_L \in K_0^L$ and $\sum_\lambda p_\lambda[\nabla(\lambda)]_R \in K_0^R$. In particular, for $M \in \widehat{\mathcal{E}^L}$, we have from (2.6.1), (2.6.2) and (2.6.3) that

$$[M] = \sum_\mu \sum_n (-1)^n \dim \mathrm{Hom}^n_{\mathcal{D}}(M,\nabla(\mu))[\Delta(\mu)]. \tag{2.6.4}$$

Lemma 2.7. *Assume that the hLCF holds for some* $\lambda \in X^+_{\mathrm{reg}}$. *Then the LCF holds for* λ *and* $L(\lambda) \in \widehat{\mathcal{E}^L}$.

Proof. Write $\lambda = x\cdot\lambda^-$ for $\lambda^- \in C^-_{\mathbb{Z}}$ and assume that (1.3.1) holds. Since $P_{y,x}$ is a polynomial in t^2, it follows that if $\mathrm{Ext}^n_G(L(x\cdot\lambda^-),\nabla(y\cdot\lambda^-) \neq 0$, then $n \equiv l(x) - l(y) = l(\lambda) - l(y\cdot\lambda^-) \bmod 2$. If $\mu \in X^+$ does not have the form $y\cdot\lambda^-$, then $\mathrm{Ext}^\bullet_G(L(\lambda),\nabla(\mu)) = 0$. Thus, Lemma 2.4 and Lemma 2.1 implies that $L(\lambda) \in \widehat{\mathcal{E}^L}$.

Therefore, (2.6.4) is applicable to $L(\lambda)$, so that

$$\begin{aligned}[L(\lambda)] &= \sum_y \sum_n (-1)^n \dim \operatorname{Ext}^n_G(L(\lambda), \nabla(y\cdot\lambda^-))[\Delta(y\cdot\lambda^-)] \\ &= \sum_y (-1)^{l(x)-l(y)} \overline{P}_{y,x}(-1)[\Delta(y\cdot\lambda^-)] \\ &= \sum_y (-1)^{l(x)-l(y)} P_{y,x}(-1)[\Delta(y\cdot\lambda^-)],\end{aligned}$$

since $P_{y,x}(-1) = P_{y,x}(1)$. Replacing $[L(\lambda)]$ (resp., $[\nabla(y\cdot\lambda^-)]$) by ch $L(\lambda)$ (resp., ch $\Delta(y\cdot\lambda^-) = \chi(y\cdot\lambda^-)$), we obtain that λ satisfies the LCF. □

3. Hecke operators and paths

Given $\lambda \in X$, there is an exact, additive projection functor $\mathrm{pr}_\lambda : G$–mod $\to G$–mod which assigns to $V \in G$–mod the largest G-submodule $\mathrm{pr}_\lambda V$ all of whose composition factors have the form $L(\mu)$ for some $\mu \in X^+ \cap W_p \cdot \lambda$. If $\operatorname{Ext}^1_G(L(\nu), L(\mu)) \neq 0$ for $\nu, \mu \in X^+$, then $\nu \in W_p \cdot \mu$. Hence, given $V \in G$–mod, $V \cong \oplus_{\lambda\in\overline{C}^+_{\mathbb{Z}}} \mathrm{pr}_\lambda V$.

Now let $\lambda, \mu \in \overline{C}^+_{\mathbb{Z}}$ and let ν_1 be the unique element in $W(\mu-\lambda)\cap X^+$. Define $T^\mu_\lambda : G$–mod $\to G$–mod by putting

$$T^\mu_\lambda(V) = \mathrm{pr}_\mu(L(\nu_1)\otimes \mathrm{pr}_\lambda V). \tag{3.0.1}$$

The functor T^μ_λ is an exact, additive functor with adjoint given by T^λ_μ. As such, T^μ_λ acts naturally on the category $\mathcal{D}$. By [26, Prop. 7.11], if $\lambda, \mu \in X^+_{\mathrm{reg}}$, T^μ_λ preserves the full subcategories $\widehat{\mathcal{E}^L}$ and $\widehat{\mathcal{E}^R}$.

We will also need to work with the category G_1B–mod. For $\lambda \in X$, the projection functor $\widehat{\mathrm{pr}}_\lambda : G_1B$–mod $\to G_1B$–mod is defined by putting $\widehat{\mathrm{pr}}_\lambda(V)$, $V \in G_1B$–mod, equal to the largest submodule of V all of whose G_1B-composition factors have the form $\widehat{L}_1(\mu)$ for some $\mu \in W_p\cdot\lambda$. Then $V = \oplus_{\lambda\in\overline{C}^+_{\mathbb{Z}}} \widehat{\mathrm{pr}}_\lambda(V)$, since given $\lambda, \mu \in X$, if $\operatorname{Ext}^1_{G_1B}(\widehat{L}(\nu), \widehat{L}(\mu)) \neq 0$, then $\nu \in W_p \cdot \mu$ [26, Prop. 9.20]. In addition,

$$\operatorname{res}^G_{G_1B} \circ \mathrm{pr}_\lambda = \widehat{\mathrm{pr}}_\lambda \circ \operatorname{res}^G_{G_1B}. \tag{3.0.2}$$

As discussed in [26, Ch. 9], given $\lambda, \mu \in \overline{C}^+_{\mathbb{Z}}$, the translation functor $\widehat{T}^\mu_\lambda : G_1B$–mod $\to G_1B$–mod is defined on $V \in G_1B$–mod by

$$\widehat{T}^\mu_\lambda(V) = \widehat{\mathrm{pr}}_\mu(\widehat{L}_1(\nu_1)\otimes \widehat{\mathrm{pr}}_\lambda V), \tag{3.0.3}$$

where $\nu_1 \in X^+$ is as in (3.0.1).

While the translation functors have been defined for G–mod and G_1B-mod, they carry over with the same definition for the larger categories G–Mod and G_1B–Mod.

For any $\lambda \in X$, we have

$$\mathrm{pr}_\lambda \circ \operatorname{ind}^G_{G_1B} = \operatorname{ind}^G_{G_1B} \circ \widehat{\mathrm{pr}}_\lambda. \tag{3.0.4}$$

In fact, given $\tau = p\tau_1 + \tau_0 \in X$, with $\tau_0 \in X_1^+$, Lemma 1.2 implies that $\operatorname{ind}_{G_1B}^G \widehat{L}(\tau) \cong \nabla(\tau_1)^{(1)} \otimes L(\tau_0)$. Here we interpret $\nabla(\tau_1) = 0$ if $\tau_1 \notin X^+$. If $\tau_1 \in X^+$, the composition factors $L(\xi)$ of $\nabla(\tau_1)^{(1)} \otimes L(\tau_0)$ satisfy $\xi \in W_p \cdot \tau$. It follows by induction on the length of a composition series that, given $V \in G$–mod, and $\nu, \mu \in \overline{C^+}_{\mathbb{Z}}$ with $\nu \notin W_p \cdot \mu$, we have $\mathrm{pr}_\nu \operatorname{ind}_{G_1B}^G \widehat{\mathrm{pr}}_\mu V = 0$. Hence,

$$\begin{aligned}\mathrm{pr}_\lambda \circ \operatorname{ind}_{G_1B}^G &= \mathrm{pr}_\lambda \circ \operatorname{ind}_{G_1B}^G \circ \widehat{\mathrm{pr}}_\lambda \\ &= \operatorname{ind}_{G_1B}^G \circ \widehat{\mathrm{pr}}_\lambda .\end{aligned}$$

Now (3.0.4) follows directly.

For $\omega \in X$, let $t_{p\omega} \in \widetilde{W}_p$ by translation by the weight $p\omega$. We also let $t_{p\omega}$ denote the functor G_1B–mod $\to G_1B$–mod given by $V \mapsto p\omega \otimes M$. We have the following useful result.

Lemma 3.1. *(a) Let $\lambda, \mu \in \overline{C}_{\mathbb{Z}}^-$ and $\omega \in X$. Let n be the image of $t_{p\omega}$ under the quotient map $\widetilde{W}_p \to \widetilde{W}_p/W_p = N$. Then*

$$\widehat{T}_\lambda^\mu \circ t_{p\omega} = t_{p\omega} \circ \widehat{T}_{n^{-1}\cdot\lambda}^{n^{-1}\cdot\mu}.$$

(b) Let $\nu, \nu' \in \overline{C}_{\mathbb{Z}}^-$. Then

$$\begin{cases} T_\nu^{\nu'} \circ \operatorname{ind}_{G_1B}^G & = \operatorname{ind}_{G_1B}^G \circ \widehat{T}_\nu^{\nu'} \\ T_\nu^{\nu'} \circ \operatorname{res}_{G_1B}^G & = \operatorname{res}_{G_1B}^G \circ \widehat{T}_\nu^{\nu'} . \end{cases}$$

(c) Suppose that $p \geq h$ and that λ, λ' are restricted dominant weights lying in the same alcove C. Suppose that $\nu \in X^+$ and $L(\lambda) \otimes \nabla(\nu)^{(1)} \in \widehat{\mathcal{E}^R}$. Then $L(\lambda') \otimes \nabla(\nu)^{(1)} \in \widehat{\mathcal{E}^R}$. Dually, if the $L(\lambda) \otimes \Delta(\nu)^{(1)} \in \widehat{\mathcal{E}^L}$, then $L(\lambda') \otimes \Delta(\nu)^{(1)} \in \widehat{\mathcal{E}^L}$.

Proof. We first prove (a).If $\delta \in \overline{C}_{\mathbb{Z}}^+$, then $\xi \in W_p \cdot \delta$ implies that $p\omega + \xi \in W_p \cdot (n \cdot \delta)$. Therefore, using the notation of (3.0.1),

$$\begin{aligned}\widehat{T}_\lambda^\mu \circ t_{p\omega}(V) &\cong \widehat{\mathrm{pr}}_\mu(\widehat{\mathrm{pr}}_\lambda(t_{p\omega}(V)) \otimes \widehat{L}_1(\nu_1)) \\ &\cong \widehat{\mathrm{pr}}_\mu(t_{p\omega} \circ \widehat{\mathrm{pr}}_{n^{-1}\cdot\lambda}(V) \otimes \widehat{L}_1(\nu_1)) \\ &\cong t_{p\omega} \circ \widehat{\mathrm{pr}}_{n^{-1}\cdot\mu}(\widehat{\mathrm{pr}}_{n^{-1}\cdot\lambda}(V) \otimes \widehat{L}_1(\nu_1)).\end{aligned} \tag{3.1.1}$$

On the other hand, write $n^{-1} = xt_{-p\omega}$, for $x \in W$. Then

$$\begin{aligned}n^{-1} \cdot \mu - n^{-1} \cdot \lambda &= xt_{-p\omega} \cdot \mu - xt_{-p\omega} \cdot \lambda \\ &= x(\mu) - x \cdot p\omega - x(\lambda) + x \cdot p\omega \\ &= x(\mu - \lambda).\end{aligned} \tag{3.1.2}$$

The desired formula in (a) follows by combining (3.1.1) and (3.1.2) with the definition of the translation functors.

The first part of assertion (b) follows immediately from (3.0.4), using the definitions (translation) and (3.0.3) of translation. The second part is adjoint to the first part.

To see (c), let $\tau = \lambda + p\nu$ and $\tau' = \lambda' + p\nu$. Clearly, τ and τ' are both regular weights lying in the same alcove. At the level of G_1B–mod, the translation functor $\widehat{T}_\tau^{\tau'}$ takes $\widehat{L}_1(\tau)$ to $\widehat{L}_1(\tau')$. (See [26, Prop. 9.19(4)].) Hence, by (1),

$$\begin{aligned} T_\tau^{\tau'}(L(\lambda) \otimes \nabla(\nu)^{(1)}) &\cong T_\tau^{\tau'}(\operatorname{ind}_{G_1B}^G L_1(\tau)) \\ &\cong \operatorname{ind}_{G_1B}^G(\widehat{T}_\tau^{\tau'}(\widehat{L}_1(\tau))) \\ &\cong \operatorname{ind}_{G_1B}^G(\widehat{L}_1(\tau')) \\ &\cong L(\lambda') \otimes \nabla(\nu)^{(1)}. \end{aligned}$$

Since the translation functors define an equivalence between the block containing $L(\tau)$ and that containing $L(\tau')$, it follows that $L(\lambda') \otimes \nabla(\nu)^{(1)} \in \widehat{\mathcal{E}^R}$ provided that $L(\lambda) \otimes \nabla(\nu)^{(1)} \in \widehat{\mathcal{E}^R}$. This proves the first assertion in (b), and the second follows dually. ☐

Now assume that $p \geq h$. We will require a variation of wall-crossing functors [26, II.7.2.1]. For each simple reflection $s \in S_p$, fix $\mu = \mu_s \in \overline{C^-}_{\mathbb{Z}}$ to lie on the s-face of $\overline{C^-}$. We will define a functor $\theta_s : G$–mod $\to G$–mod. Let $\mathcal{B}$ be a block in G–mod. If the irreducible modules $L(\tau)$ contained in $\mathcal{B}$ have singular high weights τ, we set $\theta_s \equiv 0$ on $\mathcal{B}$. Otherwise, suppose that the irreducible modules in $\mathcal{B}$ have high weights in $W_p \cdot \tau$ for $\tau \in C^-$, then we put $\theta_s(M) = T_\mu^\tau \circ T_\tau^\mu$ on $\mathcal{B}$. Since θ_s is defined on each block $\mathcal{B}$, it is defined on G–mod.

For any $M \in G$–mod, the adjointness of the T_μ^λ and T_λ^μ defines morphisms

$$M \overset{\delta(M)}{\to} \theta_s M \overset{\epsilon(M)}{\to} M. \tag{3.1.3}$$

For certain $M \in G$–mod, (3.1.3) is a complex, and hence defines an element, denoted $\beta_s M$, in the derived category $\mathcal{D} = D^b(G\text{–mod})$. In particular, $\beta_s M$ is defined when $M = L(\lambda), \Delta(\lambda)$, or $\nabla(\lambda)$. When $M = L(\lambda)$ and $\lambda < \lambda s$, $\beta_s L(\lambda)$ is a complex with cohomology concentrated in degree 0. (If $\lambda s < \lambda$, then $\theta_s L(\lambda) = 0$.)

At the level of G_1B-mod, there are similar constructions. In particular, on modules with composition factors $\widehat{L}_1(\lambda)$ with $\lambda \in W_p \cdot \tau$, where $\tau \in C^-$, then $\widehat{\theta}_s \equiv \widehat{T}_\mu^\tau \circ \widehat{T}_\tau^\mu$. In this case, we have an analogous sequence

$$M \overset{\widehat{\delta}(M)}{\to} \widehat{\theta}_s M \overset{\widehat{\epsilon}(M)}{\to} M. \tag{3.1.4}$$

in G_1B-mod for any G_1B-module M. In some cases, (3.1.4) is a complex. In these cases, let $\widehat{\beta}_s(M)$ denote the corresponding complex in $D^b(G_1B\text{-mod})$. For example, suppose λ is a regular weight such that $\lambda < \lambda s$. Then the complex $\widehat{\beta}_s(\widehat{L}_1(\lambda))$ is defined, and it has cohomology concentrated in degree 0.

Although the functors $\theta_s, \widehat{\theta}_s$ have been defined on the categories G–mod and G_1B-mod of finite dimensional modules, the definitions make sense for the larger categories G-Mod and G_1B-Mod of arbitrary dimensional modules.

Theorem 3.2. *Assume that $p \geq h$ and let $s \in S_p$.*

(a) For $n \in \mathbb{N}$, we have $\theta_s \circ R^n \operatorname{ind}_{G_1B}^G = R^n \operatorname{ind}_{G_1B}^G \circ \widehat{\theta}_s$ for all $s \in S_p$;

(b) Let $\lambda \in X^+_{\rm reg}$ satisfy $\lambda < \lambda s$. Identify $\widehat{\beta}_s \widehat{L}_1(\lambda)$ with its cohomology. For $n > 0$,

$$R^n \operatorname{ind}_{G_1B}^G \widehat{\beta}_s(\widehat{L}_1(\lambda)) = 0.$$

Proof. We first prove (a). Because θ_s and $\widehat{\theta}_s$ are exact functors preserving injective modules (in the categories G–Mod and G_1B–Mod), it suffices to prove (a) when $n = 0$. But the $n = 0$ case follows immediately from Lemma 3.1(a).

Now we prove (b). Write $\lambda = \lambda_0 + p\lambda_1$, where $\lambda_0 \in X_1^+$. The definition of θ_s, together with [26, (9.19(4))], implies that $\widehat{\theta}_s \widehat{L}_1(\lambda) \cong p\lambda_1 \otimes M$, where M is a rational G-module (see Lemma (3.1)(a) and (3.0.2). In fact, $M = \theta_{s'} L(\lambda_0)|_{G_1B} \cong \widehat{\theta}_{s'} \widehat{L}_1(\lambda_0)$ for some $s' \in S_p$. It follows that $\widehat{\beta}_s \widehat{L}_1(\lambda)$ has the form $p\lambda_1 \otimes N$ for a rational G-module N. (Tensor the defining sequence for $\widehat{\beta}_s \widehat{L}_1(\lambda)$ with $-p\lambda_1$, and use the fact that B-module morphisms of G-modules are always G-module morphisms). Therefore, $R^n \operatorname{ind}_{G_1B}^G \widehat{\beta}_s \widehat{L}_1(\lambda) \cong (R^n \operatorname{ind}_B^G \lambda_1)^{(1)} \otimes N$ by Lemma 1.2, which vanishes if $n > 0$ by Kempf's theorem. ☐

Corollary 3.3. *Let $\omega, \lambda \in X^+$ with $\lambda \in X^+_{\rm reg}$. Assume that $s \in S_p$ so that $\lambda + p\omega < (\lambda + p\omega)s$. (Equivalently, $\lambda < \lambda s'$, where $s' = n \cdot \lambda$ with n the image of $t_{p\omega}$ in $N = \widetilde{W_p/W_p}$.) Then β_s is defined for $M = \Delta(\omega)^{(1)} \otimes L(\lambda)$, and $\beta_s M$ has cohomology concentrated in degree 0.*

Proof. In the category G_1B–mod, (3.1.4) defines a complex

$$\widehat{L}_1(p\omega + \lambda) \to \widehat{\theta}_s \widehat{L}_1(p\omega + \lambda) \to \widehat{L}_1(p\omega + \lambda)$$

which has cohomology concentrated in degree 0. In fact, this cohomology has the form $p\omega \otimes N$ for a rational G-module N. The corollary now follows from Theorem 3.2 by applying induction $\operatorname{ind}_{G_1B}^G$ to this sequence. Note that $R^1 \operatorname{ind}_{G_1B}^G$ vanishes on $\widehat{L}_1(p\omega + \lambda)$ and well as on $p\omega \otimes N = 0$. ☐

Lemma 3.4. *Suppose that $M \in G$–mod is such that (3.1.3) is a complex with cohomology concentrated in degree 0 and that $M \in \widehat{\mathcal{E}^L}$. Then for any $\nu \in X^+_{\rm reg}$,*

$$\operatorname{Hom}^n_{\mathcal{D}}(\beta_s M, \nabla(\nu)) \cong \operatorname{Hom}^n_{\mathcal{D}}(M, \beta_s \nabla(\nu)).$$

If, in addition, all the composition factors of M have regular high weights, then $\beta_s M \in \widehat{\mathcal{E}^L}$ also.

Proof. The argument given in [15, Thm. 4.8.3] applies (with M replacing the irreducible object $L(\zeta)$ introduced above (4.8.3.2) there—the irreduciblity is not used). ☐

We conclude this subsection with the following result involving paths of linked dominant weights.

Proposition 3.5. *Let* $\xi_0 \in X^+ \cap \overline{C}^+$ *and suppose there is a path* $\xi_0 < \xi_1 < \cdots < \xi_m = \xi$ *in* $X^+ \cap W_p \cdot \xi_0$ *with* ξ_{i-1} *adjacent to* ξ_i *for* $0 < i \leq m$ *and with each* $\operatorname{Ext}^1_G(L(\xi_{i-1}), L(\xi_i)) \neq 0$. *(See Fig. 2.) Then* $L(\xi) \in \widehat{\mathcal{E}^L}$.

Proof. By induction, we can assume that $L(\xi_{n-1}) \in \widehat{\mathcal{E}^L}$. Write $\xi_n = \xi_{n-1}s$. It is well-known [26, Lemma 7.20(b)] that $L(\xi_n)$ has multiplicity 1 as a composition factor of $\beta_s L(\tau s)$. If $\operatorname{Ext}^1_G(L(\xi_{n-1}), L(\xi_n)) \neq 0$, $L(\xi_n)$ appears in the head of the self-dual module $\beta_s L(\xi_{n-1})$; cf. [26, Lemma 7.20(c)] . Hence, $L(\xi_n)$ is a direct summand of $\beta_s L(\xi_{n-1})$. However, since $L(\xi_{n-1}) \in \widehat{\mathcal{E}^L}$, Lemma (3.4) implies that $\beta_s L(\xi_{n-1}) \in \widehat{\mathcal{E}^L}$. Here we use the long exact sequence of cohomology applied to the explicit formulas for $\beta_s \nabla(\lambda)$ and $\beta_s \Delta(\lambda)$ given in [17, 4.2]. Hence, $L(\xi_n) \in \widehat{\mathcal{E}^L}$, too. $\square$

4. TWISTED MODULES

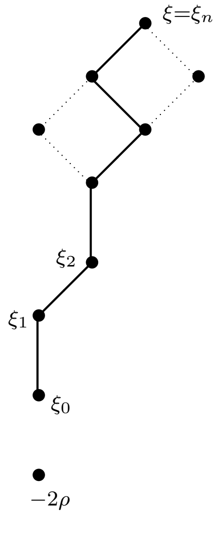

FIGURE 2

We consider whether the twisted modules $\Delta(\lambda)^{(1)}$ (resp., $\nabla(\lambda)^{(1)}$), $\lambda \in X^+$, belong to $\widehat{\mathcal{E}^L}$ (resp., $\widehat{\mathcal{E}^R}$). By Proposition 1.6, $\Delta^{\mathrm{red}}(p\lambda) \cong \Delta(\lambda)^{(1)}$ and $\nabla_{\mathrm{red}}(p\lambda) \cong \nabla(\lambda)^{(1)}$ for all $\lambda \in X^+$.

An easy spectral sequence argument shows that for $\mu \in X^+$, if $H^\bullet(G_1, \nabla(\mu)) \neq 0$, then $\mu \in \widetilde{W}_p \cdot 0$. (One can also argue from the linkage principle [26, II.9.1.9(1)].) The following result provides more precise information on such cohomology modules. The restriction that $p > h$ is required in order to be able to apply the results of [29].

Write

$$\prod_{\alpha \in \Phi^+} (1 - te^\alpha)^{-1} = \sum_{\sigma \in X} \sum_{n=0}^{\infty} \mathfrak{p}_n(\sigma) e^\sigma t^n.$$

Lemma 4.1. *Assume that* $p > h$.

(a) For any non-negative integer n, the rational G-module $H^n(G_1, \nabla(\mu))^{(-1)}$ *is nonzero only if* $\mu = w \cdot 0 + p\xi \in \widetilde{W}_p \cdot 0 \cap X^+$ *(*$w \in W$, $\xi \in X$*) with* $n \equiv l(w)$ *mod* 2.*(Necessarily,* $w \cdot 0 + p\xi \in X^+$ *implies* $\xi \in X^+$.*)*

(b) Assume $H^n(G_1, \nabla(\mu))^{(-1)} \neq 0$. *Then* $H^n(G_1, \nabla(\mu))^{(-1)}$ *has a G-filtration with sections* $\nabla(\lambda)$, $\lambda \in X^+$. *The multiplicity of* $\nabla(\lambda)$ *as a section is such a* ∇*-filtration is given by*

$$\begin{aligned}&[H^n(G_1,\nabla(\mu))^{(-1)}:\nabla(\lambda)]\\ &\qquad=\begin{cases}\sum_{x\in W}(-1)^{l(x)}\mathfrak{p}_{\frac{n-l(w)}{2}}(x\cdot\lambda-\xi) & \text{if}\, n\equiv l(w) \bmod 2;\\ 0, \quad \text{otherwise.}\end{cases}\end{aligned}\tag{4.1.1}$$

Proof. For $w\in W$ and $\alpha\in\Pi$, $|(w\cdot 0,\alpha^\vee)|=|(w\rho,\alpha^\vee)-1|\leq h$. Hence, if $\mu=w\cdot 0+p\xi\in X^+$ with $w\in W$, then the restriction on p forces $\xi\in X^+$.

Let $\mathfrak{u}$ be the Lie algebra of the unipotent radical $R_u(B)$ of B. Recall that a root subgroup $U_\alpha\subset B^-$ if and only if $-\alpha\in\Phi^+$. By [29, Thm. 2],

$$H^n(G_1,\nabla(\mu))^{(-1)}\cong H^0(G/B,S^{\frac{n-l(w)}{2}}(\mathfrak{u}^*)\otimes\xi)$$

for any $n\geq 0$, where the right hand side is interpreted as 0 if $n\not\equiv l(w)$ mod 2. For $\gamma\in X$, assume that there exists $x\in W$ such that $\gamma=x\cdot\gamma^+$ for some $\gamma^+\in X^+$. If k_γ is the 1-dimensional B- module defined by the character γ on B, $\chi(k_\gamma)$ is independent of characteristic, so $\chi(k_\gamma)=(-1)^{l(x)}\operatorname{ch}\Delta(\gamma^+)$. If γ does not have the form $x\cdot\gamma^+$ for some dominant weight γ^+ and some $w\in W$, then $\chi(k_\gamma)=0$. By [29, Thm. 2], $\operatorname{ch} H^0(G/B,S^{\frac{n-l(w)}{2}}(\mathfrak{u}^*)\otimes\xi)=\chi(S^{\frac{n-l(w)}{2}}(\mathfrak{u}^*)\otimes\xi)$, and [29, Thm. 8] implies that $H^0(G/B,S^{\frac{n-l(w)}{2}}(\mathfrak{u}^*)\otimes\xi)$ has a ∇-filtration. Because $\chi(M)=\chi(N)+\chi(M/N)$, whenever N is a B-submodule of M, the multiplicity of $\nabla(\lambda)$ as a section in such a ∇-filtration is given by (4.1.1). □

By Lemma 1.1(b), for any $\lambda\in X^+$,

$$\operatorname{ch}\Delta(\lambda)^{(1)}=\sum_{\mu\in X^+}p_{\mu,\Delta(\lambda)^{(1)}}(-1)\operatorname{ch}\Delta(\mu),$$

where $p_{\mu,\Delta(\lambda)^{(1)}}$ is the Poincare polynomial defined in (1.0.4) for $M=\Delta(\lambda)^{(1)}$.

Proposition 4.2. *Assume that $p>h$ and that $\lambda,\mu\in X^+$. Then $p_{\mu,\Delta(\lambda)^{(1)}}=0$ unless $\mu=w\cdot 0+p\xi$, $w\in W$, $\xi\in X$. In this case,*

$$p_{\mu,\Delta(\lambda)^{(1)}}=\sum_{n=0}^{\infty}\sum_{x\in W}(-1)^{l(x)}\mathfrak{p}_{\frac{n-l(w)}{2}}(x\cdot\lambda-\xi)t^n.$$

where the sum is restricted to those integers n such that $n\equiv l(w)$ mod 2.

Proof. There is a Hochschild-Serre spectral sequence

$$E_2^{s,t}=\operatorname{Ext}^s_G(\Delta(\lambda),H^t(G_1,\nabla(\mu))^{(-1)})\Rightarrow\operatorname{Ext}^{s+t}_G(\Delta(\lambda)^{(1)},\nabla(\mu)).\tag{4.2.1}$$

If $H^t(G_1,\nabla(\mu))\neq 0$, then Lemma 4.1 implies that $\mu=w\cdot 0+p\xi$ and $t\equiv l(w)$ mod 2. Also, $H^t(G_1,\nabla(\mu)^{(-1)})$ has a ∇-filtration, so in (4.2.1), $E_2^{s,t}=0$ unless $s=0$ by Lemma 1.1(a). Thus, using Lemma 4.1

$$\begin{aligned}\dim\operatorname{Ext}^n_G(\Delta(\lambda)^{(1)},\nabla(\mu))&=\dim\operatorname{Hom}_G(\Delta(\lambda),H^n(G_1,\nabla(\mu))^{(-1)})\\ &=[H^n(G_1,\nabla(\mu)^{(1)}:\nabla(\lambda)].\end{aligned}\tag{4.2.2}$$

Now apply (4.1.1). ☐

Now we can answer the question posed by this section.

Theorem 4.3. *Assume that* $p > h$. *Let* $\lambda \in X^+$. *Then*

$$\Delta^{\rm red}(p\lambda)[-l(p\lambda)] \in \mathcal{E}^L \quad \text{and} \quad \nabla(\lambda)^{(1)}[-l(p\lambda)] \in \mathcal{E}^R.$$

(Here $l(p\lambda) := l(t_{p\lambda}) = \sum_{\alpha\in\Phi^+}(\lambda, \alpha^\vee)$.*)*

Proof. We prove that $\Delta(\lambda)^{(1)}[-l(t_{p\lambda})] \in \mathcal{E}^L$; that $\nabla(\lambda)^{(1)}[-l(t_{p\lambda})] \in \mathcal{E}^R$ is handled similarly. The composition factors $L(\tau)$ of $\Delta(\lambda)^{(1)}$ satisfy $\tau \in \widetilde{W}_p \cdot 0$ and $l(\tau) \equiv l(p\lambda) := l(t_{p\lambda}) \bmod 2$. We must show that if $\mu \in X^+$, then

$$\text{(4.3.1)} \qquad \operatorname{Ext}^n_G(\Delta(\lambda)^{(1)}, \nabla(\mu)) \neq 0 \implies n \equiv l(p\lambda) - l(\mu) \bmod 2.$$

We must have $\mu = w \cdot 0 + p\xi$, and $n \equiv l(w) \bmod 2$. So, to conclude the proof, we must determine that $l(p\lambda) \equiv l(p\xi)$ mod 2. If $\operatorname{Ext}^n_G(\Delta(\lambda)^{(1)}, \nabla(\mu)) \neq 0$, then $p\lambda$ and $\mu = w\cdot 0 + p\xi$ belong to the same W_p-linkage class. Hence, $p\lambda - p\xi \in \mathbb{Z}\Phi$. Since $p > h$, $X/\mathbb{Z}\Phi$ has no p-torsion, so $p\lambda = p\xi + p\delta$, with $\delta \in \mathbb{Z}\Phi$. Then $t_{p\lambda} = t_{p\xi}t_{p\delta}$. Since $l(t_{p\delta})$ is even, $l(p\lambda) \equiv l(p\xi)$ mod 2, as required. ☐

Corollary 4.4. *Assume that* $p > h$. *Suppose that* $\lambda \in X^+$ *and that* $\Delta(\lambda) \cong L(\lambda)$. *Then* $L(\lambda)^{(1)}[-l(p\lambda)] = L(p\lambda)[-l(p\lambda)] \in \mathcal{E}^L \cap \mathcal{E}^R$.

In type A_{n-1} (i. e., $G = SL_n(k)$), there is a determination of all λ for which $\Delta(\lambda) = L(\lambda)$ given in [26, (8.21)].

5. Quantum groups and some integral representation theory

We consider when $\lambda \in X^+_{\rm reg}$ satisfies the hLCF, as defined in (1.3.1). We will say that $X \in \mathcal{D}$ satisfies the $\widehat{E}^L$ (resp., $\widehat{E}^R$) condition provided that $X \in \widehat{\mathcal{E}^L}$ (resp., $X \in \widehat{\mathcal{E}^R}$). Given $\lambda \in X^+_{\rm reg}$, write $\lambda \in$ LCF (resp., $\lambda \in$ hLCF, $\lambda \in \widehat{E}^L$, $\lambda \in \widehat{E}^R$) provided that $L(\lambda)$ satisfies the LCF (resp., the hLCF, the $\widehat{E}^L$ condition, $\widehat{E}^R$ condition). Of course, $\lambda \in \widehat{E}^L \iff \lambda \in \widehat{E}^R$.

Theorem 5.1. *For* $\lambda \in X^+_{\rm reg}$, $\lambda \in$ hLCF *if and only if* $\lambda \in$ LCF *and* $\lambda \in \widehat{E}^L$.

Proof. First, suppose that $\lambda \in$ hLCF. Write $\lambda = w \cdot \lambda^-$, $\lambda^- \in \overline{C}^-_{\mathbb{Z}}$. If $\mu \notin W_p \cdot \lambda^-$, then $\operatorname{Ext}^\bullet_G(L(\lambda), \nabla(\mu)) = 0$ by the linkage principle. Hence, if the hLCF holds, then for $\mu = y \cdot \lambda^-$, we have, by (1.3.1) that

$$p_{y\cdot\lambda^-, L(w\cdot\lambda^-)}(-1) = (-1)^{l(w)-l(y)}\overline{P}_{y,w}(-1) = (-1)^{l(w)-l(y)}P_{y,w}(-1).$$

Therefore, Lemma 1.1(b) implies that ch $L(\lambda) = \chi_{\rm KL}(\lambda)$, so that $\lambda \in$ LCF. Also, since $P_{y,w}$ is a polynomial in $q = t^2$, the validity of (1.3.1) implies that if $\operatorname{Ext}^n_G(L(\lambda), \nabla(\mu)) \neq 0$, then $l(\lambda) - l(\mu) \equiv n \bmod 2$. So, $L(\lambda)[-l(\lambda)] \in \mathcal{E}^L$ and $\lambda \in \widehat{E}^L$.

To prove the reverse direction, assume that $\lambda \in \mathrm{LCF}$ and $\lambda \in \widehat{E}^L$. Then $\lambda \in \mathrm{hLCF}$ provided that

$$\dim \operatorname{Ext}^n_{U_K}(L_\zeta(\lambda), \nabla_\zeta(\mu)) = \dim \operatorname{Ext}^n_G(L(\lambda), \nabla(\mu)) \tag{5.1.1}$$

holds for any $\mu \in X^+_{\mathrm{reg}}$ and all non-negative integers n. The left hand side of (5.1.1) is computed in the category of integrable, type 1 U_K-modules. See Remark 1.4.

Write $\widetilde{L}_\zeta(\lambda) = \widetilde{L}^{\min}_\zeta(\lambda) = \widetilde{U}_\zeta \cdot v^+$, and choose an admissible lattice $\widetilde{\nabla}_\zeta(\mu)$ for $\nabla_\zeta(\mu)$. We can assume that $\widetilde{\nabla}_\zeta(\mu)/\pi\widetilde{\nabla}_\zeta(\mu) \cong \nabla(\mu)$; see [20, p. 159] (which makes use of results of [4]). Because the LCF holds for λ, $L(\lambda) \cong \widetilde{L}_\zeta(\lambda)/\pi\widetilde{L}_\zeta(\lambda)$. Thus, we have a short exact sequence

$$0 \to \widetilde{\nabla}_\zeta(\mu) \xrightarrow{p} \widetilde{\nabla}_\zeta(\mu) \to \nabla(\mu) \to 0.$$

By (1.4.4), $\operatorname{Ext}^\bullet_{\widetilde{U}_\zeta}(\widetilde{L}_\zeta(\lambda), \nabla(\mu)) \cong \operatorname{Ext}^\bullet_G(L(\lambda), \nabla(\mu))$.

Therefore, by the long exact sequence of Ext for the functor $\operatorname{Hom}_{\widetilde{U}_\zeta}(\widetilde{L}_\zeta(\lambda), -)$, we obtain, for any non-negative integer n, a long exact sequence

$$\cdots \to \operatorname{Ext}^n_{\widetilde{U}_\zeta}(\widetilde{L}_\zeta(\lambda), \widetilde{\nabla}_\zeta(\mu)) \xrightarrow{\pi} \operatorname{Ext}^n_{\widetilde{U}_\zeta}(\widetilde{L}_\zeta(\lambda), \widetilde{\nabla}_\zeta(\mu)) \to \operatorname{Ext}^n_G(L(\lambda), \nabla(\mu)) \to \cdots . \tag{5.1.2}$$

But, by hypothesis, $L(\lambda) \in \widehat{\mathcal{E}^L}$, while $L_\zeta(\lambda)$ belongs to the analogous category for U_K. If $n \not\equiv l(\lambda) - l(\mu) \bmod 2$, then

$$\operatorname{Ext}^n_{\widetilde{U}_\zeta}(\widetilde{L}_\zeta(\lambda), \widetilde{\nabla}_\zeta(\mu)) \xrightarrow{\pi} \operatorname{Ext}^n_{\widetilde{U}_\zeta}(\widetilde{L}_\zeta(\lambda), \widetilde{\nabla}_\zeta(\mu))$$

is surjective. Since $\operatorname{Ext}^n_{\widetilde{U}_\zeta}(\widetilde{L}_\zeta(\lambda), \widetilde{\nabla}_\zeta(\mu))$ is a finite $\mathcal{O}$-module, $\operatorname{Ext}^n_{\widetilde{U}_\zeta}(\widetilde{L}_\zeta(\lambda), \widetilde{\nabla}_\zeta(\mu)) = 0$ in this case. It follows that, when $n \equiv l(\lambda) - l(\mu) \bmod 2$, the above long exact sequences provides a short exact sequence

$$0 \to \operatorname{Ext}^n_{\widetilde{U}_\zeta}(\widetilde{L}_\zeta(\lambda), \widetilde{\nabla}_\zeta(\mu)) \xrightarrow{\pi} \operatorname{Ext}^n_{\widetilde{U}_\zeta}(\widetilde{L}_\zeta(\lambda), \widetilde{\nabla}_\zeta(\mu)) \to \operatorname{Ext}^n_G(L(\lambda), \nabla(\mu)) \to 0.$$

If n does not satisfy the congruence $n \equiv l(\lambda) - l(\mu) \bmod 2$, the terms of (5.1.2) vanish. Thus, the finite $\mathcal{O}$-module $\operatorname{Ext}^n_{\widetilde{U}_\zeta}(\widetilde{L}_\zeta(\lambda), \widetilde{\nabla}_\zeta(\mu))$ is torsion-free (and possibly 0), so free. Therefore, (5.1.1) follows from (1.4.3). □

Corollary 5.2. *Assume that $p > h$. For $\lambda \in X^+$, write $p\lambda = x \cdot \tau^-$, $x \in W_p$ and $\tau^- \in \overline{C}^-_{\mathbb{Z}}$. Then $\Delta(\lambda)^{(1)}$ satisfies the* hLCF *condition, in the sense that*

$$t^{l(x)-l(y)}\overline{P}_{y,x} = \sum_{n=0}^{\infty} \dim \operatorname{Ext}^n_G(\Delta(\lambda)^{(1)}, \nabla(y \cdot \tau^-))t^n. \tag{5.2.1}$$

Proof. Clearly, $\widetilde{L}^{\min}_\zeta(p\lambda)$ satisfies $\widetilde{L}^{\min}_\zeta(p\lambda)/\pi\widetilde{L}^{\min}_\zeta(p\lambda) \cong \Delta(\lambda)^{(1)}$ (from the universal mapping property of $\Delta(\lambda)$). By Theorem 4.3, $\Delta(\lambda)^{(1)} \in \mathcal{E}^L$. The result (5.2.1) now follows as in the proof of the theorem. ☐

For $\lambda \in X^+_{\rm reg}$, let $t = -1$ in (5.2.1). Then Lemma 1.1(b) gives that

$$\mathrm{ch}\ \Delta(\lambda)^{(1)} = \chi(\lambda)^{(1)} = \chi_{\rm KL}(p\lambda),$$

which is just the special $\lambda_0 = 0$ case of Lemma 1.3. Note that neither Lemma 1.3 nor Corollary 5.2 depend in any way on the assumption that the LCF holds in an ideal.

The above discussion provides some evidence for a potentially far reaching question involving the category of rational G-modules. We begin with the following definition.

Definition 5.3. The left (resp., right) *homological lattice property* hLP^L (resp., hLP^R) holds for $\lambda \in X^+_{\rm reg}$ provided that $L_\zeta(\lambda)$ has an admissible lattice $\widetilde{L}_\zeta(\lambda)$ (resp., $\widetilde{L}'_\zeta(\lambda)$) such that $\mathrm{Ext}^\bullet_{\widetilde{U}_\zeta}(\widetilde{L}_\zeta(\lambda), \widetilde{\nabla}_\zeta(\mu))$ (resp., $\mathrm{Ext}^\bullet_{\widetilde{U}_\zeta}(\widetilde{\Delta}_\zeta(\mu), \widetilde{L}'_\zeta(\lambda))$) is $\mathcal{O}$-torsion-free for any dominant weight $\mu \leq \lambda$.

The two lattices $\widetilde{L}_\zeta(\lambda)$ and $\widetilde{L}'_\zeta(\lambda)$ appearing in the conditions hLP^L and hLP^R may not be the same. Equality does hold only when $\widetilde{L}_\zeta(\lambda)/\pi\widetilde{L}_\zeta(\lambda)$ and $\widetilde{L}'_\zeta(\lambda)/\pi\widetilde{L}'_\zeta(\lambda)$ are irreducible (and hence isomorphic to $L(\lambda)$), as discussed in more detail below. This irreducibility condition holds, along with hLP^L and hLP^R, when the LCF holds on $\Gamma \cap X^+_{\rm reg}$ for some finite ideal Γ.

The $n = 0, 1$ case of the hLP^L implies there is an exact sequence

$$0 \to \mathrm{Hom}_{\widetilde{U}_\zeta}(\widetilde{L}_\zeta(\lambda), \widetilde{\nabla}_\zeta(\mu)) \xrightarrow{\pi} \mathrm{Hom}_{\widetilde{U}_\zeta}(\widetilde{L}_\zeta(\lambda), \widetilde{\nabla}_\zeta(\mu)) \to \mathrm{Hom}_G(\widetilde{L}_\zeta(\lambda)/\pi\widetilde{L}_\zeta(\lambda), \nabla(\mu)) \to 0$$

similar to (5.1.2). Thus, $\widetilde{L}_\zeta(\lambda)/\pi\widetilde{L}_\zeta(\lambda)$ has simple head $L(\lambda)$. By Nakayama's lemma, $\widetilde{L}_\zeta(\lambda)$ is the lattice in $L_\zeta(\lambda)$ generated by some vector $v^+ \in \widetilde{L}_\zeta(\lambda)\backslash\pi\widetilde{L}_\zeta(\lambda)$ of weight λ, i. e., $\widetilde{L}_\zeta(\lambda) = \widetilde{L}^{\min}_\zeta(\lambda)$.

Similarly, if the hLP^R holds for λ, then the required lattice $\widetilde{L}_\zeta(\lambda)$ is unique up to isomorphism, and can be taken to be $L^{\max}_\zeta(\lambda)$.

Theorem 5.4. *Assume that* $p > h$.

(a) For $\lambda \in X^+_{\rm reg}$, hLP^L *(resp.* hLP^R*) holds if and only if* $\Delta^{\rm red}(\lambda)[-l(\lambda)] \in \mathcal{E}^L$ *(resp.,* $\nabla_{\rm red}(\lambda)[-l(\lambda)] \in \mathcal{E}^R$*).*

(b) Suppose that hLP^L *(resp.,* hLP^R*) holds for* $\lambda, \mu \in X^+_{\rm reg}$*. Then*

$$\dim \mathrm{Ext}^n_G(\Delta^{\rm red}(\lambda), \nabla_{\rm red}(\mu)) = \sum_{m=0}^{n}\sum_{\nu} \dim \mathrm{Ext}^m_G(\Delta^{\rm red}(\lambda), \nabla(\nu)) \cdot \dim \mathrm{Ext}^{n-m}_G(\Delta(\nu), \nabla_{\rm red}(\mu)).$$

Furthermore, if $\lambda = x \cdot \lambda^-$*, where* $\lambda^- \in C^-_{\mathbb{Z}}$*, then*

$$\begin{aligned} t^{l(x)-l(y)}\overline{P}_{y,x} &= \sum_{n=0}^{\infty} \dim \mathrm{Ext}^n_G(\Delta^{\mathrm{red}}(\lambda), \nabla(y\cdot\lambda^-))t^n \\ &= \sum_{n=0}^{\infty} \dim \mathrm{Ext}^n_G(\Delta(y\cdot\lambda^-), \nabla_{\mathrm{red}}(\lambda))t^n. \end{aligned}$$

In particular,

$$\dim \mathrm{Ext}^n(\Delta^{\mathrm{red}}(\lambda), \nabla_{\mathrm{red}}(\mu)) = \dim \mathrm{Ext}^n_{\mathcal{C}_\zeta}(L_\zeta(\lambda), L_\zeta(\mu)),$$

as given in (1.4.2).

Proof. We first prove (a). If $\Delta^{\mathrm{red}}(\lambda)[-l(\lambda)] \in \mathcal{E}^L$, then, just as in the proof of Theorem 5.1, there is for every integer n a short exact sequence

$$0 \to \mathrm{Ext}^n_{\widetilde{U}_\zeta}(\widetilde{L}_\zeta(\lambda), \widetilde{\nabla}_\zeta(\mu)) \xrightarrow{\pi} \mathrm{Ext}^n_{\widetilde{U}_\zeta}(\widetilde{L}_\zeta(\lambda), \widetilde{\nabla}_\zeta(\mu)) \to \mathrm{Ext}^n_G(\Delta^{\mathrm{red}}(\lambda), \nabla(\mu)) \to 0$$

which implies that hLP^L holds for λ. Conversely, if hLP^L holds for λ, then we obtain the same short exact sequence, so that $L(\lambda)[-l(\lambda)] \in \mathcal{E}^L$, using (1.4.3) and Remark 1.4. A similar argument applies for the other half of (a).

Finally, (b) follows immediately from (a) and [13, Thm. 3.5]. See also Remark 1.4. □

Remark 5.5. Suppose that $p > h$. Suppose $\lambda = x\cdot\lambda^-$, $\lambda^- \in C^-_{\mathbb{Z}}$, satisfies the hLP^L (resp., hLP^R). Let $\mu = x\cdot\mu^-$ for $\mu^- \in C^-_{\mathbb{Z}}$. Then μ satisfies the hLP^L (resp., hLP^R).

To see this, consider the assertion for hLP^L; the other case is similar. Let $\mathcal{B}$ (resp., $\mathcal{B}'$) be the block (in the category of rational G-modules) determined by the linkage class $W_p\cdot\lambda^- \cap X^+$ (resp., $W_p\cdot\mu^-$). Let $T = T^\mu_\lambda : \mathcal{B} \to \mathcal{B}'$ be highest weight category equivalence defined by translation. Then T takes the category the full subcategory $\mathcal{E}^L(\mathcal{B})$ of $\mathcal{E}^L$ to the analogous full subcategory $\mathcal{E}^L(\mathcal{B}')$. The functor T can be defined for the categories of integrable, type 1 U_ζ-modules [4, §8.3] and integrable, type 1 $\widetilde{U}_\zeta$-modules.[3] Furthermore, it commutes with base change $-\otimes_{\mathcal{O}} k$, so it carries $\Delta^{\mathrm{red}}(x\cdot\lambda)$ to $\Delta^{\mathrm{red}}(x\cdot\mu)$.

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[3] The definition in [4, §8.3] applies for this latter category, after some additional observations. In the definition [4, §8.3], one tensors with a certain costandard module and then takes a projection onto a linkage class. the costandard module has an analogue "over $\mathcal{O}$" (even over the algebra $\mathcal{A}$) in [4]which integrable, type 1. Tensor products of integrable, type 1 $\widetilde{U}_\zeta$-modules are (obviously) integrable, type 1. Projection onto a linkage class makes sense in the category of integrable, type 1 $\widetilde{U}_\zeta$-modules: First, one can talk about "composition factors" for $\widetilde{U}_\zeta$-modules and their weights. The category of all finite rank, integrable, type 1 modules is an $\mathcal{O}$-finite highest weight category $\widetilde{\mathcal{C}}$ in the sense of [20, §2]. The category $\widetilde{\mathcal{C}}[\Gamma]$ of all modules in $\widetilde{\mathcal{C}}$ whose composition factors have only high weights belonging to a fixed finite poset ideal Γ in X^+ has enough projective modules, and its projective indecomposable modules reduce modulo p to projective indecomposable modules [20, §§2,3]. Projections onto linkage classes can now be made in the category $\widetilde{\mathcal{C}}[\Gamma]$ for any sufficiently large Γ.

6. Some general results and conjectures

In this section, we begin with several conjectures concerning the families

$$\{\Delta^{\rm red}(\lambda)\}_{\lambda\in X^+_{\rm reg}} \text{ and } \{\nabla_{\rm red}(\lambda)\}_{\lambda\in X^+_{\rm reg}}$$

of indecomposable rational G-modules. We will see in Theorem 6.7 that the first three conjectures are theorems, if one assumes the validity of (a form of) the Lusztig conjecture in positive characteristic p. This latter conjecture is itself a theorem, though, presently, only for p very large (size unknown, depending on the root system) [3]. Still, we are able to present some striking applications, handling smaller primes separately, in section 7. The remaining two conjectures we formulate in this section are proved in part under the same hypothesis (of a valid Lusztig conjecture) in Theorem 6.8.

Independently of the validity of these conjectures, the modules above have characters given by

$$\text{(6.0.1)}\qquad {\rm ch}\,\Delta^{\rm red}(\lambda) = {\rm ch}\,\nabla_{\rm red}(\lambda) = \sum_{y\cdot\lambda^-\in X^+}(-1)^{l(x)-l(y)}P_{y,x}(1){\rm ch}\,\Delta(y\cdot\lambda^-),$$

where $\lambda = x\cdot\lambda^-$ with $\lambda^-\in C^-$. (Recall the assumption in Remark 1.4.)

Conjecture 6.1. Assume that $p > h$. The left and right homological lattice properties hold for all $\lambda\in X^+_{\rm reg}$. In other words, for each $\lambda\in X^+_{\rm reg}$, the lattices $\widetilde{L}^{\min}_\zeta(\lambda)$ and $\widetilde{L}^{\max}_\zeta(\lambda)$ of $L_\zeta(\lambda)$ have the property that the $\mathcal{O}$-modules ${\rm Ext}^\bullet_{\widetilde{U}_\zeta}(\widetilde{L}^{\min}_\zeta(\lambda),\widetilde{\nabla}_\zeta(\mu))$ and ${\rm Ext}^\bullet_{\widetilde{U}_\zeta}(\widetilde{\Delta}(\lambda),\widetilde{L}^{\max}_\zeta(\lambda))$ are torsion-free for all $\mu\in X^+_{\rm reg}$.

Conjecture 6.2. Assume that $p > h$. For each $\lambda\in X^+_{\rm reg}$, $\Delta^{\rm red}(\lambda)[-l(\lambda)]\in\mathcal{E}^L$ and $\nabla_{\rm red}(\lambda)[-l(\lambda)]\in\mathcal{E}^R$. (Equivalently, for $\lambda\in X^+_{\rm reg}$, $\Delta^{\rm red}(\lambda)\in\widehat{\mathcal{E}^L}$ and $\nabla_{\rm red}(\lambda)\in\widehat{\mathcal{E}^R}$.)

The parenthetic comments follows from Lemma 2.1. Also, by Theorem 5.4, we have the following observation.

Proposition 6.3. *Conjecture 6.1* $\iff$ *Conjecture 6.2.*

There is a variation on Conjecture 6.2:

Conjecture 6.4. Assume that $p > h$. For $\lambda\in X^+_{\rm reg}$, write $\lambda=\lambda_0+p\lambda_1$, with $\lambda_0\in X^+_1$ and $\lambda_1\in X^+$, and set

$$\Delta^{\rm red}(\lambda)' := L(\lambda_0)\otimes\Delta(\lambda_1)^{(1)},\quad \nabla_{\rm red}(\lambda)' := L(\lambda_0)\otimes\nabla(\lambda_1)^{(1)}.$$

Then $\Delta^{\rm red}(\lambda)'[-l(\lambda)]\in\mathcal{E}^L$ and $\nabla_{\rm red}(\lambda)'[-l(\lambda)]\in\mathcal{E}^R$.

Conjecture 6.5. Assume that $p > h$. For $\lambda\in X^+$, $\Delta(\lambda)$ (resp., $\nabla(\lambda)$) has a $\Delta^{\rm red}$-filtration, i. e., a filtration as a G-module with sections of the form $\Delta^{\rm red}(\nu)$ (resp., $\nabla_{\rm red}(\nu)$), $\nu\in X^+$.

It seems likely that it is enough to check Conjecture 6.5 in the special case when $\lambda \in X^+_{\rm reg}$ where $\lambda \in W_p \cdot (-2\rho) \cap X^+$, since the conclusion should behave well with respect to translation within an alcove, or translation to a wall. (We have not checked this assertion carefully.) Also, as far as we know, the assumption $p \geq h$ is sufficient in any of the above conjectures.[4]

We can also formulate the following variation of the last conjecture, using the modules introduced in Conjecture 6.4.

Conjecture 6.6. For $\lambda \in X^+$, $\Delta(\lambda)$ (resp., $\nabla(\lambda)$) has a $\Delta^{\rm red'}$-filtration (resp., a $\nabla'_{\rm red}$-filtration), i. e., a filtration as a G-module with sections of the form $\Delta^{\rm red}(\nu)'$ (resp., $\nabla_{\rm red}(\nu)'$), $\nu \in X^+_{\rm reg}$.

Theorem 6.7. *Let $p > h$.*

(a) Assume that the LCF holds for all regular weights in X^+_1. Then Conjectures 6.1, 6.2 and 6.4 are true.

(b) If Conjecture 6.4 holds and $p \geq 2h-2$, then the LCF holds for all regular weights in X^+_1.

Proof. First, we prove (a). By Proposition 1.6, the hypothesis implies that if $\lambda \in X^+_{\rm reg}$, then $\Delta^{\rm red}(\lambda) = \Delta^{\rm red}(\lambda)' \cong L(\lambda)$ and $\nabla_{\rm red}(\lambda) = \nabla_{\rm red}(\lambda)' \cong L(\lambda)$. Suppose that $\lambda, \lambda s \in X^+_1$ are regular restricted weights and that $\lambda < \lambda s$, where $s \in S_p$. Suppose for $\omega \in X^+$, that $\Delta^{\rm red}(\lambda + p\omega) = L(\lambda) \otimes \Delta(\omega)^{(1)} \in \widehat{\mathcal{E}^L}$. Let the wall separating the dominant weights $\lambda + p\omega$ and $\lambda s + p\omega$ be of type s'. By Corollary 3.3 and Lemma 3.4, $\beta_{s'}$ is defined at $M := L(\lambda) \otimes \Delta(\omega)^{(1)}$ and $\beta_{s'} M \in \widehat{\mathcal{E}^L}$. It suffices to prove that $L(\lambda s) \otimes \Delta(\omega)^{(1)}$ is a direct summand of $\beta_{s'}(L(\lambda) \otimes \Delta(\omega)^{(1)})$. But in the category G_1B–mod,

$$\widehat{\beta}_{s'}(p\omega \otimes \widehat{L}_1(\lambda)) \cong p\omega \otimes \widehat{\beta}_s \widehat{L}_1(\lambda s),$$

for some $s \in S_p$, using Lemma 3.1. The LCF holds for regular restricted weights, so that $L(\lambda s)$ is a direct summand of $\beta_s L(\lambda)$. Also, $\widehat{\beta}_s \widehat{L}_1(\lambda) \cong \beta_s L(\lambda)|_{G_1B}$. We conclude that $p\omega \otimes \widehat{L}_1(\lambda s)$ is a direct summand of $\widehat{\beta}_{s'}(p\omega \otimes \widehat{L}_1(\lambda))$. Now apply the induction functor ${\rm ind}_{G_1B}^G$ and Lemma 3.1 to conclude that $L(\lambda s) \otimes \Delta(\omega)^{(1)}$ is a direct summand of $\beta_{s'}(L(\lambda) \otimes \Delta(\omega)^{(1)})$. By Theorem 4.3, for $\omega \in X^+$, $\Delta^{\rm red}(p\omega) \cong \Delta(\omega)^{(1)} \in \widehat{\mathcal{E}^L}$. Since any regular restricted weight can be connected to a weight in $C^+_{\mathbb{Z}}$ by a series of adjacent restricted weights[5], we can now apply Remark 5.5 to conclude that (a) holds.

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[4] But Conjecture 6.5 cannot be extended to all primes p. We are grateful to Will Turner for indicating that there is a type A counterexample when $p = 2$.

[5] To see this fact, we argue as follows: Given a restricted regular weight x in an alcove C, draw a (straight) line from x to 0. The line must past through at least one reflecting affine hyperplane H, and we pick a point z which is the nearest such an intersection to x. Let B be any open ball around z. Then B clearly contains points of C, and so the intersection of C with B is a nonempty open set. Also, the point z is clearly on the boundary of C, and, as such lies in the closure of a wall F of C whose containing hyperplane H also contains z. The intersection of the ball B with F is nonempty, since $z \in \overline{F}$. Let C' be the unique alcove other than C which has F as a wall. Since any point of

(b) Suppose that Conjecture 6.4 holds and that $p \geq 2h-2$. Then if $\lambda \in X_1^+$, the well-known inequality

$$(\lambda+\rho, \alpha+0^\vee) < p(p-h+2)$$

holds. Let $\mu \in X^+_{\text{reg}}$ satisfy $\mu \leq \lambda$, and write $\mu = \mu_0 + p\mu_1$, $\mu_0 \in X_1^+$. Because $(\mu+\rho, \alpha_0^\vee) \leq (\lambda+\rho, \alpha_0^\vee)$, we see that $(\mu_1+\rho, \alpha_0^\vee) \leq p$, i. e., $\mu_1 \in \overline{C}^+_{\mathbb{Z}}$. Thus, $L(\mu) \cong \Delta^{\text{red}}(\mu)'$, so that $L(\mu)[-l(\mu)] \in \mathcal{E}^L$. Therefore, by Lemma 2.2 (taking Γ to be the ideal in the poset $(X^+, \uparrow)$ of regular weight generated by the regular restricted weights), the LCF holds for λ. $\square$

For $\lambda \in X^+_{\text{reg}}$, put

$$E_\zeta(\lambda) = \Delta_\zeta(\lambda)/\text{rad}^2\Delta_\zeta(\lambda). \tag{6.7.1}$$

Let $\widetilde{E}(\lambda)$ be the image of $\widetilde{\Delta}_\zeta(\lambda)$ in $E_\zeta(\lambda)$, and set $E(\lambda) = \widetilde{E}(\lambda)/\pi\widetilde{E}(\lambda)$. Observe that $\widetilde{E}(\lambda)$ is an admissible lattice for $E_\zeta(\lambda)$.

We are not yet able to establish Conjectures 6.4 and 6.5 under a LCF assumption like that given in Theorem 6.7(a). However, we have the following partial result.

Theorem 6.8. *Assume that $p > h$ and that the LCF holds for all regular weights in X_1^+. With $\widetilde{E}(\lambda)$ as in (6.7.1), let $\widetilde{D}(\lambda)$ be the kernel of the natural surjection*

$$\widetilde{E}(\lambda) \twoheadrightarrow \widetilde{L}^{\min}(\lambda),$$

and similarly let $D_\zeta(\lambda)$ be the kernel of $E_\zeta(\lambda) \twoheadrightarrow L_\zeta(\lambda)$. (Thus, $\widetilde{D}(\lambda)$ is an admissible lattice for $D_\zeta(\lambda)$.) Then $\widetilde{D}(\lambda)$ has a $\widetilde{\mathcal{C}}_\zeta$-filtration with (distinct) sections $\widetilde{L}^{\min}(\mu)^{\oplus n_\mu}$, where $n_\mu = n_\mu(\lambda)$ is given, for $\mu < \lambda$, by

$$n_\mu = [D_\zeta(\lambda) : L_\zeta(\mu)] = \dim \text{Ext}^1_{\mathcal{C}_\zeta}(L_\zeta(\lambda), L_\zeta(\mu)).$$

In particular, $E(\lambda)$ has a Δ^{red}-filtration with top section $\Delta^{\text{red}}(\lambda)$.

Proof. Let Γ be any ideal of dominant weights with $\Gamma \subseteq (-\infty, \lambda)$. Let $\widetilde{D}_\Gamma(\lambda)$ be the largest $\mathcal{O}$-free quotient of $\widetilde{D}(\lambda)$ such that each composition factor of $\widetilde{D}_\Gamma(\lambda) \otimes_{\mathcal{O}} K = D_{\Gamma,\zeta}(\lambda)$ has high weight in Γ. Also, let $\widetilde{E}_\Gamma(\lambda)$ be the resulting extension of $\widetilde{L}^{\min}(\lambda)$ by $\widetilde{D}_\Gamma(\lambda)$. The conclusion of the theorem has an evident analogue with $\widetilde{E}(\lambda)$ replaced by $\widetilde{E}_\Gamma(\lambda)$, $\widetilde{D}(\lambda)$ replaced by $\widetilde{D}_\Gamma(\lambda)$, and μ required to be in Γ.

We proceed by induction on $|\Gamma|$ starting with the case $\Gamma = \emptyset$. Thus, we assume that we are given Γ so that $\widetilde{D}_\Gamma(\lambda)$ has a filtration with sections $\widetilde{L}^{\min}(\mu)^{\oplus n_\mu}$ for $\mu \in \Gamma$ and $n_\mu = [D_{\Gamma,\zeta}(\lambda) : L_\zeta(\mu)] = \dim \text{Ext}^1_{\mathcal{C}_\zeta}(L_\zeta(\lambda), L_\zeta(\mu))$. (Note that $n_\mu = 0$ for all $\mu \in \Gamma$ satisfying $l(\mu) = l(\lambda)$.)

F is the limit of points of C', the intersection of B with C' is nonempty. Taking B small enough, we may assume that B is entirely contained in the open restricted parallelopiped. So the alcove C' must be restricted, and, not being on the same side of H as x, is on the side of H containing 0, and is therefore "smaller". A reflection of x into C' and induction gives the desired result.

<u>Claim 1.</u> *Let* $E_\Gamma(\lambda) = \widetilde{E}_\Gamma(\lambda)/\pi\widetilde{E}_\Gamma(\lambda)$ *and* $E_{\Gamma,\zeta}(\lambda) = \widetilde{E}_\Gamma(\lambda)\otimes_{\mathcal{O}} K$*. Suppose* $\omega < \lambda$ *has parity opposite to that of* λ*. Then* $\mathrm{Ext}^1_{\widetilde{\mathcal{C}}_\zeta}(\widetilde{E}_\Gamma(\lambda), \widetilde{L}^{\max}(\omega))$ *is torsion-free. Also,*

$$\mathrm{Ext}^1_{\widetilde{\mathcal{C}}_\zeta}(\widetilde{E}_\Gamma(\lambda), \widetilde{L}^{\max}(\omega))/\pi\mathrm{Ext}^1_{\widetilde{\mathcal{C}}_\zeta}(\widetilde{E}_\Gamma(\lambda), \widetilde{L}^{\max}(\omega)) \cong \mathrm{Ext}^1_G(E_\Gamma(\lambda), \nabla_{\mathrm{red}}(\omega)).$$

To verify this assertion, put $D_\Gamma(\lambda) = \widetilde{D}_\Gamma(\lambda)/\pi\widetilde{D}_\Gamma(\lambda)$. Since $D_\Gamma(\lambda)$ has a filtration with sections $\Delta^{\mathrm{red}}(\mu)$, μ has the same parity as ω, so that $\mathrm{Ext}^1_G(D_\Gamma(\lambda), \nabla_{\mathrm{red}}(\omega)) = 0$ by Theorem 6.7. Therefore, the long exact sequence of Ext applied to the short exact sequence

$$0 \to D_\Gamma(\lambda) \to E_\Gamma(\lambda) \to \Delta^{\mathrm{red}}(\lambda) \to 0$$

gives an exact sequence

$$\begin{aligned}0 \to \mathrm{Hom}_G(D_\Gamma(\lambda), \nabla_{\mathrm{red}}(\omega)) \to \mathrm{Ext}^1_G(\Delta^{\mathrm{red}}(\lambda), \nabla_{\mathrm{red}}(\omega)) \to \\ \mathrm{Ext}^1_G(E_\Gamma(\lambda), \nabla_{\mathrm{red}}(\omega)) \to 0.\end{aligned} \tag{6.8.1}$$

Next, $\mathrm{Ext}^1_{\mathcal{C}_\zeta}(D_{\Gamma,\zeta}(\lambda), L_\zeta(\omega)) = 0$ by the quantum analogue of Lemma 2.2, since $D_{\Gamma,\zeta}(\lambda)$ is a direct sum of irreducible modules $L_\zeta(\mu)$ with μ having the same parity as ω. Therefore, the long exact sequence of Ext applied to the short exact sequence

$$0 \to D_{\Gamma,\zeta}(\lambda) \to E_{\Gamma,\zeta}(\lambda) \to L_\zeta(\lambda) \to 0$$

gives an exact sequence

$$\begin{aligned}0 \to \mathrm{Hom}_{\mathcal{C}_\zeta}(D_{\Gamma,\zeta}(\lambda), L_\zeta(\omega)) \to \mathrm{Ext}^1_{\mathcal{C}_\zeta}(L_\zeta(\lambda), L_\zeta(\omega)) \to \\ \mathrm{Ext}^1_{\mathcal{C}_\zeta}(E_{\Gamma,\zeta}(\lambda), L_\zeta(\omega)) \to 0.\end{aligned} \tag{6.8.2}$$

Clearly, the left-hand Hom in (6.8.2) has dimension equal to the multiplicity of $L_\zeta(\omega)$ in $D_{\Gamma,\zeta}(\lambda)$. This multiplicity equals the multiplicity of $\Delta^{\mathrm{red}}(\omega)$ as a section in the Δ^{red}-filtration of $D_\Gamma(\lambda) = \widetilde{D}_\Gamma(\lambda)/\pi\widetilde{D}_\Gamma(\lambda)$ given by induction and reduction mod π. (Recall $\Delta^{\mathrm{red}}(\omega) = \widetilde{L}^{\min}(\omega)/\pi\widetilde{L}^{\min}(\omega)$, and the module $\widetilde{L}^{\min}(\omega)$ is torsion-free.) If $\Delta^{\mathrm{red}}(\tau)$ is a section this filtration, then τ and ω have the same parity, so, $\mathrm{Ext}^1_G(D_\Gamma(\lambda), \nabla_{\mathrm{red}}(\omega)) = 0$. Thus, dim $\mathrm{Hom}_G(D_\Gamma(\lambda), \nabla_{\mathrm{red}}(\omega))$ also equals this multiplicity. It follows that

$$\dim \mathrm{Hom}_G(D_\Gamma(\lambda), \nabla_{\mathrm{red}}(\omega)) = \dim \mathrm{Hom}_{\mathcal{C}_\zeta}(D_{\Gamma,\zeta}(\lambda), L_\zeta(\lambda)). \tag{6.8.3}$$

By Theorem 5.4(b) and Theorem 6.7, the middle Ext^1-terms in (6.8.1) and (6.8.2) also have the same dimension. Thus, the right-hand Ext^1-terms have the same dimension, as well. Form the exact sequence

$$\mathrm{Ext}^1_{\widetilde{\mathcal{C}}_\zeta}(\widetilde{E}_\Gamma(\lambda), \widetilde{L}^{\max}(\omega)) \xrightarrow{\pi} \mathrm{Ext}^1_{\widetilde{\mathcal{C}}_\zeta}(\widetilde{E}_\Gamma(\lambda), \widetilde{L}^{\max}(\omega)) \to \mathrm{Ext}^1_{\widetilde{\mathcal{C}}_\zeta}(\widetilde{E}_\Gamma(\lambda), \nabla_{\mathrm{red}}(\omega)).$$

By (1.4.4), $\mathrm{Ext}^1_{\widetilde{\mathcal{C}}_\zeta}(\widetilde{E}_\Gamma(\lambda), \nabla_{\mathrm{red}}(\omega)) \cong \mathrm{Ext}^1_G(E_\Gamma(\lambda), \nabla_{\mathrm{red}}(\omega))$, so that both assertions of Claim 1 now follow from (6.8.3) and (1.4.3).

<u>Claim 2.</u> If $\omega < \lambda$ $\mathrm{Ext}^1_{\mathcal{C}_\zeta}(L_\zeta(\lambda), L_\zeta(\omega)) \neq 0$ if and only if $[D_\zeta(\lambda) : L_\zeta(\omega)] \neq 0$.

Claim 3. *Assume that* $[D_\varsigma(\lambda):L_\varsigma(\omega)]\neq 0$ *(so that* $\omega<\lambda$*, and* ω *and* λ *have opposite parity). Then*

$$\omega\in\Gamma \iff \mathrm{Ext}^1_G(E_\Gamma(\lambda),\nabla_{\mathrm{red}}(\omega))=0.$$

Claim 2 is clear, so consider Claim 3. First, assume that $\omega\in\Gamma$. Then in (6.8.2), the inclusion $\iota:\mathrm{Hom}_{\mathcal{C}_\varsigma}(D_{\Gamma,\varsigma}(\lambda),L_\varsigma(\omega))\hookrightarrow\mathrm{Ext}^1_{\mathcal{C}_\varsigma}(L_\varsigma(\lambda),L_\varsigma(\omega))$ is an isomorphism, so that $\mathrm{Ext}^1_G(E_\Gamma(\lambda),\nabla_{\mathrm{red}}(\omega))=0$ by (6.8.2) and Claim 1. Conversely, if $\mathrm{Ext}^1_G(E_\Gamma(\lambda),\nabla_{\mathrm{red}}(\omega))=0$, then Claim 1 implies that ι is an isomorphism so that $\omega\in\Gamma$. This proves Claim 3.

Claim 4. *If* $\omega\in\Gamma$ *has parity opposite to* λ*, then* $\mathrm{Ext}^1_G(E_\Gamma(\lambda),L(\omega))=0$.

Since $\mathrm{Hom}_G(E_\Gamma(\lambda),\nabla_{\mathrm{red}}(\omega)/L(\omega))=0$, the claim follows from the long exact sequence of Ext provided that we know that

$$\mathrm{Ext}^1_G(E_\Gamma(\lambda),\nabla_{\mathrm{red}}(\omega))=0. \tag{6.8.4}$$

This later vanishing is true by Claim 3, if $[D_\varsigma(\lambda):L_\varsigma(\omega)]\neq 0$ Otherwise, we have that $\mathrm{Ext}^1_{\mathcal{C}_\varsigma}(L_\varsigma(\lambda),L_\varsigma(\omega))=0$ by Claim 2, so so the right-hand term of (6.8.2) vanishes. Thus, (6.8.4) holds by (1.4.3) and Claim 1. This proves Claim 4.

Claim 5. *Let* $\nu<\lambda$ *of opposite parity to* λ *with* $\nu\notin\Gamma$*. We have*

$$\mathrm{Ext}^1_G(E_\Gamma(\lambda),\nabla_{\mathrm{red}}(\nu))\cong\mathrm{Ext}^1_G(\Delta^{\mathrm{red}}(\lambda),\nabla_{\mathrm{red}}(\nu)),$$

which has dimension equal to $\dim\mathrm{Ext}^1_{\mathcal{C}_\varsigma}(L_\varsigma(\lambda),L_\varsigma(\nu))$.

To prove the isomorphism, we use the fact that $D_\Gamma(\lambda)$ has, as previously noted, a filtration with sections $\Delta^{\mathrm{red}}(\mu)$. A module $\Delta^{\mathrm{red}}(\mu)$ appears as a section only when $\mathrm{Ext}^1_{\mathcal{C}_\varsigma}(L_\varsigma(\lambda),L_\varsigma(\mu))\neq 0$, which implies that μ,ν have the same parity. Therefore, $\mathrm{Ext}^1_G(\Delta^{\mathrm{red}}(\mu),\nabla_{\mathrm{red}}(\nu))=0$, and so $\mathrm{Ext}^1_G(D_\Gamma(\lambda),\nabla_{\mathrm{red}}(\nu))=0$. Also, since $\nu\notin\Gamma$, we have $\mathrm{Hom}_G(D_\Gamma(\lambda),\nabla_{\mathrm{red}}(\nu))=0$. The first equality follows from the long exact sequence of Ext applied to the short exact sequence $0\to D_\Gamma(\lambda)\to E_\Gamma(\lambda)\to\Delta^{\mathrm{red}}(\lambda)\to 0$. The second equality is a consequence of Theorem 5.4(b). This proves Claim 5.

We note also the obvious

$$\dim\mathrm{Ext}^1_{\mathcal{C}_\varsigma}(L_\varsigma(\lambda),L_\varsigma(\nu))=[D_\varsigma(\lambda):L_\varsigma(\nu)] \tag{6.8.5}$$

for any $\nu<\lambda$.

Assume that $\widetilde{D}_\Gamma(\lambda)\neq\widetilde{D}(\lambda)$. Let $\nu\in X^+$ be minimal satisfying $[D_\varsigma(\lambda):L_\varsigma(\nu)]\neq 0$ and $\nu\notin\Gamma$. Necessarily, $\nu<\lambda$, and ν and λ have opposite parity. Without loss, we can replace Γ by $\Gamma^\natural=\{\omega\in X^+\,|\,\omega<\nu\}$ and keep $E_{\Gamma,\varsigma}(\lambda)$, $D_{\Gamma,\varsigma}(\lambda)$, $\widetilde{D}_\Gamma(\lambda)$ the same. If $\Gamma\neq\Gamma^\natural$, we could enlarge Γ to $\Gamma^\natural$. Thus, we may assume that $\Gamma=\Gamma^\natural$.

All composition factors of $\nabla_{\mathrm{red}}(\nu)=\nabla(\nu_1)^{(1)}\otimes L(\nu_0)$ have high weight ω with the same parity as ν, which is opposite to the parity of λ. Except for ν, these weights ω also belong to Γ, by the minimality of ν. By Claim 4,

$$\mathrm{Ext}^1_G(E_\Gamma(\lambda),L(\omega))=0 \tag{6.8.6}$$

for such ω. In particular,

$$\operatorname{Ext}^1_G(E_\Gamma(\lambda), L(\nu)) \cong \operatorname{Ext}^1_G(E_\Gamma(\lambda), \nabla_{\rm red}(\nu)) \tag{6.8.7}$$

by Claim 4 and the long exact sequence of cohomology. Also, there is an injection

$$\operatorname{Ext}^1_G(E_\Gamma(\lambda), \Delta^{\rm red}(\nu)) \hookrightarrow \operatorname{Ext}^1_G(E_\Gamma(\lambda), L(\nu)). \tag{6.8.8}$$

The lattice inclusion $\widetilde{L}^{\min}(\nu) \subseteq \widetilde{L}^{\max}(\nu)$ has quotient which has a finite length with composition factors of the form $L(\omega)$, $\omega < \nu$ where $L(\omega)$ a composition factor of $\nabla_{\rm red}(\nu)$. Thus, using (6.8.6) and (1.4.4), there is an identification

$$\operatorname{Ext}^1_{\widetilde{\mathcal{C}}_\zeta}(\widetilde{E}_\Gamma(\lambda), \widetilde{L}^{\min}(\nu)) \cong \operatorname{Ext}^1_{\widetilde{\mathcal{C}}_\zeta}(\widetilde{E}_\Gamma(\lambda), \widetilde{L}^{\max}(\nu)). \tag{6.8.9}$$

By Claim 1, both sides of (6.8.9) are torsion-free. So the reduction modulo π of the left-hand side injects to $\operatorname{Ext}^1_G(E_\Gamma(\lambda), \Delta^{\rm red}(\nu))$, which, we have just seen, injects into $\operatorname{Ext}^1_G(E_\Gamma(\lambda), L(\nu))$. The latter group is isomorphic to $\operatorname{Ext}^1_G(E_\Gamma(\lambda), \nabla_{\rm red}(\nu))$ by (6.8.7). By Claim 5, $\dim \operatorname{Ext}^1_G(E_\Gamma(\lambda), \nabla_{\rm red}(\nu)) = [D_\zeta(\lambda) : L_\zeta(\mu)]$, which is the rank of the right-hand side of (6.8.9). Thus,

$$\operatorname{Ext}^1_G(E_\Gamma(\lambda), \Delta^{\rm red}(\nu)) \cong \operatorname{Ext}^1_G(E_\Gamma(\lambda), L(\nu)) \cong \operatorname{Ext}^1_G(E_\Gamma(\lambda), \nabla_{\rm red}(\nu)). \tag{6.8.10}$$

Also,

$$\operatorname{Ext}^1_G(E_\Gamma(\lambda), \Delta^{\rm red}(\nu)) \cong \operatorname{Ext}^1_{\widetilde{\mathcal{C}}_\zeta}(\widetilde{E}_\Gamma(\lambda), \widetilde{L}^{\min}(\mu))/\pi\operatorname{Ext}^1_{\widetilde{\mathcal{C}}_\zeta}(\widetilde{E}_\Gamma(\lambda), \widetilde{L}^{\min}(\mu)), \tag{6.8.11}$$

and

$$\operatorname{Ext}^1_{\widetilde{\mathcal{C}}_\zeta}(\widetilde{E}_\Gamma(\lambda), \widetilde{L}^{\min}(\mu)) \text{ is a free } \mathcal{O}\text{-module of rank } n_\mu. \tag{6.8.12}$$

To complete the proof of the "inductive step," put $\Gamma' = \{\nu\} \cup \Gamma$. Form an extension

$$0 \to \widetilde{L}^{\min}(\nu)^{\oplus n_\nu} \to \widetilde{E} \to \widetilde{E}_\Gamma(\lambda) \to 0 \tag{6.8.13}$$

using a $\mathcal{O}$-basis of $\operatorname{Ext}^1_{\widetilde{\mathcal{C}}_\zeta}(\widetilde{E}_\Gamma(\lambda), \widetilde{L}^{\min}(\nu))$. By (6.8.10), (6.8.11), and (6.8.12), $\widetilde{E}$ has the same head $L(\lambda)$ as $\widetilde{E}_\Gamma(\lambda)$. (One gets the same head after reduction mod π, and even after factoring out by the radical of $\Delta^{\rm red}(\nu)^{\oplus n_\nu}$.) Thus, there is a surjection $\widetilde{\Delta}_\zeta(\lambda) \twoheadrightarrow \widetilde{E}$, which induces a surjection $\widetilde{E}(\lambda) \to \widetilde{E}$. The image $\widetilde{D}$ of $\widetilde{D}(\lambda)$ under this latter surjection is $\mathcal{O}$-free and $[D_\zeta : L_\zeta(\omega)] = [D_\zeta(\lambda) : L_\zeta(\omega)] = n_\omega$ for all $\omega \in \Gamma'$, where $D_\zeta = \widetilde{D} \otimes_{\mathcal{O}} K$. Thus, $\widetilde{D} = \widetilde{D}_{\Gamma'}(\lambda)$, and $\widetilde{E} = \widetilde{E}_{\Gamma'}(\lambda)$.

We must check that for $\mu \in \Gamma'$, $\widetilde{D}_{\Gamma'}(\lambda)$ has a $\widetilde{\mathcal{C}}_\zeta$-filtration with sections $\widetilde{L}^{\min}(\mu)^{\oplus n_\mu}$ where $n_\mu = [D_\zeta(\lambda) : L_\zeta(\mu)] = \dim \operatorname{Ext}^1_{\mathcal{C}_\zeta}(L_\zeta(\lambda), L_\zeta(\mu))$. This condition follows from (6.8.13), induction, and the calculation $[D_\zeta : L_\zeta(\omega)] = n_\omega$ above, with $\omega = \mu$. This completes the proof. □

Remark 6.9. Observe that the proof of Theorem 6.8 shows that

$$\operatorname{Ext}^1_G(D(\lambda), \nabla_{\rm red}(\omega)) = 0 \tag{6.9.1}$$

for all regular dominant weights ω satisfying $\omega < \lambda$ and $l(\omega) \not\equiv l(\lambda) \bmod 2$. In a later paper, we will show that (assuming the LCF holds for restricted regular weights) the category of all rational G-modules with composition factors having regular dominant weights of a fixed parity forms a highest weight category whose standard and costandard modules are $\Delta^{\rm red}(\mu)$ and $\nabla_{\rm red}(\mu)$, respectively. Since all composition factors of $D(\lambda)$ are regular and have parity opposite to λ and are smaller than λ, the above vanishing (6.9.1) result is precisely the standard criterion that $D(\lambda)$ (or $\widetilde{D}(\lambda)$) have a $\Delta^{\rm red}$-(or $\widetilde{L}^{\rm min}$-)filtration in the highest weight category. This remark helps provide some conceptual insight into the above proof.

One can conjecturally extend the main conclusion of Theorem 6.8(a) to the entire radical series of $\Delta_\zeta(\lambda)$. For $\lambda \in X^+_{\rm reg}$ and $n \geq r$, put

$$E^n_\zeta(\lambda) := \Delta_\zeta(\lambda)/{\rm rad}^n\Delta_\zeta(\lambda),$$

and let $\widetilde{E}^n(\lambda)$ be the image of $\widetilde{L}^{\rm min}(\lambda)$ in $E^n_\zeta(\lambda)$.

Conjecture 6.10. Assume that $p > h$ and let $\lambda \in X^+_{\rm reg}$. Let $\widetilde{D}^{n-1}(\lambda)$ be the kernel of the natural surjection $\widetilde{E}^n(\lambda) \twoheadrightarrow \widetilde{E}^{n-1}(\lambda)$ $(n \geq 2)$. Then $\widetilde{D}^{n-1}(\lambda)$ has a filtration with sections $\widetilde{L}^{\rm min}(\mu)$, $\mu \in X^+_{\rm reg}$.

7. Applications to degree one cohomology

In this section, we apply the results of the previous section to obtain new results on the bounds of 1-cohomology for finite groups $G(q)$ of Lie type. This section also contains a number of results on ${\rm Ext}^1_G(L, L')$ and $H^1(G, L)$ for the algebraic group G and irreducible modules L, L', relating these groups to quantum analogues.

Given $\lambda \in X^+$, write $\lambda = \sum_{i=0}^\infty p^i\lambda_i$, where $\lambda_i \in X^+_1$. In this section, we make no assumption on p, except those explicitly noted below. Put

$$\lambda^{(i)} = \sum_{j=i}^\infty p^{j-i}\lambda_j.$$

We recall that if G is a semisimple, simply connected group, then ${\rm Ext}^1_{G_1}(L, L) \neq 0$ for some irreducible G_1-module L if and only if $p = 2$ and G has a simple component of type A_1 or C_n. When $G = SL_2(k)$ and $p = 2$, the only irreducible G_1-module L having a non-trivial self-extension is the trivial module $L = L(0)$. When $p = 2$ and $G = Sp_{2n}(k)$, $n \geq 2$, we have, on the other hand, ${\rm Ext}^1_{G_1}(L(0), L(0)) = 0 \neq {\rm Ext}^1_{G_1}(L(\alpha_0), L(\alpha_0))$. See [1] for more details.

We have the following simple result.

Lemma 7.1. *If either $\lambda = 0$ and G has a component of type A_1, or $\lambda \neq 0$ and G has a component of type C_n, assume that p is an odd prime. Let $\lambda, \mu \in X^+$, and let j be minimal so that $\lambda_j \neq \mu_j$. (If $\lambda = \mu$, put $j = \infty$ and $\lambda^{(j)} = 0$.) Then*

$$ {\rm Ext}^1_G(L(\lambda), L(\mu)) \cong {\rm Ext}^1_G(L(\lambda^{(j)}), L(\mu^{(j)})).$$

Proof. Our hypothesis implies that $\mathrm{Ext}^1_G(L(\lambda),L(\lambda))=0$, so we can assume that $j<\infty$. It suffices to show that if $\lambda_0=\mu_0$, then

$$\mathrm{Ext}^1_G(L(\lambda),L(\mu))\cong\mathrm{Ext}^1_G(L(\lambda^{(1)}),L(\mu^{(1)})). \tag{7.1.1}$$

However, there is a spectral sequence

$$E_2^{s,t}=\mathrm{Ext}^s_{G/G_1}(L(\lambda^{(1)}),\mathrm{Ext}^t_{G_1}(L(\lambda_0),L(\lambda_0))^{(-1)}\otimes L(\mu^{(1)})\Rightarrow\mathrm{Ext}^{s+t}_G(L(\lambda),L(\mu)).$$

If $\lambda=0$, then $\lambda_0=0$, and, so, $\mathrm{Ext}^1_{G_1}(L(\lambda_0),L(\lambda_0))=0$ because p is odd if G has a component of type A_1. Otherwise, $\mathrm{Ext}^1_{G_1}(L(\lambda_0),L(\lambda_0))=0$ because p is odd if G has a component of type C_n and $\lambda\neq 0$. Now (7.1.1) follows immediately. □

Now we can prove the following result.

Theorem 7.2. *Assume that $p>h$ and that the LCF holds for all regular weights in X_1^+. Let $\lambda,\mu\in X^+$ be distinct weights with $\lambda>\mu$ and choose j minimal so that $\lambda_j\neq\mu_j$. Suppose that $\lambda^{(j)}\in X^+_{\mathrm{reg}}$. Then*[6]

$$\dim\mathrm{Ext}^1_G(L(\lambda),L(\mu))=\dim\mathrm{Ext}^1_G(L(\lambda^{(j)}),L(\mu^{(j)}))\le\dim\mathrm{Ext}^1_{\mathcal{C}_\zeta}(L_\zeta(\lambda^{(j)}),L_\zeta(L(\mu^{(j)})).$$

Proof. Using Lemma 7.1, we can assume that $\lambda_0\neq\mu_0\in X^+_{\mathrm{reg}}$. The hypothesis implies that

$$\Delta^{\mathrm{red}}(\lambda)\cong L(\lambda_0)\otimes\Delta(\lambda^{(1)})^{(1)}.$$

Therefore, the composition factors of $\Delta^{\mathrm{red}}(\lambda)$ have the form $L(\lambda_0)\otimes L(\tau)^{(1)}$, where $\tau\in X^+$ satisfies $\tau\le\lambda^{(1)}$. It follows that $\mathrm{Hom}_G(\mathrm{rad}\,\Delta^{\mathrm{red}}(\lambda),L(\mu))=0$, so that, by the long exact sequence of cohomology, we have an injection

$$\mathrm{Ext}^1_G(L(\lambda),L(\mu))\hookrightarrow\mathrm{Ext}_G(\Delta^{\mathrm{red}}(\lambda),L(\mu)).$$

Also, the inclusion $L(\mu)\hookrightarrow\nabla_{\mathrm{red}}(\mu)$ induces an inclusion

$$\mathrm{Ext}^1_G(\Delta^{\mathrm{red}}(\lambda),L(\mu))\hookrightarrow\mathrm{Ext}^1_G(\Delta^{\mathrm{red}}(\lambda),\nabla_{\mathrm{red}}(\mu))$$

since there is no nonzero morphism $\Delta^{\mathrm{red}}(\lambda)\to\nabla_{\mathrm{red}}(\mu)/L(\mu)$ because $\lambda>\mu$. Therefore, composing these inclusions gives an inclusion

$$\mathrm{Ext}^1_G(L(\lambda),L(\mu))\hookrightarrow\mathrm{Ext}^1_G(\Delta^{\mathrm{red}}(\lambda),\nabla_{\mathrm{red}}(\mu)).$$

Now, by Theorem 6.7,

$$\dim\mathrm{Ext}^1_G(\Delta^{\mathrm{red}}(\lambda),\nabla_{\mathrm{red}}(\mu))=\dim\mathrm{Ext}^1_{\mathcal{C}_\zeta}(L_\zeta(\lambda),L_\zeta(\mu)).$$

□

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[6] It is quite possible that this restriction of regularity can be removed, at least in many cases. For results and conjectures comparing Ext^1 between irreducible modules with singular high weights and Ext^1 for irreducible modules with related regular high weights, see [39, §4].

The next result establishes there exists a uniform upper bounded on 1-cohomology of irreducible modules, depending only on the root system. Observe that if $L(\mu)$ is an irreducible module for a semisimple, simply group G, then dim $H^1(G, L(\mu)) =$ dim $\mathrm{Ext}^1(L(0), L(\mu))$ is at most the multiplicity of the trivial module $L(0)$ as a composition factor of $\nabla(\mu)$. In particular, if $G \cong SL_2(k)$, because the weight spaces in $\nabla(\mu)$ are one-dimensional, dim $H^1(SL_2(k), L(\mu)) \leq 1$ for all $\mu \in X^+$.

Theorem 7.3. *There is a constant $C = C(\Phi)$, depending only on the root system Φ, such that if G is a semisimple, simply connected algebraic group over an algebraically closed field k with root system Φ, then*

$$\dim\, H^1(G, L(\mu)) \leq C,\ \forall \mu \in X^+.$$

Proof. Without loss of generality, we may assume Φ is irreducible. By remarks above, we may also assume that Φ is not of type A_1 (where $C = 1$ works). Thus, Lemma 7.1 may be applied with $\lambda = 0$. *We may also assume, without loss, that $j = 0$ and that $\mu_0 \neq 0$.* In particular, $H^0(G_1, L(\mu_0)) = 0$, and $H^1(G, L(\mu))$ injects into $H^1(G_1, L(\mu))$; in fact, $H^1(G, L(\mu)) \cong H^1(G_1, L(\mu))^{G/G_1}$. Write $\mu = \mu_0 + p\mu_1$, $\mu_1 \in X^+$. Then

$$H^1(G_1, L(\mu))^{G/G_1} \cong \mathrm{Hom}_G(L(\mu_1)^\star, H^1(G_1, L(\mu_0))^{(-1)}),$$

where $L(\mu_1)^\star$ is the module dual to $L(\mu_1)$. Hence, dim $H^1(G, L(\mu))$ is bounded by the number of G-composition factors in $H^1(G_1, L(\mu_0))$.

Since there are only a finite number of possible restricted weights μ_0, we have proved there exists a constant $C(\Phi, p)$ depending on both the root system Φ and the prime p such that if $\mu \in X^+$, then dim $H^1(G, L(\mu)) \leq C(\Phi, p)$, when G is the semisimple, simply connected group over $k = \overline{\mathbb{F}}_p$ having root system Φ.

For each root system Φ, there exists a constant $D(\Phi)$ such that if $p > D(\Phi)$, the LCF holds for all regular $\lambda \in X_1^+$; see [3]. Assume that $p > D(\Phi)$. Let $\mu \in X^+$ be so that $\mu_0 \neq 0$ and $H^1(G, L(\mu)) \neq 0$. By Theorem 7.2,

$$1 \leq \dim\, H^1(G, L(\mu)) \leq \dim\, \mathrm{Ext}^1_{\mathcal{C}_\zeta}(L_\zeta(0), L_\zeta(\mu)). \tag{7.3.1}$$

Let $\mathrm{St}_\zeta \in \mathcal{C}_\zeta$ denote the irreducible (Steinberg) module with high weight $(p-1)\rho$. Then St_ζ is well-known to be a (self-dual) projective module, as is $\mathrm{St}_\zeta \otimes \mathrm{St}_\zeta$. Therefore, $\mathrm{St}_\zeta \otimes \mathrm{St}_\zeta$ contains the projective indecomposable cover of $L_\zeta(0)$ as a direct summand. Therefore, (7.3.1) implies that $L_\zeta(\mu)$ must be a composition factor of $\mathrm{St}_\zeta \otimes \mathrm{St}_\zeta$. Thus, $\mu \leq 2(p-1)\rho$. This gives, for $x = \mu$,

$$(x + \rho, \alpha_0^\vee) \leq (2p-1)(\rho, \alpha_0^\vee) = (2p-1)(h-1). \tag{7.3.2}$$

Let R be the region in Euclidean space $\mathbb{E}$ consisting of all x satisfying the inequalities (7.3.2) and $(x + \rho, \alpha^\vee) > 0$ for all $\alpha \in \Pi$, inequalities which $x = \mu$ also satisfy. The region R can intersect at most $2h|\Phi^+|$ hyperplanes $(x + \rho, \alpha^\vee) = np$, $\alpha \in \Phi^+, n \in \mathbb{Z}$. The length $l(y)$ of $y \in W_p$ with $y \cdot (-2\rho) \in X^+$ is equal to $l(w_0) + N$, with $y = w_0 y'$ and $l(y') = N \leq 2h|\Phi^+|$ is the number of these hyperplanes separating 0 from $y \cdot (-2\rho)$.

Thus, if also $y.(-2\rho) \in R$, then $N \leq 2h|\Phi^+|$. Thus, there are at most $2h|\Phi^+|$ dominant weights μ such that $\mu_0 \neq 0$ and $\mathrm{Ext}^1_{\mathcal{C}_\zeta}(L_\zeta(0), L_\zeta(\mu))$.

The weights μ with $\mu_0 \neq 0$ and $H^1(G, L(\mu)) \neq 0$ with the underlying characteristic $p > D(\Phi)$ may be written as $y\cdot(-2\rho)$ for at most $2h|\Phi^+|$ elements of W_p, independent of p. For each such $\mu = y\cdot(-2\rho)$, dim $\mathrm{Ext}^1_{\mathcal{C}_\zeta}(L_\zeta(0), L_\zeta(\mu))$ is bounded by the coefficient of $t^{l(y)-l(w_0)-1}$ in the Kazhdan-Lusztig polynomial $P_{w_0,y}$ (see Remark 1.4 and use (1.4.2), or use (1.3.1) after dualizing the modules in its formulation), and

$$\dim H^1(G, L(\mu)) \leq \dim \mathrm{Ext}^1_{\mathcal{C}_\zeta}(L_\zeta(0), L_\zeta(\mu))$$

by Theorem 7.2 again. Let $C'(\Phi)$ to be the maximal of these finitely many coefficients.

Finally, putting things together, we see that

$$C(\Phi) := \mathrm{MAX}\,\{C'(\Phi), C(p, \Phi),\, p \leq D(\Phi).\}$$

satisfies the requirements of the theorem. □

We will now apply these results to generic cohomology for the infinite families of finite groups groups of Lie type. Let $q = p^d$. These groups fall into several classes: (1) the split groups $A_n(q), B_n(q), \ldots, E_8(q)$; (2) the (Steinberg) twisted groups ${}^2A_n(q), \ldots, {}^2E_6(q)$; (3) the Suzuki groups ${}^2B_2(2^{2n+1})$; and (4) the Ree groups ${}^2F_4(2^{2n+1})$ and ${}^2G_2(3^{2n+1})$. The reader is referred to [19, Ch. 3]. It will be convenient, however, to denote these groups as ${}^iG(q)$. For example, if $G = SL_n(k)$, then ${}^2G(q)$ denotes ${}^2A_{n-1}(q)$.[7]

Fix G and an associated infinite family $\{{}^iG(q)\}$. Here i is fixed and q is allowed for vary over appropriate powers of p, as indicated above. For any finite dimensional rational G-module V and positive integer n, the generic cohomology of V in degree n is defined to be the common limit

$$H^n_{\mathrm{gen}}(G, V) := \lim_{q\to\infty} H^n({}^iG(q), V) = \lim_{m\to\infty} H^n(G, V^{(m)}). \tag{7.3.3}$$

In [18] and [21] it is shown that this limit is achieved for the split $i = 1$ cases for q or m sufficiently large. (The characteristic p is fixed in [18], but allowed to vary in [21].) Paper [6] treats these results in the remaining cases $i > 1$.[8]

Applying Theorem 7.3 to the right-hand side of (7.3.3), we obtain:

Theorem 7.4. *The dimension* dim $H^1_{\mathrm{gen}}(G, L)$ *is, for all irreducible rational G-modules L, bounded by a constant depending only on the root system, and not on p and L.*

Remarks 7.5. (a) In [23], Guralnick conjectured that there exists a constant C such that if G is a finite group acting faithfully on an absolutely irreducible module L, then dim $H^1(G, L) \leq C$. In [24], it is suggested that, in fact, dim $H^1(G, L) \leq 2$ in all cases.

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[7] We are using the notation of [19]. Some authors would denote this group by ${}^2A_{n-1}(q^2)$.

[8] In [11] it is shown (in general) that, for any m, $H^n(G, V^{(m)}) \hookrightarrow H^n_{\mathrm{gen}}(G, V)$, so that $H^n_{\mathrm{gen}}(G, V)$ is the (finite) directed union of the cohomology of the twisted modules $V^{(m)}$.

Although this specific guess is now known to be false [38], the original conjecture on a universal bound remains open. In the conference report [25], Guralnick mentioned that the current highest dimension known is 3, and expressed the view, verbally (in his conference talk), that (even if it is found that there is no constant bound), it should be the case that the dimension grows very, very slowly (in an unspecified way). Theorem 7.4 provides very positive evidence for this philosophy and is even consistent with the original conjecture.

(b) Our results in this paper apply to the cohomology of the finite groups of Lie type with coefficients in a non-trivial irreducible module L in the *defining characteristic* p of the ambient algebraic group G. We have not considered similar results in the cross-characteristic case, where the module L is defined over a field of characteristic different from p. The generic cohomology is only partly developed in the cross-characteristic case, though there is a reasonably satisfactory theory in type A (with the homological degree in a range) [16]. Considerably more is known about cross-characteristic cohomology with trivial coefficients; see, for example, [37] for $GL_n(q)$.

8. Appendix: Examples

If $p > h$ and $\lambda \in X^+$, then the hLP^L and hLP^R both hold for $p\lambda \in X^+_{\rm reg}$ by Proposition 4.3. The following example provides further evidence for the validity of Conjecture 6.1 (or, equivalently, Conjecture 6.2). We have have not investigated the extent to which a variation of the conjecture might be plausible for singular weights or for primes $p < h$, nor have we considered the $p = h$ case (apart from the special cases considered below).

Example 8.1. Let $G = SL_2(k)$ and assume first that $p > 2$. We will give a direct verification of Conjecture 6.2 which adapts also to give the conclusion of Conjecture 6.2 when $p = 2$. Write $\varpi = \varpi_1$ for the fundamental dominant weight and $\Pi = \{\alpha\}$. We consider $\lambda \in X^+ \cap W_p \cdot 0$; thus, $\lambda = 2pa\varpi$ or $\lambda = (2pa - 2)\varpi$ for $a \in \mathbb{N}$. In case $\lambda = 2pa\varpi$, both hLP^L and hLP^R hold for λ by Proposition 4.3. Now assume that $\lambda = (2ap - 2)\varpi$. Let $\widetilde{L}_\zeta(\lambda) = \widetilde{U}_\zeta \cdot v^+$ be the lattice generated by a high weight vector in $L_\zeta(\lambda)$. Set $\Delta^{\rm red}(\lambda) = \widetilde{L}_\zeta(\lambda)/\pi\widetilde{L}_\zeta(\lambda)$. Since W_p is an infinite dihedral group, the Kazhdan-Lusztig polynomials satisfy $P_{y,x} = 1$ whenever $y \leq x$. The LCF for U_ζ is an alternating sum ch $L_\zeta(x \cdot (-2\rho)) = \sum_{y\leq x}(-1)^{l(x)-l(y)}P_{y,x}(1)$ ch $\Delta_\zeta(y \cdot (-2\rho))$. Thus, ch $L_\zeta(x \cdot (-2\rho))$ = ch $\Delta_\zeta(x \cdot (-2\rho))$ − ch $L_\zeta(xs \cdot (-2\rho))$ provided that $xs \cdot (-2\rho) \in X^+$ and $s \in S_p$ satisfy $xs < x$, since all $y \cdot (-2\rho) \in X^+$ with $y \leq x$ also then satisfy $y \leq xs$, except when $y = x$. (The fundamental reflection s is unique with $xs < x$.) In particular, if $\lambda = x \cdot (-2\rho) = (2ap - w)\varpi$, then $s = s_{\alpha,-p}$, $xs \cdot (-2\rho) = 2(a - 1)p\varpi$ and ch $\Delta^{\rm red}(\lambda)$ = ch $\Delta(\lambda)$ − ch $\Delta(2(a-1)\varpi)^{(1)}$ because ch $\Delta^{\rm red}(\lambda)$ = ch $L_\zeta(\lambda)$, ch $\Delta_\zeta(\lambda)$ = ch $\Delta(\lambda)$ and ch $L_\zeta(2p(a - 1)\varpi)$ = ch $\Delta(2(a - 1))^{(1)}$; cf. Lemma 1.3. Since $\Delta^{\rm red}(\lambda)$ is generated by a high weight vector (of weight λ), the universal mapping property of standard modules implies there is a surjection $\Delta(\lambda) \twoheadrightarrow \Delta^{\rm red}(\lambda)$ with kernel $K(\lambda)$ having the same character (and dimension) as $\Delta(2(a-1)\varpi)^{(1)}$.

We will show that $K(\lambda) \cong \Delta(2(a-1)\varpi)^{(1)}$ by a consisting of weight vectors v_i, $0 \le i \le 2ap-2$, of weight $2(ap-1-i)\varpi$ and which satisfy

$$x_{-\alpha}(t)v_i = \sum_{j=0}^{2ap-2-i} \binom{i+j}{j} t^j v_{i+j}, \tag{8.1.1}$$

where $x_{-\alpha}(t) = \left(\begin{smallmatrix} 1 & 0 \\ t & 1 \end{smallmatrix}\right)$. For $0 \le j \le 2(a-1)$, put $u_j = v_{p-1+jp}$. Since u_0 has weight $2(a-1)p\varpi$, $u_0 \in K(\lambda)$; in fact, the line ku_0 is B-stable. For $0 \le i \le 2(a-1)$, (8.1.1) gives

$$\begin{aligned} x_{-\alpha}(t)u_i &= x_{-\alpha}(t)v_{p-1+ip} \\ &= \sum_{j=0}^{(2a-1-i)p-1} \binom{p-1+ip+j}{j} t^j v_{p-1+ip+j} \\ &= \sum_{l=0}^{2a-1-i} \binom{i+l}{l} t^{pl} u_{i+l} \end{aligned} \tag{8.1.2}$$

since $\binom{p-1+ip+j}{j} \equiv 0 \bmod p$ if p does not divide j and $\equiv \binom{i+l}{l} \bmod p$ if $j = pl$. Since u_j has weight $(2(a-1)-2j)p\varpi$, the u_i span a G-submodule of $\Delta(\lambda)$ which is B-, and hence G-, isomorphic to $\Delta(2(a-1)\varpi)^{(1)}$. So $K(\lambda) \cong \Delta(2(a-1)\varpi)^{(1)}$, as required.

Thus, we obtain a short sequence

$$0 \to \Delta(2(a-1)\varpi)^{(1)} \to \Delta((2ap-2)\varpi) \to \Delta^{\rm red}((2ap-2)\varpi) \to 0. \tag{8.1.3}$$

Then (8.1.3) provides a distinguished triangle in the derived category of $\mathcal{C}_{G,0}$, which, after rotating four times, gives another distinguished triangle

$$\Delta((2ap-2)\varpi)[1] \to \Delta^{\rm red}((2ap-2)\varpi)[1] \to \Delta(2(a-1)\varpi)^{(1)}[2] \to .$$

By Proposition 4.3 and the definitions, $\Delta^{\rm red}((2ap-2)\varpi)[1] \in \mathcal{E}^L$. A dual argument applies for $\nabla_{\rm red}$. Therefore, Conjecture 6.2 holds for $SL_2(k)$ for $p > 2$. If $p = 2$, then Proposition 4.3 is not available. However if $G = SL_2(k)$ and $p = 2$, the results given in [2, (3.10)] show that Proposition 4.3 holds in this case. Also, the calculation (8.1.2) remains valid if $p = 2$, establishing that the $p = h$ version of Conjecture 6.2 holds for $G = SL_2(k)$.

Conjecture 6.2 may also be established, even allowing $p = 2$, by using (8.1.3) and a related sequence. We omit details. The regular weight case is essentially also a consequence of Theorem 6.8.

We next consider one non-trivial calculation for $p = 3$ (that is, $p = h$, when $G = SL_3(k)$).

Example 8.2. Let $G = SL_3(k)$, with $p = 3$. We consider the weight $f = 3\varpi_1 + 3\varpi_2$. We will show that, even though Proposition 4.2 does not apply, we still have $\Delta^{\rm red}(f) \in \widehat{\mathcal{E}^L}$. (It follows also that $\nabla_{\rm red}(f) \in \widehat{\mathcal{E}^R}$.

Put $d = \varpi_1 + 4\varpi_2$. The ideal Γ of dominant weights in $X^+ \cap W_p \cdot 0 = X^+ \cap W_p \cdot d$ generated by d consists of d, $a = \varpi_1 + \varpi_2$, $c = 3\varpi_2$, and 0. Each is the reduction modulo π of a quantum irreducible module, and so it follows from [13] that $L(d) \in \mathcal{E}^L$. Also, $L(d) = \Delta^{\rm red}(d) = \Delta(\varpi_2)^{(1)} \otimes L(a) = {\rm Ind}_{BG_1}^G p\varpi_2 \otimes L(a)$. Applying the results of Section 3, we have

$$\begin{aligned} L(d) &= \theta_s \cdot {\rm Ind}_{BG_1}^G \cdot t_{p\varpi_2}(L(a)) \\ &= \theta_s \cdot t_{p\varpi_2}(L(a)) \\ &= {\rm Ind}_{BG_1}^G \cdot t_{p\varpi_1} \cdot \theta_s(L(a)). \end{aligned}$$

Thus $\beta_s L(d)$ has

$${\rm Ind}_{BG_1}^G t_{p\varpi_2} L(b) = \Delta(\varpi_2)^{(1)} \otimes L(b) = L(c) \otimes L(b),$$

as a direct summand, where $b = 3\varpi_2 = as_1$.

Now, $L(c) \otimes L(b)$ is the reduction modulo π of $\widetilde{T} = \widetilde{L}^{\min}(c) \otimes \widetilde{L}^{\min}(b)$, which contains a pure submodule $\widetilde{E} = L_\zeta(f) \cap \widetilde{T}$. We have dim $L_\zeta(f) = 8$, and dim $L(c) \otimes L(b) = 9$. Also ${\rm Hom}_G(L(c) \otimes L(b),\ L(0)) \cong {\rm Hom}_G(L(c), L(d) \cong k$, and dim $L(f) = 8$. It follows that $\widetilde{E} \cong \widetilde{L}^{\min}(f)$, and we have an exact sequence

$$0 \to \Delta^{\rm red}(f) \to L(c) \otimes L(b) \to L(0) \to 0.$$

Since $L(0) \cong \Delta(0)$, we have ${\rm Ext}^n_G(L(0), \nabla(\nu)) = 0$ for $n > 0$. Hence, ${\rm Ext}^m_G(\Delta^{\rm red}(f), \nabla(\nu))$ vanishes whenever ${\rm Ext}^m_G(L(0) \otimes L(b), \nabla(\nu))$ vanishes ($m \geq 0$). It follows that $\Delta^{\rm red}(f)$ as in $\mathcal{E}^L$.

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