SOME $\mathbb{Z}/2$-GRADED REPRESENTATION THEORY

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We dedicate this paper to Michel Broué on his 60th birthday

Preliminary Version

1. Introduction

Graded structures play an important and often subtle role in representation theory. For example, in characteristic 0 theory, the existence of a $\mathbb{Z}^+$-grading—even a Koszul structure—on the finite dimensional (quasi-hereditary) algebra representing the principal block for the category $\mathcal{O}$ (associated to a complex semisimple Lie algebra $\mathfrak{g}_\mathbb{C}$) follows from classical (and deep) work of Beilinson-Ginzburg-Soergel [4] (see also [23] for another point of view). In positive characteristic p, Andersen-Jantzen-Soergel [1] prove that the principal block of the restricted enveloping algebra of a semisimple algebraic group G has a kind of Koszul structure whenever the Lusztig conjecture holds. In general, the existence of an interesting $\mathbb{Z}^+$-grading on other important algebras (e. g., Schur algebras) seems to be a very difficult problem: If the $\mathbb{Z}^+$-grading is sufficiently rich, its existence implies the validity of the character conjectures in standard Lie theory contexts [14, Rem. 2.3.5 and Ex. 3.2]. However, even the validity of the strongest non-graded versions of these conjectures yet devised do not imply the existence of a nice (compatible with the radical powers) $\mathbb{Z}^+$-grading in an abstract setting.

But the existence of an interesting $\mathbb{Z}/2$-grading appears much easier, and is potentially almost as useful.

Let A be a quasi-hereditary algebra (QHA) over a field k with weight poset Λ. Call A a $\mathbb{Z}/2$-QHA provided that A has a $\mathbb{Z}/2$-grading $A = A_{\bar{0}} \oplus A_{\bar{1}}$ with $A_{\bar{1}} \subseteq \operatorname{rad} A$, the radical of A. We will let $\mathcal{C}_{\rm gr}$ denote the category of graded A-modules. Natural examples of such $\mathbb{Z}/2$-QHA algebra structures, may, of course, be obtained from any Koszul grading (or any $\mathbb{Z}^+$-grading compatible with the radical powers) on a quasi-hereditary algebra by collecting together even and odd grades. But another algebraically natural $\mathbb{Z}/2$-QHA structure arises for any quasi-hereditary algebra whenever a function $\ell : \Lambda \to \mathbb{Z}$ is given: Put

$$t := \sum_{\lambda\in\Lambda} (-1)^{\ell(\lambda)} e_\lambda, \qquad (*)$$

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where the e_λ are a complete set of orthogonal idempotents in A such that Ae_λ is isomorphic to a direct sum of the projective indecomposable cover of the irreducible module $L(\lambda)$ indexed by λ. Clearly, $t^2 = 1$, so let $\theta \in \mathrm{Aut}(A)$ be conjugation by t. If $A_{\bar 0}$ (resp., $A_{\bar 1}$) denotes the fixed-point space (resp., -1-eigenspace) of θ, $A = A_{\bar 0} \oplus A_{\bar 1}$ defines a $\mathbb{Z}/2$-QHA structure on A (provided $p = \operatorname{char} k \neq 2$).

This paper studies $\mathbb{Z}/2$-QHAs. In particular, we introduce the notion of a $\mathbb{Z}/2$-based Kazhdan-Lusztig theory in terms of functions $\ell' : \Lambda \to \mathbb{Z}/2$. We require that $\mathrm{Hom}^n_{D^b(\mathcal{C}_{\rm gr})}(L(\lambda), \nabla(\mu)(m)) \neq 0 \implies \ell'(\lambda) - \ell'(\mu) \equiv m + n \bmod 2$ for all $\lambda, \mu \in \Lambda$. A dual condition involving $\Delta(\lambda)(m)$ and $L(\mu)$ is also assumed. (Typically, $\ell' = 0$ identically.) In Theorem 5.1, we prove that if $\mathcal{C}_{\rm gr}$ is defined by means of (*), then it has a $\mathbb{Z}/2$-based Kazhdan-Lusztig theory provided that A-mod has a Kazhdan-Lusztig theory.

The converse to Theorem 5.1 fails. A counterexample is provided in Example 5.2. Thus, the new notion is a candidate for a revised, and, perhaps more flexible, version of an abstract Kazhdan-Lusztig theory. So far, most properties previously proved by us for quasi-hereditary algebras A for which A-mod has a Kazhdan-Lusztig theory also hold for this new notion. For example, in Theorem 4.2 and Remark 4.3(a), we prove that the existence of a $\mathbb{Z}/2$-based Kazhdan-Lusztig theory is sufficient to establish character formulas for irreducible modules in the standard Lie-theoretic setting; in particular, a $\mathbb{Z}/2$-based Kazhdan-Lusztig theory is sufficient to prove the famous Lusztig conjecture (for reductive groups in positive characteristic $p \geq h$, the Coxeter number). This sufficiency also a property of the original Kazhdan-Lusztig theory. We also prove another major property of the previous notion, that all Ext^n_G groups between irreducible modules can be determined in terms of abstract Kazhdan-Lusztig polynomials in the presence of a valid $\mathbb{Z}/2$-based Kazhdan-Lusztig theory. See the remark after Definition 4.1.

Finally, we discuss, in type A, some suggestive symmetric group ring to which this investigation led. The examples suggest a possible Kazhdan-Lusztig theory—even in the original sense—for Schur algebras, involving at least some (singular) weights for smaller primes $p < h$. We speculate on a possible future verification, involving a sufficiently strong version of Broué's abelian defect group conjecture (now known as an abstract equivalence by Chuang-Rouquier [6]).

Notation. Given $\mathcal{A}$ an abelian category (with enough injectives and projectives), let $D^b(\mathcal{A})$ be the associated bounded derived category; see [3]. Let $[1] : D^b(\mathcal{A}) \to D^b(\mathcal{A})$, $X \mapsto X[1]$, be the translation functor on $D^b(\mathcal{A})$ arising from its structure as a triangulated category.

Let A be a finite dimensional algebra over k, and let $\mathcal{C} = \mathcal{C}(A)$ be the abelian category (with enough projectives and injectives) of finite dimensional left A-modules. We will often assume that A is a graded algebra, e. g., $A = \bigoplus_{\omega\in\Omega} A_\omega$, where Ω is an additive monoid, and $A_\omega A_{\omega'} \subseteq A_{\omega+\omega'}$. Usually, $\Omega = \mathbb{Z}/2, \mathbb{Z}$ or $\mathbb{N}$. In this case, let $\mathcal{C}_{\rm gr} = \mathcal{C}_{\rm gr}(A)$ be the category of finite dimensional, graded A-modules. Then $\mathcal{C}_{\rm gr}$ is an abelian category with enough projective and injectives. For $M \in \mathrm{Ob}(\mathcal{C}_{\mathrm{gr}})$, and $\omega' \in \Omega$, define $M(\omega') \in \mathrm{Ob}(\mathcal{C}_{\mathrm{gr}})$ by setting $M(\omega')_\omega := M(\omega + \omega')$. For $M, N \in \mathrm{Ob}(\mathcal{C}_{\mathrm{gr}})$, there is a graded isomorphism

$$\mathrm{Ext}^\bullet_{\mathcal{C}}(M,N) \cong \bigoplus_{\omega\in\Omega} \mathrm{Ext}^\bullet_{\mathcal{C}_{\mathrm{gr}}}(M, N(\omega)), \tag{1.0.1}$$

where in computing $\mathrm{Ext}^\bullet_{\mathcal{C}}(M,N)$ we "forget" the graded structure on M and N. Because the "twisting functor" $(\omega') : \mathcal{C}_{\mathrm{gr}} \to \mathcal{C}_{\mathrm{gr}}$, $M \mapsto M(\omega')$, is exact it defines a twisting functor $(\omega') : D^b(\mathcal{C}_{\mathrm{gr}}) \to D^b(\mathcal{C}_{\mathrm{gr}})$ on derived categories. Let $\{\omega'\} := (\omega')\circ[1] = [1]\circ(\omega') : D^b(\mathcal{C}_{\mathrm{gr}}) \to D^b(\mathcal{C}_{\mathrm{gr}})$.

Many of the objects considered in this paper will be modules for the ring $\mathcal{Z} := \mathbb{Z}[t, t^{-1}]$ of Laurent polynomials in an indeterminant t. Let $(-)^- : \mathcal{Z} \to \mathcal{Z}, f \mapsto \bar{f}$, be the unique automorphism satisfying $\bar{t} = t^{-1}$.

2. $\mathbb{Z}/2$-quasi-hereditary algebras

From now on, $\mathcal{C} = \mathcal{C}(A)$ for a quasi-hereditary algebra (QHA) A over a field k. The "weight" poset making $\mathcal{C}$ into a highest weight category (HWC) is denoted by Λ. For $\lambda \in \Lambda$, let $\Delta(\lambda)$, $\nabla(\lambda)$, and $L(\lambda)$ be the corresponding standard, costandard, and irreducible modules. Thus, $\Delta(\lambda)$ (resp., $\nabla(\lambda)$) has head (resp., socle) isomorphic to $L(\lambda)$.

Let $\mathbb{Z} \to \mathbb{Z}/2$, $i \mapsto \bar{i}$, be the quotient map. We often suppose that A has a fixed structure $A = A_{\bar 0} \oplus A_{\bar 1}$ as a $\mathbb{Z}/2$-graded algebra. Thus, $A_{\bar i}A_{\bar j} \subseteq A_{\bar i+\bar j}$ for $\bar i, \bar j \in \mathbb{Z}/2$. We will also assume that $A_{\bar 1} \subseteq \mathrm{rad}\, A$, the radical of A. When these conditions hold, we say that A is a $\mathbb{Z}/2$-quasi-hereditary algebra ($\mathbb{Z}/2$-QHA). Let $\mathcal{C}_{\mathrm{gr}} = \mathcal{C}_{\mathrm{gr}}(A)$ be the category of finite dimensional, $\mathbb{Z}/2$-graded A-modules. We call $\mathcal{C}$ is a $\mathbb{Z}/2$-highest weight category ($\mathbb{Z}/2$-HWC). When no confusion results, given $M \in \mathrm{Ob}(\mathcal{C}_{\mathrm{gr}})$ and $i \in \mathbb{Z}$, write M_i in place of $M_{\bar i}$ and $M(i)$ for $M(\bar i)$. We also write A_i for $A_{\bar i}$, $i \in \mathbb{Z}$.

For $\lambda \in \Lambda$, we regard $L(\lambda) \in \mathrm{Ob}(\mathcal{C}_{\mathrm{gr}})$ by setting $L(\lambda)_0 = L(\lambda)$. This grading is possible since $A_{\bar 1} \subseteq \mathrm{rad}\, A$ and $(\mathrm{rad}\, A)L(\lambda) = 0$. The standard and costandard modules are objects in $\mathcal{C}_{\mathrm{gr}}$ in a natural way, as we now show.

Lemma 2.1. *Let A be a $\mathbb{Z}/2$-QHA. For $\lambda \in \Lambda$, the standard module $\Delta(\lambda)$ has a unique structure $\Delta(\lambda) = \Delta(\lambda)_0 \oplus \Delta(\lambda)_1$ as a graded A-module such that $L(\lambda) = L(\lambda)_0$ is an A_0-homomorphic image of $\Delta(\lambda)_0$. A similar statement holds for $\nabla(\lambda)$.*

Proof. The proof is based on the arguments given in [8]. Because $A_1 \subseteq \mathrm{rad}\, A$, $A_0/\mathrm{rad}\, A_0 \cong A/\mathrm{rad}(A)$. Therefore, A_0 contains a maximal set E of primitive orthogonal idempotents of A. We can (temporarily) assume that Λ is totally ordered, and let $\lambda \in \Lambda$ be maximal. Then $\Delta(\lambda) \cong Ae$ for some $e \in E \subset A_0$. Thus, $\Delta(\lambda)$ is graded, with $\Delta(\lambda)_{\bar 0} = A_{\bar 0}e$ and $\Delta(\lambda)_{\bar 1} = A_{\bar 1}e$. Clearly, $L(\lambda) = L(\lambda)_0$ is an A_0-homomorphic image of $\Delta(\lambda)_0$. Of course, $L(\lambda)$ is also a homomorphic image of $\Delta(\lambda)$ in the graded category $\mathcal{C}_{\mathrm{gr}}$. Since $\Delta(\lambda)$ is a projective indecomposable module in $\mathcal{C}_{\mathrm{gr}}$, its grading is uniquely determined by these properties. Also, $J := AeA$ is a (graded) heredity ideal of A, and so $\bar{A} := A/J$ is a $\mathbb{Z}/2$-graded quasi-hereditary algebra. We can now argue by induction on $|\Lambda|$. □

Examples 2.2. (a) Assume the hypotheses of the lemma. For $\lambda \in \Lambda$, let $X(\lambda)$ be the indecomposable tilting module of high weight λ. Then it can be proved that $X(\lambda)$ has a unique structure as an object in $\mathcal{C}_{\rm gr}$ in such a way that the inclusion $\Delta(\lambda) \hookrightarrow X(\lambda)$ and the epimorphism $X(\lambda) \twoheadrightarrow \nabla(\lambda)$ are morphisms in $\mathcal{C}_{\rm gr}$, if $\Delta(\lambda)$ and $\nabla(\lambda)$ are given their $\mathbb{Z}/2$-gradings as in the lemma. We will not use this fact below.

(b) Let $A = \oplus_{n=0}^{\infty} A_n$ be a $\mathbb{N}$-graded QHA. Putting $A_{\bar 0} = \oplus_{n=0}^{\infty} A_{2n}$ and $A_{\bar 1} = \oplus_{n=1}^{\infty} A_{2n+1}$, we obtain a $\mathbb{Z}/2$-graded QHA.

(c) Suppose that A is a QHA and $\theta : A \to A$ is an involution in $\mathrm{Aut}(A)$. Set $A_{\bar 0} = A^{\theta}$, the fixed-point subalgebra and $A_{\bar 1}$ equal to the -1-eigenspace θ. Then $A = A_{\bar 0} \oplus A_{\bar 1}$ defines a $\mathbb{Z}/2$-QHA, provided that $A_{\bar 1} \subseteq \mathrm{rad}\, A$.

(d) To consider a specific case in (b), let $\ell : \Lambda \to \mathbb{Z}$ be a function. Now define t as in (*) in the introduction. Let $\theta \in \mathrm{Aut}(A)$ be conjugation by t. Since $te_\lambda a e_\mu t = (-1)^{\ell(\lambda)+\ell(\mu)} e_\lambda a e_\mu$ for $a \in A$, A_1 is contained in the radical of A. We refer to this $\mathbb{Z}/2$-grading as the $\mathbb{Z}/2$-QHA structure *induced by* ℓ. In this case, let $\mathcal{C}_{\rm gr}^{\ell}$ denote the corresponding category of graded modules.

Finally, it will be useful to record the following result. Part (a) is proved in [9, Lemma 2.2] and then (b) follows from (a) and (1.0.1).

Lemma 2.3. *(a) Let A be a* QHA. *For $\lambda, \mu \in \Lambda$, $n \in \mathbb{N}$,*

$$\dim \mathrm{Hom}_{D^b(\mathcal{C})}(\Delta(\lambda), \nabla(\mu)) = \dim \mathrm{Ext}^n_{\mathcal{C}}(\Delta(\lambda), \nabla(\mu)) = \delta_{\lambda,\mu}\delta_{n,0}.$$

*(b) Now assume that A is a $\mathbb{Z}/2$-*QHA. *For $\lambda, \mu \in \Lambda$, $m \in \mathbb{Z}$, and $n \in \mathbb{N}$,*

$$\dim \mathrm{Hom}^n_{D^b(\mathcal{C}_{\rm gr})}(\Delta(\lambda), \nabla(\mu)(m)) = \dim \mathrm{Ext}^n_{\mathcal{C}_{\rm gr}}(\Delta(\lambda), \nabla(\mu)(m)) = \delta_{\lambda,\mu}\delta_{n,0}\delta_{m,0}.$$

3. Variations on the enriched Grothendieck groups

Throughout this section A is a $\mathbb{Z}/2$-QHA and Λ is the poset for the HWC $\mathcal{C} = \mathcal{C}(A)$.

First, fix functions $\ell, \ell', r, r' : \Lambda \to \mathbb{Z}$. We call ℓ, ℓ', r, r' "length" functions on Λ, though we will only use their reductions modulo 2—hence, they are effectively $\mathbb{Z}/2$-valued. (Traditionally, $\ell(\lambda)$ is the length of a Weyl or affine Weyl group element associated to λ.) In the main applications we have in mind, $\ell = r$, and $\ell' = r'$ (and, even, $\ell' \equiv r' = 0$ identically), but these assumptions are not needed to develop some of the theory and allow for more flexibility.

We now define categories $\mathcal{E}^L, \mathcal{E}^{L'}, \mathcal{E}^{L''}$, and $\mathcal{E}^R, \mathcal{E}^{R'}, \mathcal{E}^{R''}$. Here $\mathcal{E}^L$ and $\mathcal{E}^R$ will be full subcategories of $D^b(\mathcal{C})$, while the others are full subcategories of $D^b(\mathcal{C}_{\rm gr})$.

First, $\mathcal{E}^L$ is defined in the familiar way (see [9, §2] and, more recently, [19, Appendix C]): $\mathcal{E}^L_0$ is the full subcategory consisting of all objects in the derived category $D^b(\mathcal{C})$ which are direct sums of $\Delta(\lambda)[m]$, $\lambda \in \Lambda, m \in \mathbb{Z}$, with $m \equiv \ell(\lambda)$ modulo 2. Then $\mathcal{E}^L_{i+1}$, is defined recursively as the full subcategory of all objects X for which there is a distinguished triangle $A \to X \to B \to$ in $D^b(\mathcal{C})$ with $A, B \in \mathrm{Ob}(\mathcal{E}^L_i)$. Set $\mathcal{E}^L = \bigcup_{i=0}^{\infty} \mathcal{E}^L_i$, regarded as a full subcategory of $D^b(\mathcal{C})$.

To define $\mathcal{E}^{L'}$, we replace $\mathcal{C}$ above by $\mathcal{C}_{\mathrm{gr}}$, and define $\mathcal{E}^{L'}_0$ to be the full subcategory of the derived category $D^b(\mathcal{C}_{\mathrm{gr}})$ consisting of objects which are direct sums of objects $\Delta(\lambda)(n)[m]$ with

$$n \equiv m + \ell'(\lambda) \mod 2, \forall \lambda \in \Lambda. \tag{3.0.1}$$

Then $\mathcal{E}^{L'}_{i+1}$ is defined as the full subcategory of objects X' which are part of a distinguished triangle $A' \to X' \to B' \to$ in $D^b(\mathcal{C}_{\mathrm{gr}})$ with $A', B' \in \mathrm{Ob}(\mathcal{E}^{L'}_i)$. Again, $\mathcal{E}^{L'} = \bigcup_{i=0}^{\infty} \mathcal{E}^{L'}_i$.

To define $\mathcal{E}^{L''}$, replace the condition (3.0.1) for a given $\lambda \in \Lambda$ by the condition

$$m \equiv \ell(\lambda) \text{ and } n \equiv m + \ell'(\lambda), \tag{3.0.2}$$

with the rest of the construction parallel to that of $\mathcal{E}^{L'}$.

The categories $\mathcal{E}^R$, $\mathcal{E}^{R'}$, $\mathcal{E}^{R''}$ are defined similarly, replacing $\Delta(\lambda)$ by $\nabla(\lambda)$ in each case, and the functions ℓ and ℓ' by the functions r and r', respectively.

Observe that $\mathcal{E}^{L'}$ and $\mathcal{E}^{R'}$ are closed under the operation $\{1\} = (1)[1] : D^b(\mathcal{C}_{\mathrm{gr}}) \to D^b(\mathcal{C}_{\mathrm{gr}})$.

Obviously, there are inclusions $\mathcal{E}^{L''} \subseteq \mathcal{E}^{L'}$, $\mathcal{E}^{R''} \subseteq \mathcal{E}^{R'}$ and natural forgetful functors $\mathcal{E}^{L''} \to \mathcal{E}^L$, $\mathcal{E}^{R''} \to \mathcal{E}^R$. The construction of $\mathcal{E}^{L''}$, $\mathcal{E}^{R''}$ in the case $\ell' \equiv r' = 0$ identically is reminiscent of a $\mathbb{Z}$-graded construction in [10], but the categories $\mathcal{E}^{L'}$ and $\mathcal{E}^{R'}$ are new. We will see that they support Kazhdan-Lusztig-like theories, just as $\mathcal{E}^L$ and $\mathcal{E}^R$ do.

All three of the pairs $(\mathcal{E}^L, \mathcal{E}^R)$, $(\mathcal{E}^{L'}, \mathcal{E}^{R'})$,$(\mathcal{E}^{L''}, \mathcal{E}^{R''})$, support enriched Grothendieck groups. Put $\widehat{\mathcal{E}}^L = \mathcal{E}^L \oplus \mathcal{E}^L[1]$, and define $K^L_0 = K^L_0(\mathcal{C})$ as the quotient of the free abelian groups on objects of $\widehat{\mathcal{E}}^L$ by the subgroup spanned by all relations $X + Z - Y$ where $X \to Y \to Z \to$ is a distinguished triangle $D^b(\mathcal{C})$ with terms X, Y, Z either all in $\mathrm{Ob}(\mathcal{E}^L)$ or all in $\mathrm{Ob}(\mathcal{E}^L[1])$, or a direct sum of two such triangles. Similarly, we put $\widehat{\mathcal{E}}^{L'} = \mathcal{E}^{L'} + \mathcal{E}^{L'}\{1\} = \mathcal{E}^{L'}$ and $\widehat{\mathcal{E}}^{L''} = \mathcal{E}^{L''} + \mathcal{E}^{L''}\{1\}$. We define $K^{L'}_0 = K^{L'}_0(\mathcal{C}_{\mathrm{gr}})$ and $K^{L''}_0 = K^{L''}_0(\mathcal{C}_{\mathrm{gr}})$ as the quotients of the free abelian groups on objects of $\widehat{\mathcal{E}}^{L'}$, $\widehat{\mathcal{E}}^{L''}$, respectively, by all relations $X + Z - Y$, where $X \to Y \to Z \to$ is a distinguished triangle in $D^b(\mathcal{C}_{\mathrm{gr}})$ which is a direct sum of distinguished triangles from $\mathcal{E}^{L'} = \mathcal{E}^{L'}\{1\}$, or from $\mathcal{E}^{L''}, \mathcal{E}^{L''}\{1\}$, respectively.

If X is an object in $\widehat{\mathcal{E}}^L$ (resp., $\widehat{\mathcal{E}}^{L'}, \widehat{\mathcal{E}}^{L''}$), then $[X]_L$ (resp., $[X]_{L'}, [X]_{L''}$) denotes its image in K^L_0 (resp., $K^{L'}_0, K^{L''}_0$). Often we will omit the subscript, and just write $[X]$ for $[X]_L, [X]_{L'}$, etc.

We now concentrate on $\widehat{\mathcal{E}}^L, \widehat{\mathcal{E}}^{L'}$ and $\widehat{\mathcal{E}}^{L''}$; see Remark 3.4 for the right hand versions of the results below. Fix an indeterminate t and define multiplications by t on $K^L_0, K^{L'}_0, K^{L''}_0$ by $t[X]_L = [X[-1]]_L$, $t[X]_{L'} = [X\{-1\}]_{L'}$, $t[X]_{L''} = [X\{-1\}]_{L''}$. In this way, each of $K^L_0, K^{L'}_0, K^{L''}_0$ becomes a $\mathcal{Z}$-module, which is spanned, over $\mathcal{Z}$, by the various elements $[\Delta(\lambda)]_L$, $[\Delta(\lambda)(\ell'(\lambda))]_{L'}$, $[\Delta(\lambda)(\ell'(\lambda))]_{L''}$, respectively ($\lambda \in \Lambda$). In fact, these elements give $\mathcal{Z}$ bases for their respective groups $K_0^L, K_0^{L'}, K_0^{L''}$, as noted in Cor. 3.2.

The categories $\widehat{\mathcal{E}}^R, \widehat{\mathcal{E}}^{R'}, \widehat{\mathcal{E}}^{R''}$, and enriched Grothendieck groups $K_0^R, K_0^{R'}, K_0^{R''}$ are defined by analogy with the "dual" L, L', L'' cases. As before, there are maps $[\ \ ]_R : \mathrm{Ob}(\widehat{\mathcal{E}}^R) \to K_0^R$, $[\ \ ]_{R'} : \mathrm{Ob}(\widehat{\mathcal{E}}^{R'}) \to K_0^{R'}$, and $[\ \ ]_{R''} : \mathrm{Ob}(\widehat{\mathcal{E}}^{R''}) \to K_0^{R''}$. The subscripts R, R', R'', are sometimes omitted. Again, we define $t[X]_R = [X[-1]]_R$ in K_0^R, and $t[X] = [X\{-1\}]$, as before, in $K_0^{R'}$ and $K_0^{R''}$. We put $\nabla'(\mu) = \nabla(\mu)(r'(\mu))$, and again find that $[\nabla'(\mu)]$, $\mu \in \Lambda$ a basis for $K_0^R, K_0^{R'}, K_0^{R''}$, respectively.

For the category $\mathcal{C}_{\rm gr}$, it is convenient to write $\Delta'(\lambda) = \Delta(\lambda)(\ell'(\lambda)) \in \mathrm{Ob}(\mathcal{C}_{\rm gr})$ and $\nabla'(\lambda) = \nabla((r'(\lambda))$. Observe that $\Delta'(\lambda) \in \mathrm{Ob}(\mathcal{E}^{L'})$ and $\nabla'(\lambda) \in \mathrm{Ob}(\mathcal{E}^{R'})$. If $\ell(\lambda) \equiv 0 \bmod 2$, then $\Delta'(\lambda) \in \mathrm{Ob}(\mathcal{E}^{L''})$. If $\ell(\lambda) \equiv 1 \bmod 2$, then $\Delta'(\lambda) \in \mathrm{Ob}(\mathcal{E}^{L''}\{1\})$.

Proposition 3.1. *(a) $M \in \mathrm{Ob}(\mathcal{E}^L)$. Suppose*

$$[M] = \sum_{\mu\in\Lambda} \bar{p}_{M,\mu}[\Delta(\mu)]_L, \quad (p_{M,\mu} \in \mathcal{Z}) \tag{3.1.1}$$

in K_0^L. Then

$$p_{M,\mu}(t) = \sum_{n\in\mathbb{Z}} \dim \mathrm{Hom}^n_{D^b(\mathcal{C})}(M, \nabla(\mu))t^n. \tag{3.1.2}$$

(b) Let $M \in \mathrm{Ob}(\widehat{\mathcal{E}}^{L'})$ (resp., $\widehat{\mathcal{E}}^{L''}$). Suppose

$$[M] = \sum_{\mu\in\Lambda} \bar{p}_{M,\mu}[\Delta'(\mu)], \quad p_{M,\mu} \in \mathcal{Z}) \tag{3.1.3}$$

in $K_0^{L'}$ (resp., $K_0^{L''}$). Then

$$p_{M,\mu}(t) = \sum_{n\in\mathbb{Z}} \dim \mathrm{Hom}^n_{D^b(\mathcal{C}_{\rm gr})}(M, \nabla(\mu)(\ell'(\mu)+n))t^n. \tag{3.1.4}$$

Also, for any $m, n \in \mathbb{Z}$,

$$\mathrm{Hom}^n_{D^b(\mathcal{C}_{\rm gr})}(M, \nabla(\mu)(\ell'(\mu)+m))) = 0 \quad \text{if } m \not\equiv n \bmod 2. \tag{3.1.5}$$

Proof. The proof of (a) is similar to the proof of [9, (3.0.4)]. We leave this to the reader, and we prove (b) in the case of $M \in \mathcal{E}^{L'}$. We first address (3.1.5). The property (3.1.5) for M is obviously true on $\mathcal{E}_0^{L'}$ by Lemma 2.3, and follows for any $\mathcal{E}_i^{L'}$ $(i > 0)$ from its inductive construction and the long exact sequences for $\mathrm{Hom}^\bullet_{D^b(\mathcal{C}_{\rm gr})}(-, \nabla'(\mu)(m)$ associated to distinguished triangles in $D^b(\mathcal{C}_{\rm gr})$ $(i \geq 0)$.

It follows now from (3.1.5) that $p_{M,\mu}$ is additive in M over distinguished triangles in $\mathcal{E}^{L'}$. Hence, with $p_{M,\mu}$ as in (3.1.4), the additivity implies there is a morphism

$$K_0^{L'} \to \mathcal{Z}, \quad [M]_{L'} \mapsto p_{M,\mu}, \tag{3.1.6}$$

of abelian groups.

The additivity implies that $[M]_{L'} \mapsto p_{M,\mu}$, with $p_{M,\mu}$ given as in 3.1.4), is a $\mathbb{Z}$-homomorphism on $K_0^{L'}$. For $m \in \mathbb{Z}$ and $[M]_{L'} = t^{-m}[\Delta'(\mu)]_{L'}$, we have

$$\begin{aligned} p_{M,\mu}(t) &= \sum_{n\in\mathbb{Z}} \dim \operatorname{Hom}^{n-m}_{D^b(\mathcal{C}_{\rm gr})}(\Delta(\mu)(m+\ell'(\mu)), \nabla(\mu)(\ell'(\mu)+n)t^n \\ &= t^m. \end{aligned}$$

Then (3.1.6) implies that (3.1.4) holds when (3.1.3) does. This proves (b) for $M \in \operatorname{Ob}(\mathcal{E}^{L'})$.

A similar argument establishes (b) when $M \in \operatorname{Ob}(\widehat{\mathcal{E}}^{L''})$. □

Corollary 3.2. *The $\mathcal{Z}$-module K_0^L (resp., $K_0^{L'}, K_0^{L''}$) is a free with bases $\{[\Delta(\nu)]_L\}_{\nu\in\Lambda}$ (resp., $\{[\Delta'(\nu)]_{L'}\}_{\nu\in\Lambda}$, $\{[\Delta'(\nu)]_{L''}\}_{\nu\in\Lambda}$). The natural forgetful functor $\widehat{\mathcal{E}}^{L''} \to \widehat{\mathcal{E}}^L$ and the inclusion $\widehat{\mathcal{E}}^{L''} \subseteq \widehat{\mathcal{E}}^{L'}$ induce isomorphisms $K_0^{L''} \cong K_0^L$ and $K_0^{L''} \cong K_0^{L'}$ of $\mathcal{Z}$-modules.*

We have the following corollary of the above proof.

Corollary 3.3. *For a fixed $\mu \in \Lambda$, there is a well defined $\mathcal{Z}$-linear $K_0^L \to \mathcal{Z}$ (resp., $K_0^{L'} \to \mathcal{Z}$, $K_0^{L''} \to \mathcal{Z}$) satisfying $[M]_L \mapsto \overline{p}_{M,\mu}$ (resp., $[M]_{L'} \mapsto \overline{p}_{M,\mu}$, $[M]_{L''} \mapsto \overline{p}_{M,\mu}$), where $p_{M,\mu}$ is as given (in each case) as in the statement of the above proposition. Also,*

$$p_{M,\mu}(t) = \sum_{n\in\mathbb{Z}} \dim \operatorname{Hom}^n_{D^n(\mathcal{C})}(M, \nabla(\mu))t^n.$$

(In other words, we can the ungraded Hom^n *in each of the possible three cases: $M \in \operatorname{Ob}(\widehat{\mathcal{E}}^L)$, $M \in \operatorname{Ob}(\widehat{\mathcal{E}}^{L'})$, $M \in \operatorname{Ob}(\widehat{\mathcal{E}}^{L''})$.)*

Proof. The result is clear for $M \in \widehat{\mathcal{E}}^L$, so we may take $M \in \widehat{\mathcal{E}}^{L'}$ or $\widehat{\mathcal{E}}^{L''}$ (which is contained in $\widehat{\mathcal{E}}^{L'}$). By (1.0.1), for M in $D^b(\mathcal{C}_{\rm gr})$,

$$\operatorname{Hom}^n_{D^b(\mathcal{C})}(M, \nabla(\mu)) = \operatorname{Hom}^n_{D^b(\mathcal{C}_{\rm gr})}(M, \nabla(\mu)) \oplus \operatorname{Hom}^n_{D^b(\mathcal{C}_{\rm gr})}(M, \nabla(\mu)(1)).$$

The result now follows from (3.1.5). □

Remark 3.4. The "right" analog of Proposition 3.1 holds, with $[\nabla'(\mu)]$ replacing $[\Delta'(\mu)]$ and $\operatorname{Hom}^n_{\mathcal{D}}(\Delta(\mu)(n+r'(\mu), M)$ replacing $\operatorname{Hom}^n_{\mathcal{D}}(M, \nabla(\mu)(n+\ell'(\mu))$. However, the right-hand version of $p_{M,\mu}$, which we will call $p_{\mu,M}$, should be introduced in the right-hand version of the proposition by $[M] = \sum p_{\mu,M}[\nabla'(\mu)]$, without any "bar" over $p_{\mu,M}$. Of course, each right enriched Grothendieck group replaces its left analog.

The right analogs of the two corollaries requires similar replacements. All the proofs, of the proposition and corollaries, are entirely analogous in the right-hand case to the left-hand versions, and are omitted.

We can now state

Theorem 3.5. *(a) Suppose* $\overline{\ell(\lambda)} = \overline{r(\lambda)}$ *for all* $\lambda \in \Lambda$ *(i. e.,* $\ell - r$ *has a constant value modulo 2). Then the Laurent polynomial*

$$\langle M, N\rangle = \sum_{n\in\mathbb{Z}} \dim\, \mathrm{Hom}^n_{D^b(\mathcal{C})}(M, N)t^n,$$

defined for $M \in \mathrm{Ob}(\widehat{\mathcal{E}}^L)$*,* $N \in \mathrm{Ob}(\widehat{\mathcal{E}}^R)$*, depends only on* $[M] \in K_0^L$*,* $[N] \in K_0^R$*.*

(b) Similarly, if $\overline{\ell'(\lambda)} = \overline{r'(\lambda)}$ *for all* $\lambda \in \Lambda$*, then the "same" Laurent polynomial (using* $D^b(\mathcal{C})$*)*

$$\langle M', N'\rangle = \sum_{n\in\mathbb{Z}} \dim \mathrm{Hom}^n_{D^b(\mathcal{C})}(M', N')t^n,$$

defined for $M' \in \mathrm{Ob}(\widehat{\mathcal{E}}^{L'})$*,* $N' \in \mathrm{Ob}(\widehat{\mathcal{E}}^{R'})$*, depends only on* $[M'] \in K_0^{L'}$*,* $[N'] \in K_0^{R'}$*.*

Proof. First, assume $\overline{\ell(\lambda)} = \overline{r(\lambda)}$ for all $\lambda \in \Lambda$. If we replace $r(\lambda)$ with $\ell(\lambda)$, then $\widehat{\mathcal{E}}^R$ remains the same, since $\ell(\lambda) \equiv r(\lambda) + \epsilon$ modulo 2, $\epsilon = 0$ or 1. This reduces (a) to a result of [9]. We may also reduce (b) to the case $\ell'(\lambda) = r'(\lambda)$ by replacing $\widehat{\mathcal{E}}^{R'}$, N' with $\widehat{\mathcal{E}}^{R'}(\epsilon')$, $N'(\epsilon)$, where $\epsilon' = 0$ or 1, $\epsilon' \equiv \ell'(\lambda) - r'(\lambda)$. □

Lemma 3.6. *Assume* $\overline{\ell'(\lambda)} = \overline{r'(\lambda)}$ *for all* $\lambda \in \Lambda$*. Let* $M' \in \mathrm{Ob}(\widehat{\mathcal{E}}^{L'})$*,* $N' \in \mathrm{Ob}(\mathcal{E}^{R'})$*. Then, for any* $m, n \in \mathbb{Z}$*,*

$$\mathrm{Hom}^n_{D^b(\mathcal{C}_{\mathrm{gr}})}(M', N'(m)) \neq 0 \implies n \equiv m \pmod 2.$$

Proof. Because of the construction of $\mathcal{E}^{L'}$ as $\bigcup \mathcal{E}_i^{L'}$ and that of $\mathcal{E}^{R'}$ as $\bigcup \mathcal{E}_i^{R'}$, the long exact sequence of $\mathrm{Hom}^\bullet_{D^b(\mathcal{C}_{\mathrm{gr}})}$ shows that it suffices to assume that $M' = \Delta'(\lambda)(t)[t]$ and $N' = \nabla'(\mu)(s)[s]$ for integers s, t. Now apply Lemma 2.3. □

As a corollary, we have, for $n \in \mathbb{Z}$, $M' \in \mathrm{Ob}(\widehat{\mathcal{E}}^{L'})$, $N' \in \mathrm{Ob}(\widehat{\mathcal{E}}^{L'})$, and still assuming $\overline{\ell'(\lambda)} = \overline{r'(\lambda)}$ for all $\lambda \in \Lambda$,

$$\mathrm{Hom}^n_{D^b(\mathcal{C})}(M', N') \cong \begin{cases} \mathrm{Hom}^n_{D^b(\mathcal{C}_{\mathrm{gr}})}(M', N') & \text{if } n \equiv 0 \pmod 2, \\ \mathrm{Hom}^n_{D^b(\mathcal{C}_{\mathrm{gr}})}(M', N'(1)) & \text{if } n \not\equiv 0 \pmod 2. \end{cases}$$

In either case, dim $\mathrm{Hom}^n_{D^b(\mathcal{C})}(M', N')$ is additive over distinguished triangles in $\mathcal{E}^{L'} = \widehat{\mathcal{E}}^{L'}$ or $\mathcal{E}^{R'} = \widehat{\mathcal{E}}^{R'}$. That is, dim $\mathrm{Hom}^n_{D^b(\mathcal{C})}(M', N')$ depends only on the classes $[M']_{L'} \in K_0^{L'}$, $[N']_{R'} \in K_0^{R'}$.

Definition 3.7. Under the hypotheses of Theorem 3.5 (part (a) or part (b) depending on circumstances), we set $\langle [M]_L, [N]_L\rangle = \langle M, N\rangle$ (resp., $\langle [M]_{L'}, [N]_{L'}\rangle = \langle M, N\rangle$)) for $M \in \mathrm{Ob}(\mathcal{E}^L)$, $N \in \mathrm{Ob}(\mathcal{E}^R)$ (resp. $M \in \mathrm{Ob}(\widehat{\mathcal{E}}^{L'})$, $N \in \mathrm{Ob}(\widehat{\mathcal{E}}^{R'})$).

Given $M \in \mathcal{E}^L$ (resp., $\mathcal{E}^{L'}$, $\mathcal{E}^{L''}$), we will often simply write $[M]$ for $[M]_L$ (resp., $[M]_{L'}$, $[M]_{L''}$). A similar notation will be observed for $\mathcal{E}^R$, $\mathcal{E}^{R'}$ and $\mathcal{E}^{R''}$.

Proposition 3.8. *Under the hypotheses Theorem 3.5(a) (resp., (b)), the form* $\langle\ \ ,\ \ \rangle : K_0^L \times K_0^R \to \mathcal{Z}$ *(resp.,* $\langle\ \ ,\ \ \rangle : K_0^{L'} \times K_0^{R'} \to \mathcal{Z}$*) are* $\mathcal{Z}$*-sesquilinear, in the sense that they are bi-additive and satisfy*

$$\langle f(t^{-1}[M],[N]\rangle = \langle [M], f(t)[N]\rangle = f(t)\langle [M],[N]\rangle,$$

for all $f(t) \in \mathcal{Z}$ *and* $M \in \mathrm{Ob}(\widehat{\mathcal{E}}^{L'})$, $N \in \mathrm{Ob}(\widehat{\mathcal{E}}^{R'})$.

Proof. The result is clear, since

$$\mathrm{Hom}^n_{D^b(\mathcal{C})}(M,N) \cong \mathrm{Hom}_{D^b(\mathcal{C})}(M[-n]),N) \cong \mathrm{Hom}_{D^b(\mathcal{C})}(M,N[n])$$

for all $n \in \mathbb{Z}$. The bi-additivity is an easy consequence of the definition, since $\langle M,N\rangle$ is obviously additive over direct sum decompositions of M or N. □

Corollary 3.9. *Under the hypotheses of Theorem 3.5, we have*

$$\langle [M],[N]\rangle = \sum_{\mu\in\Lambda} p_{M,\mu}(t)p_{\mu,M}(t),$$

for $M \in \mathrm{Ob}(\widehat{\mathcal{E}}^L)$, $N \in \mathrm{Ob}(\widehat{\mathcal{E}}^R)$, *or* $M \in \mathrm{Ob}(\widehat{\mathcal{E}}^{L'})$, $N \in \mathrm{Ob}(\widehat{\mathcal{E}}^{R'})$.

Proof. The result is clear from the definition of $p_{M,\mu}$ and $p_{\mu,N}$, together with the sesquilinearity of $\langle\ \ ,\ \ \rangle$. □

Another useful way to say the same thing is the following version.

Corollary 3.10. *Under the hypotheses of the theorem, let* $n \in \mathbb{Z}$. *Then*

$$\dim\,\mathrm{Hom}^n_{D^b(\mathcal{C})}(M,N) = \sum_{r\in Z} \dim\,\mathrm{Hom}^r_{D^b(\mathcal{C})}(M,\nabla(\mu))\dim\,\mathrm{Hom}^{n-r}_{D^b(\mathcal{C})}(\Delta(\mu),N),$$

for each $M \in \mathrm{Ob}(\widehat{\mathcal{E}}^L)$, $N \in \mathrm{Ob}(\widehat{\mathcal{E}}^R)$, *or* $M \in \mathrm{Ob}(\widehat{\mathcal{E}}^{L'})$, $N \in \mathrm{Ob}(\widehat{\mathcal{E}}^{R'})$. *The sum on the right-hand side is always finite.*

The first part of the following "recognition" theorem essentially restates an old result [9, (2.4)], but the second part is new.

Theorem 3.11. *(a) Assume that* A *is a* QHA. *Let* $X \in D^b(\mathcal{C})$. *Then* $X \in \mathrm{Ob}(\mathcal{E}^L)$, *if and only if for each* $n \in \mathbb{Z}$ *and* $\nu \in \Lambda$,

$$\mathrm{Hom}^n_{D^b(\mathcal{C})}(X,\nabla(\nu)) \neq 0 \implies n \equiv \ell(\nu) \pmod 2.$$

Similarly, $X \in \mathrm{Ob}(\mathcal{E}^R)$, *if and only if for each* $n \in \mathbb{Z}$ *and* $\nu \in \Lambda$,

$$\mathrm{Hom}^n_{D^b(\mathcal{C})}(\Delta(\nu),X) \neq 0 \implies n \equiv r(\nu) \pmod{2}.$$

(b) Now assume that A *is a* $\mathbb{Z}/2$-QHA. *Let* $Y \in D^b(\mathcal{C}_{\rm gr})$. *Then* $Y \in \mathrm{Ob}(\mathcal{E}^{L'})$ *if and only if for each* $m,n \in \mathbb{Z}$ *and* $\nu \in \Lambda$,

$$\mathrm{Hom}^n_{D^b(\mathcal{C}_{\rm gr})}(Y,\nabla(\nu)(m)) \neq 0 \Rightarrow n \equiv m + \ell'(\nu) \pmod 2.$$

Similarly, $Y \in \mathrm{Ob}(\mathcal{E}^{R'})$ *if and only if for each* $m, n \in \mathbb{Z}$ *and* $\nu \in \Lambda$*,*

$$\mathrm{Hom}^n_{D^b(\mathcal{C}_{\mathrm{gr}})}(\Delta(\nu)(m), Y) \neq 0 \Rightarrow n \equiv m + r'(\nu) \pmod 2.$$

Proof. Since part (a) is essentially known, we prove only part (b). (In any case, the proof we give for part (b) works easily for part (a).)

Let Γ be the ideal of weights $\nu \in \Lambda$ such that $\nu \leq \omega$ for some ω with $L(\omega)$ a composition factor of $H^\bullet(Y)$.

By (1.0.1), the inclusion $D^b(\mathcal{C}_{\mathrm{gr}}[\Gamma]) \to D^b(\mathcal{C}_{\mathrm{gr}})$ is a full embedding, since the corresponding inclusion $D^b(\mathcal{C}[\Gamma]) \to D^b(\mathcal{C})$ is a full embedding. Here $\mathcal{C}_{\mathrm{gr}}[\Gamma]$ is the category of all graded submodules of $\mathcal{C}_{\mathrm{gr}}$ with composition factors of the form $L(\nu)(s)$, $\nu \in \Gamma$, and $s \in \mathbb{Z}/2$.

A standard argument, inductively building Y using distinguished triangles, shows Y is isomorphic to an object in $D^b(\mathcal{C}_{\mathrm{gr}}[\Gamma])$.

Let $\gamma \in \Gamma$ be maximal, and let $P_\Gamma(\gamma) = \Delta(\gamma)$ be the projective cover in $\mathcal{C}_{\mathrm{gr}}[\Gamma]$ of $L(\gamma) = L(\gamma)_0$. Put

$$E = \bigoplus_{\substack{m \in \mathbb{Z}/2 \\ n \in \mathbb{Z}}} \Delta(\gamma)(m)[-n]^{[H^n(r):L(\gamma)(m)]}.$$

This is regarded as a complex of projective modules in $\mathcal{C}_{\mathrm{gr}}$ with zero differential. Observe that

$$\begin{aligned}[H^n(\gamma) : L(\gamma)(m)] &= \dim \mathrm{Hom}^{-n}_{D^b(\mathcal{C}_{\mathrm{gr}})}(Y, \nabla(\gamma)(m)) \\ &= \dim \mathrm{Hom}^{-n}_{D^b(\mathcal{C}_{\mathrm{gr}})}(E, \nabla(\gamma)(m)),\end{aligned}$$

by construction. The Hom's in each case may also be taken in the underlying homotopy category of $\mathbb{Z}/2$-graded complexes.

We can easily give a complex map $E \to Y$ which induces an isomorphism

$$\mathrm{Hom}^{-n}_{D^b(\mathcal{C}_{\mathrm{gr}})}(Y, \nabla(\gamma)(m)) \xrightarrow{\sim} \mathrm{Hom}^{-n}_{D^b(\mathcal{C}_{\mathrm{gr}})}(E, \nabla(\gamma)),$$

for each $n \in \mathbb{Z}$ and $m \in \mathbb{Z}/2$. Indeed, we just filter the n-cocycles $Z^n(Y)$ by a composition series in $\mathcal{C}_{\mathrm{gr}}$ that passes through the n-coboundaries $B^n(Y)$

$$B^n(Y) = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_a = Z^n(Y).$$

There will be $[H^n(Y) : L(\gamma)(m)]$ terms F_{i_s}/F_{i-1_s} isomorphic to $\Delta(\gamma)(m)$. We choose a summand $D_s \cong \Delta(\gamma)(m)$ of the degree n-cochains $C^n(E) = Z^n(E)$, and a map $D_s \to Z^n(Y) \subseteq C^n(Y)$ with image in F_{i_s}, subjective to $F_{i_s}/F_{0_{s-1}}$. The various D_s, $s = 1, \ldots, [H^n(Y) : L(\gamma)(m)]$ are taken to be the full set of $[H^n(Y) : L(\gamma)(m)]$ distinct summands of $C^n(E)$ isomorphic of $\Delta(\gamma)(m)$.

Adding all the resulting maps over all $n \in \mathbb{Z}$ and $m \in \mathbb{Z}/2$ we get a cochain map $E \to Y$ inducing an injection

$$\sum \mathrm{Hom}^{-n}_{D^b(\mathcal{C}_{\mathrm{gr}})}(Y, \nabla(\gamma)(m)) \to \sum \mathrm{Hom}^{-n}_{D^b(\mathcal{C}_{\mathrm{gr}})}(E, \nabla(\gamma)),$$

which must also be an isomorphism.

Now, form a distinguished triangle $E \to Y \to Y' \to$ in $D^b(\mathcal{C}_{\rm gr})$. We have

$$\mathrm{Hom}^n_{D^b(\mathcal{C}_{\rm gr})}(Y', \nabla(\gamma)(m)) = 0,$$

for all $n \in \mathbb{Z}$ and $m \in \mathbb{Z}/2$. Also, for $\mu \in \Gamma$ with $\mu \neq \gamma$,

$$\mathrm{Hom}^n_{D^b(\mathcal{C}_{\rm gr})}(Y', \nabla(\mu)(m)) \cong \mathrm{Hom}^n_{D^b(\mathcal{C}_{\rm gr})}(Y, \nabla(\mu)(m)),$$

for all $n \in \mathbb{Z}$ and $m \in \mathbb{Z}/2$. The theorem now follows for $\mathcal{E}^L$ easily by induction (on, say,

$$\sum_{\substack{\lambda \in \Lambda \\ n \in \mathbb{Z}, m \in \mathbb{Z}/2}} \dim \mathrm{Hom}^n_{D^b(\mathcal{C}_{\rm gr})}(Y, \nabla(\lambda)(m)), \quad Y \in \mathcal{E}^L).$$

The statement for $\mathcal{E}^R$ is obtained in a similar, but dual, manner, and we omit further details. □

4. $\mathbb{Z}/2$-based Kazhdan-Lusztig theories

We have previously introduced the notion of a Kazhdan-Lusztig theory for a HWC $\mathcal{C} = \mathcal{C}(A)$ in [9]. It requires a length function $\ell : \Lambda \to \mathbb{Z}$ (which we take here to also be $r(\lambda)$, $\lambda \in \Lambda$). Explicitly, $\mathcal{C}$ has a Kazhdan-Lusztig theory with respect to ℓ provided, for $n \geq 0$ in $\mathbb{Z}$ and $\lambda, \mu \in \Lambda$, that

$$\mathrm{Ext}^n_{\mathcal{C}}(\Delta(\lambda), L(\mu)) \neq 0 \implies n \equiv \ell(\lambda) - \ell(\mu) \pmod 2, \tag{4.0.1}$$

and also

$$\mathrm{Ext}^n_{\mathcal{C}}(L(\mu), \nabla(\lambda)) \neq 0 \implies n \equiv \ell(\lambda) - \ell(\mu) \pmod 2. \tag{4.0.2}$$

When $\mathcal{C}$ has a strong duality in the sense of [7], it is sufficient that one of the two conditions above hold in order for $\mathcal{C}$ to have a Kazhdan-Lusztig theory with respect to ℓ.

As a consequence,

$$\dim \mathrm{Ext}^n_{\mathcal{C}}(L(\mu), L(\nu)) = \sum_{\substack{r \in \mathbb{Z} \\ \lambda \in \Lambda}} \dim \mathrm{Ext}^r_{\mathcal{C}}(L(\mu), \nabla(\lambda)) \dim \mathrm{Ext}^{n-r}_{\mathcal{C}}(\Delta(\lambda), L(\nu)), \tag{4.0.3}$$

when $\mathcal{C}$ has a Kazhdan-Lusztig theory [9, Thm. 3.5]. As shown in this reference this expresses the left-hand side of (4.0.3) in terms of Kazhdan-Lusztig polynomials.

We will describe an alternative theory which leads to such a formula.

Definition 4.1. A $\mathbb{Z}/2$-HWC $\mathcal{C} = \mathcal{C}(A)$ with weight poset Λ has a $\mathbb{Z}/2$-*based Kazhdan-Lusztig theory* with respect to a length function $\ell' : \Lambda \to \mathbb{Z}$ provided that, for all $\lambda, \mu \in \Lambda$ and $m, n \in \mathbb{Z}$,

$$\mathrm{Ext}^n_{\mathcal{C}_{\rm gr}}(\Delta(\lambda), L(\mu)(m)) \neq 0 \implies n - m \equiv \ell'(\lambda) - \ell'(\mu) \pmod 2,$$

and

$$\mathrm{Ext}^n_{\mathcal{C}_{\rm gr}}(L(\mu)(m), \nabla(\lambda)) \neq 0 \implies n - m = \ell'(\lambda) - \ell'(\mu) \pmod 2.$$

Taking $r' := \ell' : \Lambda \to \mathbb{Z}$, the conditions in the above definition imply that

$$L(\mu) \in \mathrm{Ob}(\mathcal{E}^{L'}) \cap \mathrm{Ob}(\mathcal{E}^{R'}), \quad \forall \mu \in \Lambda.$$

This gives, by Corollary 3.10, the formula (4.0.3) from an entirely different starting point.

We saw in [9] that the existence of a Kazhdan-Lustzig theory—even just the $n = 1$ part of the defining property—was sufficient to prove the Lusztig or Kazhdan-Lusztig conjecture in standard cases arising in Lie theory. This fact remains true for the $\mathbb{Z}/2$-based theory, as can be seen from the following result.

Theorem 4.2. *Suppose $\ell' : \Lambda \to \mathbb{Z}$ and, for each $\lambda, \mu \in \Lambda$,*

$$\mathrm{Ext}^1_{\mathcal{C}_{\mathrm{gr}}}(\Delta(\lambda), L(\mu)(m)) \neq 0 \implies m \equiv \ell'(\lambda) - \ell'(\mu) + 1 \ (\mathrm{mod}\ 2),$$

and

$$\mathrm{Ext}^1_{\mathcal{C}_{\mathrm{gr}}}(L(\mu)(m), \nabla(\lambda)) \neq 0 \implies m \equiv \ell'(\lambda) - \ell'(\mu) + 1 \ (\mathrm{mod}\, 2).$$

Then, for $\tau, \nu \in \Lambda$, the natural maps $\mathrm{Ext}^1_{\mathcal{C}}(L(\tau), L(\nu)) \to \mathrm{Ext}^1_{\mathcal{C}}(\Delta(\tau), L(\nu))$, and $\mathrm{Ext}^1_{\mathcal{C}}(L(\tau), L(\nu)) \to \mathrm{Ext}^1_{\mathcal{C}}(L(\tau), \nabla(\nu))$ are surjective.

Proof. If $\mathrm{Ext}^1_{\mathcal{C}}(\Delta(\tau), L(\nu)) \neq 0$, then $\mathrm{Ext}^1_{\mathcal{C}_{\mathrm{gr}}}(\Delta(\tau), L(\nu)(m)) \neq 0$ for some m. In fact, such an m is unique in $\mathbb{Z}/2$ satisfying $m \equiv \ell'(\nu) - \ell(\nu) + 1$. We also know $\tau < \nu$, since $\mathcal{C}$ is a HWC. Let E be any graded extension of $\Delta(\lambda)$ by $L(\nu)(m)$, and let E' be the smallest graded homomorphic image of E which contains $L(\nu)(m)$ as a submodule.

Since $\mathrm{soc}(E')$ is graded, $\mathrm{soc}(E') = L(\nu)(m)$. Also, the head of E' is $L(\tau)$. We claim E' has a composition series of length two.

If not, let ω be the largest element of Λ for which $L(\omega)(s)$ is a composition factor of E', for some $s \in \mathbb{Z}/2$, $\omega \neq \tau$, $\omega \in \nu$. Then there are graded homomorphisms

$$\Delta(\omega)(s) \to E'/L(\nu)(m),$$

$$\mathrm{rad}\, E' \to \nabla(\omega)(s),$$

resulting in nontrivial elements of $\mathrm{Ext}^1_{\mathcal{C}_{\mathrm{gr}}}(\Delta(\omega)(s), L(\omega)(m))$ and $\mathrm{Ext}^1_{\mathcal{C}_{\mathrm{gr}}}(L(\tau), \nabla(\omega)(s))$, respectively. It follows from the hypothesis of the theorem that

$$\begin{aligned} m - s \ &\equiv \ \ell'(\omega) - \ell'(\nu) + 1 \ (\mathrm{mod}\, 2); \\ &\equiv \ \ell'(\tau) - \ell'(\omega) + 1 \ (\mathrm{mod}\, 2). \end{aligned}$$

Adding these two congruences gives

$$m \equiv \ell'(\tau) - \ell'(\nu) \quad \mathrm{mod}\, 2,$$

contradicting the previously derived congruence on m.

This proves the claim, and it follows that

$$\mathrm{Ext}^1_{\mathcal{C}_{\mathrm{gr}}}(L(\tau), L(\nu)(m)) \to \mathrm{Ext}^1_{\mathcal{C}_{\mathrm{gr}}}(\Delta(\tau), L(\nu)(m))$$

is surjective. Since the right-hand Ext^1 group is zero if m is replaced by $m+1$ (mod 2), the map of ungraded Ext^1 groups between the same modules is surjective, as claimed in the theorem.

Similarly, one can prove the surjectivity of

$$\mathrm{Ext}^1_{\mathcal{C}}(L(\tau), L(\nu)) \to \mathrm{Ext}^1_{\mathcal{C}}(L(\tau), \Delta(\omega)),$$

for each $\tau, \nu \in \Lambda$. The details are similar, but with dual arguments, and are left to the reader. □

Remarks 4.3. (a) As a consequence, *the existence of a $\mathbb{Z}/2$-based Kazhdan-Lusztig theory implies the validity of the Lusztig or Kazhdan-Lusztig conjecture* in the standard cases where such a conjecture is stated, giving a formula for characters of irreducible modules in terms of characters of standard modules and Kazhdan-Lusztig polynomials. This is a consequence of [9, (5.4)] together with the above theorem.

(b) In many of the above cases in which a valid Lusztig or Kazhdan conjecture is known, the existence of a Koszul grading is also known. Together with the existence of a Kazhdan-Lusztig theory in our previous sense [9], this implies the existence of a Koszul grading for which, for $n, m \in \mathbb{Z}$ and $\lambda, \mu \in \Lambda$,

$$\mathrm{Ext}^1_{\mathcal{C}_{\mathrm{gr}}}(\Delta(\lambda), L(\mu)(m)) \neq 0 \implies n = m,$$

and

$$\mathrm{Ext}^1_{\mathcal{C}_{\mathrm{gr}}}(L(\mu)(-m), \nabla(\lambda)) \neq 0 \implies n = m.$$

In particular, *there is a $\mathbb{Z}/2$-based Kazhdan-Lusztig theory in these cases* (even a $\mathbb{Z}/2$-graded Kazhdan-Lusztig theory—see the note at the end of this section for a definition—and a graded Kazhdan-Lusztig theory in the sense of [9]).

(c) However, for abstract HWCs, it is possible to have a $\mathbb{Z}/2$-based Kazhdan-Lusztig theory in our previous sense, or a Koszul grading on the underlying quasihereditary algebra. This is demonstrated by the following example.

Note on terminology. We prefer to reserve the term $\mathbb{Z}/2$-*graded Kazhdan-Lusztig theory* for the case where we have *both* a length function $\ell : \Lambda \to \mathbb{Z}$ for $\mathcal{E}^L, \mathcal{E}^R$ and a length function $\ell' : \Lambda \to \mathbb{Z}$ for $\mathcal{E}^{L'}, \mathcal{E}^{R'}$ such that both the above theories work: $L(\mu) \in \mathrm{Ob}(\mathcal{E}^L) \cap \mathrm{Ob}(\mathcal{E}^R) \cap \mathrm{Ob}(\mathcal{E}^{L'}) \cap \mathrm{Ob}(\mathcal{E}^{R'}) = \mathrm{Ob}(\mathcal{E}^{L''}) \cap \mathrm{Ob}(\mathcal{E}^{R''})$ for all $\mu \in \Lambda$.

5. Kazhdan-Lusztig theory versus $\mathbb{Z}/2$-based Kazhdan-Lusztig theory

In this section, we investigate the relationship between the existence of a Kazhdan-Lusztig theory and the existence of a $\mathbb{Z}/2$-based Kazhdan-Lusztig theory. Let A be a QHA with weight poset Λ, and consider the $\mathbb{Z}/2$-QHA structure induced by a fixed function $\ell : \Lambda \to \mathbb{Z}$ as defined in Example 2.2(d).

Theorem 5.1. *Suppose that $\mathcal{C}$ has a Kazhdan-Lusztig theory with respect to $\ell : \Lambda \to \mathbb{Z}$. Then $\mathcal{C}^{\ell}_{\mathrm{gr}}$ has a $\mathbb{Z}/2$-based Kazhdan-Lusztig theory taking $\ell' \equiv r' = 0$ identically.*

Proof. We have to prove that, under the stated hypothesis, given $n \in \mathbb{N}, m \in \mathbb{Z}$,

$$\mathrm{Ext}^n_{\mathcal{C}^\ell_{\rm gr}}(\Delta(\lambda), L(\mu)(m)) \neq 0 \implies n \equiv m \bmod 2. \tag{5.1.1}$$

and

$$\mathrm{Ext}^n_{\mathcal{C}^\ell_{\rm gr}}(L(\mu)(m), \nabla(\lambda)) \neq 0 \implies n \equiv m \bmod 2. \tag{5.1.2}$$

We will prove (5.1.1), leaving the similar argument for (5.1.2) to the reader.

Fix $\lambda \in \Lambda$, and let

$$\cdots \to P_2 \to P_1 \to P_0 \to \Delta(\lambda) \to 0 \tag{5.1.3}$$

be a minimal projective resolution of $\Delta(\lambda)$ in $\mathcal{C}$. For a given $i \geq 0$, P_i is a direct sum of various projective indecomposable modules $P(\mu)$, say $P_i = \oplus_{\mu\in\Omega_i} P(\mu)^{\oplus c_i(\mu)}$, where $\Omega_i \subseteq \Lambda$ and each $c_i(\mu) > 0$ for $\mu \in \Omega_i$. If $\mu \in \Omega_i$, then $\mathrm{Ext}^i_{\mathcal{C}}(\Delta(\lambda), L(\mu)) \neq 0$. Since $\mathcal{C}$ has a Kazhdan-Lusztig theory with respect to ℓ, it follows that $i \equiv \ell(\lambda) - \ell(\mu) \bmod 2$.

Now we grade P_i by writing

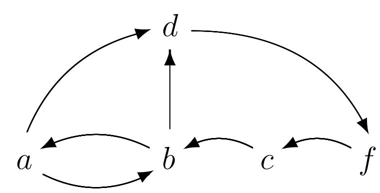
$$P_i = \bigoplus_{\mu\in\Omega_i} P(\mu)(\ell(\mu) - \ell(\lambda))^{\oplus c_i(\mu)}, \tag{5.1.4}$$

where $P(\mu)$ is $\mathbb{Z}/2$-graded with its degree 1 summand contained in rad $P(\mu)$. (That is $P(\mu)$ is given the standard grading with "head in degree 0". In P_i this grading is shifted by $\ell(\mu) - \ell(\lambda)$.) Each idempotent e_μ further decomposes as a sum $e_\mu = e_{\mu,1} + \cdots + e_{\mu,d_\mu}$ of primitive orthogonal idempotents so that $P(\mu) \cong Ae_{\mu,1} \cong \cdots \cong Ae_{\mu,d_\mu}$. Since any morphism $Ae_{\mu,i} \to Ae_{\nu,j}$ is right multiplication by $e_\mu x e_\nu$ for some $x \in A$, it follows that the differential $P_i \to P_{i-1}$ in (5.1.3) defines a morphism in $\mathcal{C}_{\rm gr}$.

Thus, if $\mathrm{Ext}^n_{\mathcal{C}_{\rm gr}}(\Delta(\lambda), L(\mu)(m)) \neq 0$, (1.0.1) and the assumption that $\mathcal{C}$ has a Kazhdan-Lusztig theory implies that $\ell(\lambda) - \ell(\mu) \equiv n \bmod 2$. Thus, all summands of P_n have heads in degree $\ell(\lambda) - \ell(\mu)$. Hence, $m \equiv \ell(\mu) - \ell(\lambda) \bmod 2$. So, $m \equiv n \bmod 2$, as required. ☐

On the other hand, the following example shows that the converse to the above theorem fails.

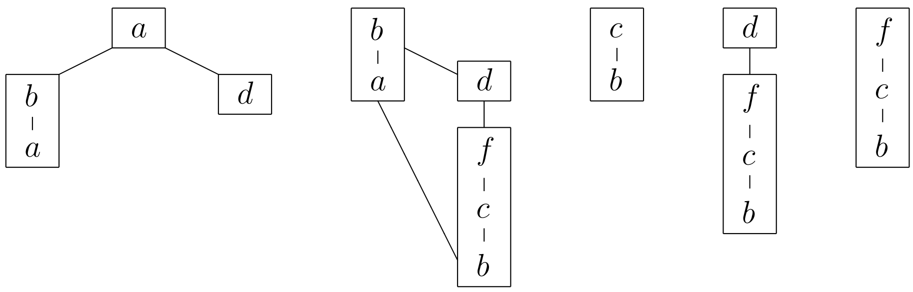
Example 5.2. Consider the quiver



with relations

$$\begin{aligned} e_{fd}e_{da} &= e_{da}e_{ab} = e_{ab}e_{bc} = e_{db}e_{bc} = 0, \\ e_{ba}e_{ab} &= e_{bc}e_{cf}e_{fd}e_{db}. \end{aligned}$$

Here e_{ij} represents the unique arrow in the diagram from i to j, and compositions are read from right to left. The algebra A with such generators, together with idempotents $e_i = e_{ii}$, $i \in \{a, b, c, d, f\}$ satisfying the above relations, together with $e_i e_{ij} = e_{ij} = e_{ij} e_{jj}$ (and $e_i^2 = e_i$), whenever there is an arrow from i to j in the diagram, is 19-dimensional over any field. It is quasi-hereditary with respect to the poset $\Lambda = \{a, b, c, d, f\}$ with $a \leq b$; $b \leq a$, $b \leq d$, $c \leq f$, $d \leq f$. This gives the same standard modules $\Delta(\lambda)$, $\lambda \in \Lambda$, as the simpler, linear order $a \leq b \leq c \leq d \leq f$. Diagrams for the PIM's $P(\lambda)$ are given by the following diagrams. These diagrams are to be viewed as directed graphs, with all edges pointing downward. There is an edge from λ to μ precisely when $\mathrm{Ext}^1_A(L(\lambda), L(\mu))$ is nonzero (in which case it is 1-dimensional) and is realized by a section in the ambient PIM. The nodes are all labeled by various elements λ of Λ, and represent composition factors of the ambient PIM. The rectangles in the diagram represent standard modules, exhibiting the filtration by standard modules of the ambient PIM.



Certainly, A has no Kazhdan-Lusztig theory in the sense of the parity conditions of [9], since the groups $\mathrm{Ext}^1_A(L(a), L(b)), \mathrm{Ext}^1_A(L(a), L(d))$ and $\mathrm{Ext}^1_A(L(b), L(d))$ are all nonzero.

Also, A has no Koszul grading. This is fairly evident from the structure of $P(b)$ above, and we leave further verification of this fact to the reader.

However, we will show that A has a $\mathbb{Z}/2$-based Kazhdan-Lusztig theory with respect to a suitable $\mathbb{Z}/2$-grading and "length" function $\ell' : \Lambda \to \mathbb{Z}$. For the $\mathbb{Z}/2$-grading, we assign e_{ab} and e_{ba} degree 1 in $\mathbb{Z}/2$, and each of the other generators e_{ij} or e_i degree 0. For the length function, we set $\ell'(a) = \ell'(b) = \ell'(f) = 0$, and $\ell'(c) = \ell'(d) = 1$ in $\mathbb{Z}/2$.

Next observe the following minimal projective resolutions, which are also minimal $\mathbb{Z}/2$-graded projective resolutions:

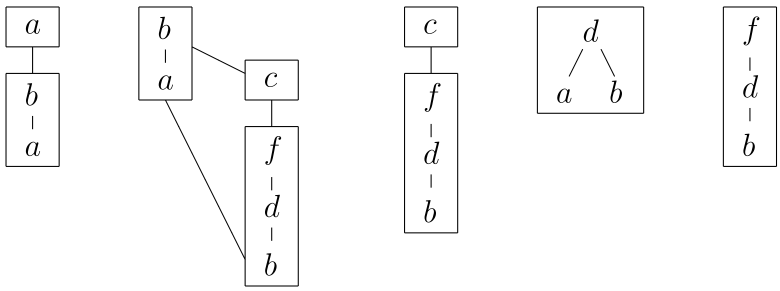
$$\begin{aligned} 0 \to (P(d)(1) \oplus P(f)) \to (P(b)(1) \oplus P(d)) \to P(a) \to \Delta(a) &\to 0, \\ 0 \to P(d) \to P(b) \to \Delta(b) &\to 0, \\ 0 \to P(c) \to \Delta(c) &\to 0, \\ 0 \to P(f) \to P(d) \to \Delta(d) &\to 0, \\ 0 \to P(f) \to \Delta(f) &\to 0. \end{aligned}$$

Computing $\mathrm{Ext}^n_A(\Delta(\lambda, L(\mu)(m))$ with these resolutions, we find, for $n \in \mathbb{Z}$, $m \in \mathbb{Z}/2$ and $\lambda, \mu \in \Lambda$,

$$\mathrm{Ext}^n_{\mathcal{C}_{\mathrm{gr}}}(\Delta(\lambda), L(\mu)(m)) \neq 0 \implies n - m \equiv \ell'(\lambda) - \ell'(\mu) \pmod 2.$$

Here $\mathcal{C}_{\mathrm{gr}}$ denotes the category of $\mathbb{Z}/2$-graded A-modules.

Next, we consider the dual condition involving costandard modules. Using linear duality, this amounts to the above standard module condition, but with A replaced by A^{op}. The PIM's for A^{op} are found to be



"Quiver" generators e^{op}_{ij} can be found for A^{op} by taking $e^{\mathrm{op}}_{ij} \in \mathrm{End}_A({}_AA, {}_AA)$ to be a graded map from $P(i)$ to $P(j)$ with nonzero image in $\mathrm{rad}\, P(i)/\mathrm{rad}^2 P(0)$ if such a nonzero map exists. The algebra $A^{\mathrm{op}} \cong \mathrm{End}_A({}_AA, {}_AA)$ has a natural $\mathbb{Z}/2$-graded structure, and we find that e^{op}_{ij} has degree 1 in $\mathbb{Z}/2$ for $i = a$, $j = b$, or $i = b$, $j = a$, and has degree 0 otherwise.

Minimal graded projective resolutions of the standard modules $\Delta^{\mathrm{op}}(\lambda)$ are given as follows:

$$\begin{aligned} 0 \to P^{\mathrm{op}}(c)(1) \to P^{\mathrm{op}}(b)(1) \to P^{\mathrm{op}}(a) \to \Delta^{\mathrm{op}}(a) &\to 0, \\ 0 \to P^{\mathrm{op}}(c) \to P^{\mathrm{op}}(b) \to \Delta^{\mathrm{op}}(b) &\to 0, \\ 0 \to P^{\mathrm{op}}(f) \to P^{\mathrm{op}}(c) \to \Delta^{\mathrm{op}}(c) &\to 0, \\ 0 \to P^{\mathrm{op}}(d) \to \Delta^{\mathrm{op}}(d) &\to 0, \\ 0 \to P^{\mathrm{op}}(f) \to \Delta^{\mathrm{op}}(f) &\to 0. \end{aligned}$$

Computing with these resolutions, we find, for $n \in \mathbb{Z}$, $m \in \mathbb{Z}/2$, and $\lambda, \mu \in \Lambda$,

$$\mathrm{Ext}^n_{\mathcal{C}_{\mathrm{gr}}}(\Delta^{\mathrm{op}}(\lambda), L^{\mathrm{op}}(\mu)(m)) \neq 0 \implies n - m \equiv \ell'(\lambda) - \ell'(\mu) \pmod 2.$$

Equivalently,

$$\mathrm{Ext}^n_{\mathcal{C}_{\mathrm{gr}}}(L(\mu)(m), \nabla(\lambda)) \neq 0 \implies n - m \equiv \ell'(\lambda) - \ell'(\mu) \pmod 2.$$

This completes our proof that $\mathcal{C} = \mathcal{C}(A)$ has a $\mathbb{Z}/2$-based Kazhdan-Lusztig theory, though it has neither a Kazhdan-Lusztig theory in the sense of [9] or a graded Koszul structure.

6. Homological duals

If A is a finite dimensional algebra over the field k, let $L_1, \ldots, L_n$ be the distinct (up to isomorphism) irreducible A-modules, and put

$$L \equiv L(A) := \bigoplus_{i=1}^{n} L_i \tag{6.0.1}$$

and

$$A^! := \operatorname{Ext}^\bullet_A(L, L), \tag{6.0.2}$$

the *homological dual* of A. If $\mathcal{C} \cong A$-mod, then the homological dual of $\mathcal{C}$ is defined to be $\mathcal{C}^! = A^!$-mod. This category is unique up to Morita equivalence.

In [11], it proved that if $\mathcal{C}$ is a highest weight category having a Kazhdan-Lusztig theory, then the homological dual $\mathcal{C}^!$ is also a HWC. The main ingredient in the proof is the formula (4.0.3), which, as we have remarked following (4.1), is available here. The surjectivity [9, (4.3)] is also used, but is a consequence of (4.0.3). Finally, there is an inductive derived category description of objects satisfying the even-odd vanishing properties used in the proof, but there is a completely parallel description here, from the constructions in section 3. With this briefly sketched proof, we claim the following $\mathbb{Z}/2$-graded version of this result on the homological dual.

Theorem 6.1. *Assume that $\mathcal{C} = \mathcal{C}(A)$ is a $\mathbb{Z}/2$-HWC with weight poset Λ which has a $\mathbb{Z}/2$-based Kazhdan-Lusztig theory with respect to a length function $\ell : \Lambda \to \mathbb{Z}$. Then $\mathcal{C}^!$ is a HWC.*

7. Kazhdan-Lusztig theory and Broué's conjecture

One motivation for studying $\mathbb{Z}/2$-based Kazhdan-Lusztig theories is the possibility that such theories might be easier to verify for algebraic categories involving singular weights. The idea is that exact functors, such as translation to a wall or facet, might more easily be found to preserve a $\mathbb{Z}/2$-grading than the parity in a length function. This philosophy has, so far, not been substantiated, but has lead, nevertheless, to the discovery of unanticipated examples in the singular case. These examples have a length function satisfying the original formulation of a Kazhdan-Lusztig theory [9], though it is doubtful the authors would have had the courage to investigate them without the $\mathbb{Z}/2$-based theory in hand.

To illustrate the approach, we begin by proving a special case of the Lusztig (and James) conjecture, using only symmetric group theory existing prior to [6]: The (new) result is that all $L(\lambda)$ for $SL_p(k)$ which have regular high weight $\lambda \leq p\varpi_1$ satisfy the Lusztig character formula (LCF). From the classical viewpoint of [19] (say), the original conjecture is noted to be true only for A_3 and in all rank 2 cases. Indeed, we will not settle the conjecture for all regular weights of SL_p in the Jantzen region, but our point of view will be to focus on a smaller, but non-trivial collection of regular weights.

To sketch the proof, the modular theory of the symmetric group $\mathfrak{S}_p$ ($p > 2$) of degree p gives a resolution of the Specht module S_{1^p} by Young modules Y_λ (the indecomposable components of permutation modules associated with all partitions $\lambda \vdash p$):

(7.0.1) $$0 \to S_{1^p} \to Y_{1^p} \to Y_{2,1^{p-2}} \to \cdots \to Y_{p-1,1} \to Y_p \to 0.$$

This resolution comes from Brauer's defect 1 theory and its subsequent interpretation by James, but its resemblance to the Solomon resolution [28] (and, more recently, [15], [21], and [24]) is striking, Moreover, all images of differentials in this resolution are themselves Specht modules, associated to the partition of the containing Young module. In this way, we obtain resolutions by Young modules of all Specht modules in the principal block of $\mathfrak{S}_p$. The contravariant version of Schur-Weyl duality (see [24], [13], or the contravariant equivalence [25, (5.0.3)]) gives a minimal projective resolutions of all $\Delta(\varpi)$ lying in the principal block of the Schur algebra $S(p,p)$ (so $\varpi \leq p\varpi_1$). Visibly, the parity conditions for a Kazhdan-Lusztig theory in the sense of [CPS17] hold, with length function equal to the number of parts of a partition. The results [9] thus imply that the (LCF) holds for these weights (with the length function assigned by Lusztig)!

We can pass to a similar resolution as (7.0.1) of Specht modules by Young modules for the direct product $E = \mathfrak{S}_p \times \cdots \times \mathfrak{S}_p$ ($n < p$ factors). And, though Specht and Young modules are not officially defined for the group $N = \mathfrak{S}_p \wr \mathfrak{S}_n = E \rtimes \mathfrak{S}_n = N_{\mathfrak{S}_{np}}(E)$, we at least get resolutions of the modules $S_\lambda \uparrow_E^N$ by modules $Y_\lambda \uparrow_E^N$, which are again direct summands of natural permutation modules. The Broué conjecture guarantees that the derived categories of $\mathfrak{S}_{np}$ and N are equivalent. Moreover, formulations by Broué-Rickard [5] provide, in this equivalence, a correspondence between objects represented by complexes of direct summands of permutation modules. Projective modules of the Schur algebras $S(m, np)$ are obtained by contravariant Schur-Weyl duality from such modules.

We speculate here that, with a deeper study of Chuang-Rouquier [6], it might be possible to reproduce the essential ingredients of the above result for SL_p instead for SL_{np}, $n < p$. In addition, SL_m with $p^2 > m$, should be similarly attackable, and the method could yield a proof, in general, of the James conjecture at pth roots of unity [18]. We have checked that the required Specht module resolutions exist in the case $n = 2$, $p = 3$. Here the symmetric group is $\mathfrak{S}_6$. The two most complicated resolutions are:

$$0 \to S_{2,1^4} \to Y_{2,1^4} \to Y_{s^3} \oplus Y_{3,1^3} \to Y_{3,2,1} \oplus Y_{5,1} \to Y_{3,3} \to 0$$
$$0 \to S_{1^6} \to Y_{1^6} \to Y_{2,1^4} \oplus Y_{3,2,1} \to Y_{2^3} \oplus Y_{3,1^3} \oplus Y_{4,1,1} \to Y_{5,1} \oplus Y_{3,2,1} \to Y_{3,3} \to 0.$$

Resolutions in all cases are given in Table 1 at the of this section. They were obtained with standard symmetric group representation theory, together with the methods of [27]. A key property is that the partitions indexing the Y_λ's in a given degree alternate in parity, if we declare $(5,1)$, $(3,2,1)$, $(2,1^4)$ to be "odd" and all others "even". This discussion provides the first example of *any* kind of Kazhdan-Lusztig theory for a Schur algebra (or any reductive algebraic group) for $p < h$ (the Coxeter number, which m for SL_m). All weights, when $p < h$ are, of course, singular in the sense of alcove geometry.

Specht Resolutions by Young Modules ($p = 3$ Principal Block of S_6)	**Specht Dimensions**
$0 \to S_6 \to Y_6 \to 0$	1
$0 \to S_{5,1} \to Y_{5,1} \to Y_6 \to 0$	5
$0 \to S_{3,3} \to Y_{3,3} \to Y_{5,1} \to Y_6 \to 0$	5
$0 \to S_{4,1,1} \to Y_{4,1,1} \to Y_{5,1} \to Y_6 \to 0$	10
$0 \to S_{3,2,1} \to Y_{3,2,1} \to Y_{4,1,1} \oplus Y_{3,3} \to Y_{5,1} \to Y_6 \to 0$	16
$0 \to S_{3,1^3} \to Y_{3,1^3} \to Y_{3,2,1} \oplus Y_{3,3} \to 0$	10
$0 \to S_{2^3} \to Y_{2^3} \to Y_{3,2,1} \oplus Y_{5,1} \to Y_{3,3} \oplus Y_{4,1} \to Y_{5,1} \to Y_{3,3} \to 0$	5
$0 \to S_{2,1^4} \to Y_{2,1^4} \to Y_{2^3} \oplus Y_{3,1^3} \to Y_{3,2,1} \oplus Y_{5,1} \to Y_{3,3} \to 0$	5
$0 \to S_{1^6} \to Y_{1^6} \to Y_{2,1} \oplus Y_{3,2,1} \to Y_{2^3} \oplus Y_{3,1^3} \oplus Y_{4,1,1} \to Y_{5,1} \to Y_{3,2,1} \to Y_{3,3} \to 0$	1

Young Module Specht Filtrations and Dimensions

S_6	S_6 $S_{5,1}$	$S_{5,1}$ $S_{3,3}$	$S_{5,1}$ $S_{4,1,1}$	$S_{5,1}$ $S_{3,3}$ $S_{4,1,1}$ $S_{3,2,1}$ projective	$S_{4,1,1}$ $S_{3,2,1}$ $S_{3,1^3}$ projective	S_6 $S_{5,1}$ $S_{3,2,1}$ S_{2^3} projective	$S_{3,2,1}$ S_{2^3} $S_{3,1^3}$ $S_{2,1^4}$ projective	$S_{3,3}$ $S_{2,1^4}$ $S_{3,2,1}$ S_{1^6} projective
1	6	10	15	36	36	27	36	27
Y_6	$Y_{5,1}$	$Y_{3,3}$	$Y_{4,1,1}$	$Y_{3,2,1}$	$Y_{3,1^3}$	Y_{2^3}	$Y_{2,1^4}$	Y_{1^6}

Irreducible Modules and Dimensions

				4	6	1	4	1
				$D_{5,1}$	$D_{4,1,1}$	D_6	$D_{3,2,1}$	$D_{3,3}$

References

[1] H. Andersen, J. Jantzen, W. Soergel, *Representations of quantum groups at a pth root of unity and of semisimple groups in characteristic p*, Astérique **220** (1994).

[2] H. Andersen, P. Polo, K. Wen, Representations of quantum algebras, *Invent. math.* **104** (1991), 571–604.

[3] A. Beilinson, J. Bernstein, and P. Deligne, *Fasiceaux pervers,* Astérisque **100** (1982).

[4] A. Beilinson, V. Ginzburg, and W. Soergel, Koszul duality patterns in representation theory, *J. Amer. Math. Soc.* **9** (1996), 473–528.

[5] Broué, M. Rickard equivalences and block theory, Groups '93 Galway/St. Andrews, Vol. 1 (Galway, 1993), 58–79, London Math. Soc. Lecture Note Ser. **211**, Cambridge Univ. Press, Cambridge, 1995.

[6] J. Chuang and R. Rouquier, Derived equivalences for symmetric groups and $\mathrm{sl}_2$2-categorification, *Annals of Math.,* to appear.

[7] E. Cline, B. Parshall, and L. Scott, Duality in highest weight categories, *Comtemp. Math.* **82** (1989), 7-22.

[8] E. Cline, B. Parshall and L. Scott, Integral and graded quasi-hereditary algebras, I, *J. Algebra* **131** (1990), 126–160.

[9] E. Cline, B. Parshall, and L. Scott, Abstract Kazhdan-Lusztig theories, *Tôhoku Math. J.* **45** (1993), 511-534.

[10] E. Cline, B. Parshall and L. Scott, Infinitesimal Kazhdan-Lusztig theories, *Cont. Math.* **139** (1992), 43–73.

[11] E. Cline, B. Parshall, and L. Scott, The homological dual of a highest weight category, *Proc. London Math. Soc.* **68** (1994), 294–316.

[12] E. Cline, B. Parshall, and L. Scott, Simulating perverse sheaves in modular representation theory, *Proc. of Symposia in Pure Math.* **56**(1) (1994), 63–104.

[13] E. Cline, B. Parshall, and L. Scott, Stratifying endomorphism algebras, *Memoirs Amer. Math. Soc.* **591** (1996), 119+iv pages.

[14] E. Cline, B. Parshall, and L. Scott, Graded and non-graded Kazhdan-Lusztig theories, *Camb. Univ. Press* (1997).

[15] E. Cline, B. Parshall, and L. Scott, Generic and q-rational rerpesentation theory, *Publ. Res. Inst. Math. Sci.* (Kyoto) **35(1)** (1999), 31–90.

[16] J. Du and L. Scott, Lusztig conjectures, old and new, I, *J. reine Angew. math.* **455** (1994), 141–182.

[17] D. Happel, *Triangulated Categories in the Representation of Finite Dimensional Algebras* (London Mathematical Society Lecture Note Series) **119** (1988).

[18] G. James, The decomposition matrices of $GL_n(q)$ for $n \leq 10$, *Proc. London Math. Soc.* **60** (1990), 225–265.

[19] J.C. Jantzen, *Representations of algebraic groups, 2nd ed.* American Mathematical Society (2003).

[20] S. I. Kato, On Kazhdan-Lusztig polynomials for affine Weyl groups, *Adv. Math.* **55** (1985), 103–130.

[21] M. Linckelmann and S. Schroll, A two-sided q-analogue of the Coxeter complex, *J. Algebra* **289** (2005), 128–134.

[22] G. Lusztig, Some problems in the representation theory of finite Chevalley groups, Proc. Symp. Pure Math. **37** (1980), 313–317.

[23] B. Parshall and L. Scott, Koszul algebras and the Frobenius endomorphism, *Quart. J. Math. Oxford* **46** (1995), 345–384.

[24] B. Parshall and L. Scott, Quantum Weyl reciprocity for cohomology, *Proc. London Math. Soc.* **90**(3) (2005), 655-688.

[25] B. Parshall and L. Scott, Extensions, Levi subgroups and character formulas, *J. Algebra*, in press.

[26] B. Parshall and L. Scott, Beyong the Jantzen region, Oberwolfach Report No. 15/2006 (2006), 26–30.

[27] L. Scott, Modular permutation representations, *Trans. Amer. Math. Soc.* **175** (1973), 101–121.

[28] L. Solomon, A decomposition of the group algebra of a finite Coxeter group, *J. Algebra* **9** (1968), 220–239.

[29] T. Tanisaki, Character formulas of Kazhdan-Lusztig type, *Representations of finite dimensional algebras and related topics in Lie theory and geometry*, Fields Inst. Commun. **40**, Amer. Math. Soc., Providence, RI (2004) 261–276.

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