Average c-Efficiency

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Abstract

This paper introduces, for an experimental design with design matrix X, the optimal design criterion based on the expectation of the square root of the quadratic form $u'(X'X)^{-1}u$. This provides a geometric and statistical measure of the size of a confidence ellipsoid, namely, the expected radius of an ellipsoid. Properties of the criterion are derived and examples are given to demonstrate the procedure. These include the determination of points of influence in a design.

1. Introduction

A fundamental problem in statistical analysis is the selection of an appropriate design. For a linear statistical model $Y = X\beta + \epsilon$ with the errors independent, identically distributed normal random variables and $cov(\epsilon) = \sigma^2 I$, standard criteria in common usage for selecting a model include D-optimality and A-optimality, for which the design criteria to minimize are $C_D(X) = \left|(X'X)^{-1}\right|$ and $C_A(X) = tr((X'X)^{-1})$, respectively, where $|B|$ denotes the determinant of the matrix B and $tr(B)$ is the trace of B. It is well-known that D-optimality guarantees smaller generalized variances and volumes of elliptical confidence regions, whereas A-optimality guarantees smaller average variances for the Gauss-Markov estimators $\hat{\beta}_1, \ldots, \hat{\beta}_p$. A D-optimal design may be associated with a greatly elongated confidence ellipse, reflecting highly imprecise information about one or more linear parametric functions. A less D-optimal design may give regions of greater regularity. Indeed, orthogonal designs will give spherical confidence regions. Different design criteria typically lead to different experimental designs, as there are no globally optimal designs apart from special cases.

The problem to be addressed is how to compare design matrices, say X and Z, for a linear model. The matrices are assumed to be of full rank with dimensions $(n \times p)$. For each design there are the least-squared estimators $\hat{\beta}_X$ and $\hat{\beta}_Z$, respectively. We will give a practical solution to the problem of choosing between designs.

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In Section 2, we give properties of the design criterion which we call ACE for "average c-efficiency". In Section 3, we show the relationship between ACE and the expected radius of an ellipsoid (see Formula 3.4). In Section 4, we give the algorithm for computing ACE. In Section 5, we extend ACE to allow for prior information. And in Section 6, we show how ACE can be used to determine points of influence. Section 7 summarizes the results.

To fix the notation, we will consider the second-order response model

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_{11} X_1^2 + \beta_{22} X_2^2 + \beta_{12} X_1 X_2 + \epsilon, \tag{1.1}$$

together with two designs with experimental design matrices X and Z of order (9×6). The basic designs give points in the (X_1, X_2)-plane. The first is the standard 3^2 factorial design with design points $\{(0,0), (\pm 1, 0), (0, \pm 1), (\pm 1, \pm 1)\}$; and the second design derives from the first by rotating points in the (X_1, X_2)-plane counter-clockwise through 45 degrees.

For a linear parametric function $u \in \mathbf{R}^6$ and a design W, the standard error for $u'\hat{\beta}_W$ is given by $s.e.(u'\hat{\beta}_W) = (\sigma^2 u'(W'W)^{-1}u))^{1/2}$. We denote

$$C_c(W; u) = (u'(W'W)^{-1}u)^{1/2}. \tag{1.2}$$

For a family Ω of designs and a given u, a design $W_0 \in \Omega$ is said to be *c-optimal* (see, for example, Pukelsheim(1981)) for u if $C_c(W_0; u)$ achieves the minimum value over the family Ω. For the linear parametric function $u_1 = (0,0,0,1,-1,0)'$ which is associated with a test of equality of the quadratic terms, and for $u_2 = (0,0,0,0,0,1)'$, which is associated with a test for the presence of an interaction effect, $C_c(X; u_1) = 1 > C_c(Z; u_1) = 0.5$, and $C_c(X; u_2) = 0.5 < C_c(Z; u_2) = 1$. Thus the X design is preferred when the test for interaction is important, and the Z design is preferred when the test for equality of the quadratic terms is important. Note that although the two designs are not c-equivalent, they are, however, D-equivalent with $C_D(X) = C_D(Z) = 1/5184 = .0001929$.

We will now enlarge the designs under study to the family of designs Φ, which will include X and Z above, and, in addition, will include the equiradial design consisting of the origin and eight equally spaced points on a circle with center 0. To normalize the designs, we will require that the sum of the squared distances from the origin to the design points in the (X_1, X_2)-plane to be the constant 12 for all the designs. In Appendix A, we show that this normalization forces equality of the variances for the main effects. Thus, the radius of the equiradial design will have radius $r = \sqrt{3/2} = 1.2247$. We denote this equiradial design by Δ. The family of designs Φ consists of the designs given by the nine points in the (X_1, X_2)-plane $\{(0,0), (\pm a, 0), (0, \pm a), (\pm \frac{\sqrt{2}}{2} b, \pm \frac{\sqrt{2}}{2} b)\}$with $a^2 + b^2 = 3$ and $0 < a < \sqrt{3}$. The designs in Φ are symmetric, reflexive, and have balanced variances for the main effects. A design in the family Φ is determined by the value of a since $b = \sqrt{3 - a^2}$. The X design has $a = 1$; the Z design has $a = \sqrt{2}$; the Δ design has $a = \sqrt{3/2}$. To find the D-optimal design in the family Φ, one can graph $C_D(X(a))$. As Figure 1 shows, the graph has two global minima at $a = 0.8580$ and $a = 1.5046$ $\left(a = \sqrt{\frac{3}{2} \pm \frac{1}{6}\sqrt{21}}\right)$.

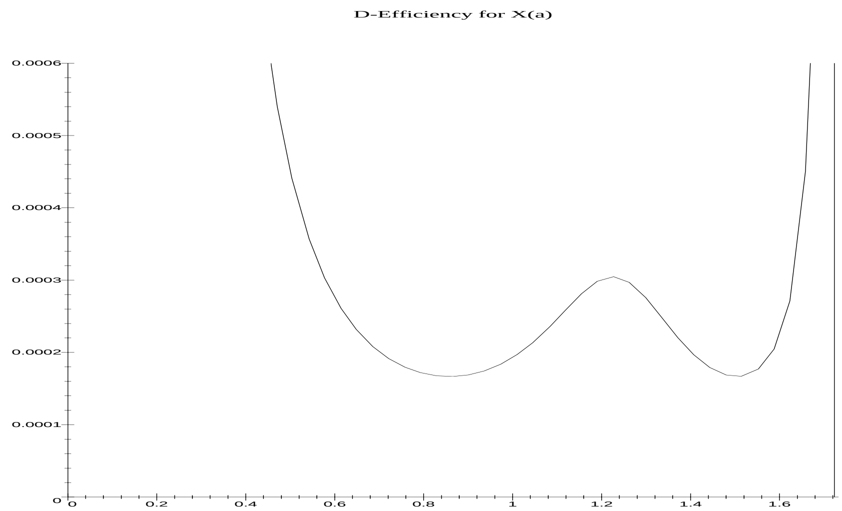

Figure 1 with the a parameter along the horizontal axis

The two most D-efficient designs are star-shaped regions in the (X_1, X_2)-plane with $a = 0.8580$ and $a = 1.5046$ and both are shown in Figure 2. Note that the two designs are congruent under a 45 degree rotation.

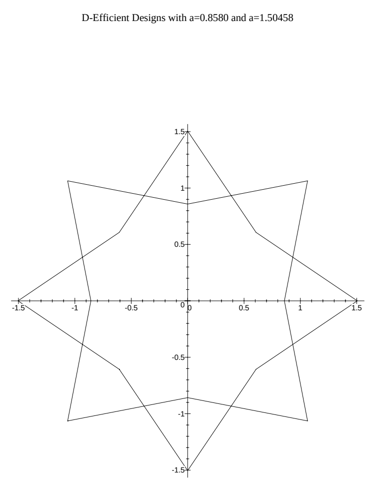

Figure 2

We now choose $u_3 = (-1, 0, 0, 2, -4, 3)'$. With this choice for u, $C_c(X; u_3) = C_c(Z; u_3) = 3.3871$ and $C_c(\Delta; u_3) = 2.8868$. Note that none of the three designs, X, Z, and Δ, are globally optimal over the set of linear parametric functions $\{u_1, u_2, u_3\}$.

If we are concerned with testing the family of hypotheses $H_0 : u'\beta = 0$, then it is convenient to scale u so that its length is one; for example, say:

$$\left(C_c(X; \frac{u_1}{(u_1' u_1)^{1/2}}) + C_c(X; \frac{u_2}{(u_2' u_2)^{1/2}}) + C_c(X; \frac{u_3}{(u_3' u_3)^{1/2}}) \right) /3 \tag{1.3}$$

We normalize the linear parametric functions to balance the contribution of the individual components. The value of a that minimizes Equation 1.3 is $a = 1.2871$. Recall that the equiradial design Δ has $a = 1.2247$.

Generally, we do not wish to restrict ourselves to some particular linear parametric functions. Thus, if no prior information is to be used, it is natural to average over *all* of the possible linear parametric functions with unit length. Thus the average c-efficiency ACE is defined to be the average over all the vectors u on the unit sphere in $\mathbf{R}^{p\cdot}$. (We will discuss the model where there is prior information in Section 5.) Thus we define for a design matrix the *average c-efficiency* to be the expected value of $C_c(X; u)$ where u is uniformly distributed over the unit sphere Ω in $\mathbf{R}^p$;

$$ACE((X^{'}X)^{-1}) = E_\omega(C_c(X; u)) = E_\omega((u'(X'X)^{-1}u)^{1/2}). \tag{1.4}$$

The graph of $ACE((X^{'}(a)X(a))^{-1})$ for the family of designs Φ is shown in Figure 3.

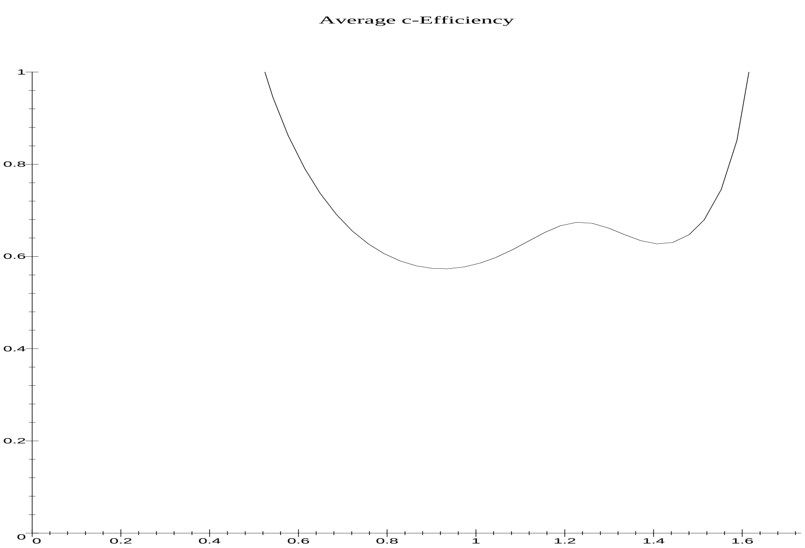

Figure 3 with the a parameter along the horizontal axis

The ACE-optimal design has $a = 0.9256$ and is shown is Figure 4. Recall that the 3^2 factorial design has $a = 1$.

ACE design with a=0.9256

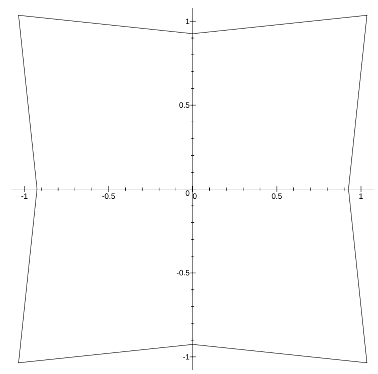

Figure 4

2. Properties of ACE

For a linear parametric function $u \in \mathbf{R}^p$ and a design W, the standard error $s.e.(u'\hat{\beta}_W)$ $= (\sigma^2 u'(W'W)^{-1}u))^{1/2}$, and we denote $C_c(W;u) = (u'(W'W)^{-1}u)^{1/2}$ and $ACE((W^{'}W)^{-1}) = E_\omega(C_c(W;u)) = E_\omega((u'(W'W)^{-1}u)^{1/2})$. Let $\Sigma_W = (W'W)^{-1}$, the inverse of the moment matrix $W'W$. In a manner parallel to Equation 1.4, we define for a positive semidefinite matrix Σ,

$$ACE(\Sigma) = E_\omega((u'\Sigma u)^{1/2}). \tag{2.1}$$

Properties of ACE are given next. The proofs are straight-forward.

Proposition 2.1. *Properties of ACE are:*
(1) For an orthogonal matrix P, $ACE(\Sigma) = ACE(P'\Sigma P)$.
(2) $ACE(\Sigma) \geq 0$.
(3) $ACE(\Sigma) = 0$ if and only if $\Sigma = 0$.
(4) $ACE(I) = 1$.
(5) $ACE(c\Sigma) = \sqrt{c}ACE(\Sigma)$, with $c \geq 0$.
(6) If each of the ordered eigenvalues of Σ_1 is less than or equal to the corresponding ordered eigenvalues of Σ_2, then $ACE(\Sigma_1) \leq ACE(\Sigma_2)$. Thus ACE is monotone with respect to the Loewner ordering.
(7) $ACE(\Sigma_1 + \Sigma_2) \leq ACE(\Sigma_1) + \|\Sigma_2\|_\infty^{1/2}$, with $\|\Sigma_2\|_\infty$ being the spectral radius (largest eigenvalue) of Σ_2 (recall Weyl's Theorem, Horn and Johnson (1985));
(8) For a design matrix X, let X_ϵ denote the X design with the columns permuted by the permutation ϵ, then $ACE((X^{'}X)^{-1}) = ACE((X_\epsilon^{'}X_\epsilon)^{-1})$. Thus ACE is invariant

under permutations of the treatment effects.
(9) If the design X is replicated k times to form $X_{(k)}$ with $X^{'}_{(k)} = (X'|\ldots|X')$, then $ACE((X^{'}_{(k)}X_{(k)})^{-1}) = ACE((X^{'}X)^{-1})/\sqrt{k}$.

The rationale for the ACE criterion is motivated by the least absolute value (LAV) or L_1 norm. The importance of LAV is becoming widely accepted in the statistical community as an alternative to the least-squares procedures, since the LAV procedures are robust, being less sensitive to outliers and the inappropriateness of the Gaussian assumptions (see Lawrence and Arthur(1990) and Birkes and Dodge(1993)). Often the researcher wishes to use a LAV procedure, but, solving an L_1 problem may be awkward because of the associated numerical difficulties, see Dodge (1987). In our case, the numerical complexity in computing Equation 1.4 involves the radical in the expectation. We show in Section 3 how to convert the difficult multiple integral in Equation 1.4 into a relatively simple univariate integral on $[0, 1]$.

We note here that the trace or A criterion has the same form as ACE but without the radical, and so its associated multiple integral can be easily computed (using the double expectation theorem) from $C_A(\Sigma) = tr(\Sigma) = p\, E_\omega(U'\Sigma U)$.

Vining and Meyers (1991) proposed evaluating response surface designs in terms of the average error of prediction over the surface of a sphere of radius ρ. This is the same principle that we are proposing with ACE. The main difference is that Vining and Myers (1991) used the mean squared errors of prediction which involve the trace of the corresponding matrix. Since we can now compute the mean standard errors, we are using this in our methodology.

The integrated mean squared error $IMSE$ criterion (see Studden (1977) and Nishii (1993)) is similar in form to ACE. There is an important difference in that ACE is an integrated average in the parameter space while $IMSE$ is an integrated average in the design space.

3. Expected Radius of an Ellipsoid

Equation 1.4 for $ACE(\Sigma)$ can be viewed as measuring the size of the ellipsoid associated with Σ, say $\{z'\Sigma^{-1}z \le c^2\}$. Consider an ellipse in $\mathbf{R}^2$. A natural geometric measure of the size of the ellipse is the expected shadow of the ellipse. The shadow of an ellipse is constructed by rotating the ellipse and measuring the projection onto the x-axis; or, indeed, any fixed line through the center. In higher dimensions, we do the following.

Let $u \in \mathbf{R}^p$ with $u'u = 1$. The shadow of the ellipsoid onto the half-line which extends from the origin in the u-direction is given by the projection of the ellipsoid onto the line containing the vector u, and has length given by

$$\text{Projection of } \{z'\Sigma^{-1}z \le c^2\} = c\,(u'\Sigma u)^{1/2} \tag{3.1}$$

by the extended Cauchy-Schwarz inequality, see Johnson and Wichern (1988). For a linear model $Y = X\beta + \epsilon$, with X of full rank with dimension $(n \times p)$, the covariance of $\hat{\beta}$, $cov(\hat{\beta}) = \sigma^2(X'X)^{-1} = \sigma^2\Sigma_X$. A $100(1-\alpha)\%$ confidence region for β is given by

$$(\beta - \hat{\beta})'(X'X)(\beta - \hat{\beta}) \leq p\, s^2 F_{p,n-p}(\alpha), \tag{3.2}$$

with s^2 the sample variance. Thus we define the *expected radius* of the $100(1-\alpha)\%$ confidence region for β to be

$$\mathcal{E}(r) = (p\, s^2 F_{p,n-p}(\alpha))^{1/2} E_\omega(u'\Sigma_X u)^{1/2} = (p\, s^2 F_{p,n-p}(\alpha))^{1/2}\, ACE(\Sigma_X). \tag{3.3}$$

Thus minimizing $ACE(\Sigma)$ is equivalent to minimizing the expected radius of the associated ellipsoid.

Let Λ be a diagonal matrix $diag(\gamma_1, \cdots, \gamma_p)$. Each point on the surface of the ellipsoid $\{z'\Lambda^{-1}z = z_1^2/\gamma_1 + \cdots z_p^2/\gamma_p = 1\}$ is canonically associated with a corresponding point u on the surface of the hypersphere by the relationship $z_i = (\gamma_i)^{1/2} u_i$, $(1 \leq i \leq p)$. The motivation for calling $\mathcal{E}(r)$ the expected radius is that the expected length of z (namely, the expected radius of the ellipsoid) satisfies

$$E_\omega(|z|) = E_\omega((z'z)^{1/2}) = E_\omega((u'\Lambda u)^{1/2}) = ACE(\Lambda) \tag{3.4}$$

with the expectations taken uniformly over the hypersphere.

Thus we have shown that the quantity $(u'(X'X)^{-1}u)^{1/2}$ has two geometric interpretations: it is proportional to:

(1) the length of the shadow of the ellipsoid onto the line in the u direction with

$$\text{Projection on } u \text{ of } \{z'\Sigma^{-1}z \leq c^2\} = \frac{c}{u'u}(u'\Sigma u)^{1/2} u \tag{3.5}$$

and

(2) the length of a radius in the ellipsoid.

4. Numerical Algorithm for ACE

In a recent paper, Dunkl and Ramirez (1994a), we gave an efficient algorithm to compute the surface measure of an ellipsoid. Our methods include the evaluation of the multiple integral of $(\gamma_1 u_1^2 + \cdots + \gamma_p u_p^2)^{1/2}$ over the hypersphere. We assume throughout that $\gamma_1 \geq \cdots \geq \gamma_p > 0$. This is a multivariate elliptic integral. The Fortran algorithm is given in Dunkl and Ramirez (1994b), and the source code is available from ACM-TOMS (http://www.netlib.org/toms/736). The $(p-1)$-dimensional multiple integral was converted, using Lauricella F_D functions, into a univariate integral on $[0, 1]$,

$$ACE(\Sigma) = \sqrt{\gamma_1}\frac{1}{pB\left(\frac{1}{2}, \frac{p+1}{2}\right)} \int_0^1 u^{-1/2}(1-u)^{(p-1)/2}\varphi(u)du, \tag{4.1}$$

where

$$\varphi(u) = \sum_{i=1}^{p} \frac{1 - x_i}{1 - ux_i} \prod_{i=1}^{p}(1 - ux_i)^{-1/2}, \tag{4.2}$$

with $x_i = 1 - \gamma_i/\gamma_1, 1 \leq i \leq p$.

To remove singularities, we converted the integral into

$$ACE(\Sigma) = \sqrt{\gamma_1}\frac{20}{pB\left(\frac{1}{2}, \frac{p+1}{2}\right)}\int_0^1 x^3(1+2x+3x^2+4x^3)^{-1/2}\psi(x)dx, \qquad (4.3)$$

where

$$\psi(x) = \left(\prod_{j=2}^{p}\frac{5x^4-4x^5}{y_j+x_j(5x^4-4x^5)}\right)^{1/2}\left(1+\sum_{j=2}^{p}\frac{y_j}{y_j+x_j(5x^4-4x^5)}\right) \qquad (4.4)$$

with $y_j = 1 - x_j, 2 \le j \le p$. This is the univariate integral on which our calculations are based.

It is important to note that the numerical complexity of the D criterion is about the same as using ACE (which requires the calculation of the eigenvalues of the covariance matrix). The calculation of ACE usually requires only about 64 function evaluations and is negligible compared to the calculation of the eigenvalues.

5. Prior informative ACE

Previously we were averaging the c-efficiencies $C_c(X;u)$ over the unit sphere in $\mathbf{R}^p$. This is equivalent to assuming that the vectors u are distributed as the standardized multivariate normal distribution $N_p(0,I)$. That is, ACE is (apart from a constant multiple) the expectation of $(u'\Sigma_X u)^{1/2}$ with respect to $N_p(0,I)$. (See Equation 5.2 with $\Xi = I$.) Now to include prior information, we will allow averaging of the c-efficiencies $C_c(X;u)$ with u having distribution $\mathcal{L}(U) = N_p(0,\Xi)$.

The ACE for a design matrix Σ with respect to the prior covariance structure matrix Ξ is

$$ACE(\Sigma \,|\, \Xi) = ACE(\Xi^{1/2}\Sigma\Xi^{1/2}). \qquad (5.1)$$

In particular, $ACE(\Sigma \,|\, I) = ACE(\Sigma)$. This allows for a Bayesian prior for the distribution of the linear parametric functions. The constant of proportionality between the expectation over the sphere and the expectation over $\mathbf{R}^p$ with respect to U with $\mathcal{L}(U) = N_p(0,\Xi)$ is

$$\begin{aligned} E((u'\Sigma u)^{1/2} \,|\, N_p(0,\Xi)) &= \frac{2^{1/2}\Gamma(\frac{p+1}{2})}{\Gamma(\frac{p}{2})}E_\omega((u'\Xi^{1/2}\Sigma\Xi^{1/2}u)^{1/2})) \qquad (5.2)\\ &= \frac{2^{1/2}\Gamma(\frac{p+1}{2})}{\Gamma(\frac{p}{2})}ACE(\Xi^{1/2}\Sigma\Xi^{1/2}). \end{aligned}$$

Details are provided in Appendix B.

6. Applications to Points of Influence

We begin by suggesting that when the researcher is using the determinant of the covariance matrix to measure the size of the confidence ellipsoid, ACE should also be considered as a possible measure of the size of this confidence ellipsoid. As we have shown in Section 3, the expected radius (equivalently, ACE) gives a natural geometric and statistical measure of the size of a confidence ellipsoid. We demonstrate this by applying ACE to the problem of finding points of influence in a design.

To fix the notation, let $I_s = \{i_1, \ldots, i_s\}$ be a multi-subset of $\{1, \ldots, n\}$(repetitions are allowed). Then $X[i_1, \ldots, i_s]$ is the result of appending rows $\{i_1, \ldots, i_s\}$ to X. We denote

$$\Sigma[I_s] = \Sigma[i_1, \ldots, i_s] = (X[i_1, \ldots, i_s]' X[i_1, \ldots, i_s])^{-1}. \tag{6.1}$$

To measure the influence of the rows $[i_1, \ldots, i_s]$, Jensen and Ramirez (1993) computed the eigenvalues of

$$\Gamma[I_s] = \Gamma[i_1, \ldots, i_s] = \Sigma^{1/2} \Sigma[i_1, \ldots, i_s]^{-1} \Sigma^{1/2}. \tag{6.2}$$

Small eigenvalues of $\Gamma[i_1, \ldots, i_s]$ occur when the rows $\{i_1, \ldots, i_s\}$ are influential. This measure is equivalent to the measure introduced by Ghosh (1989). His measure is in the spirit of Cook's (1977) distance. If one uses the D criterion as the measure of size, then Equation 6.2 is the determinantal covariance ratio

$$DCR(I_s) = \frac{\left| s^2_{I_s} (X[i_1, \ldots, i_s]' X[i_1, \ldots, i_s])^{-1} \right|}{\left| s^2 (X'X)^{-1} \right|}, \tag{6.3}$$

with $s^2_{I_s}$ and s^2 the corresponding sample variances when data has been collected. If no data has been collected (i.e., before the experiment has been completed), then replace these values by σ^2. An efficient algorithm for computing Equation 6.3 has been developed by Barrett and Gray (1992). In a like manner, we will compute the ACE covariance ratio

$$ACECR(I_s) = \frac{ACE((X[i_1, \ldots, i_s]' X[i_1, \ldots, i_s])^{-1})}{ACE((X'X)^{-1})}. \tag{6.4}$$

We study the case with $s = 2$ for the 3^2 factorial design X. Denote the points of the design matrix in $\mathbf{R}^2$ by

$$\begin{array}{ccc} 3\bullet & 6\bullet & 9\bullet \\ 2\bullet & 5\bullet & 8\bullet \\ 1\bullet & 4\bullet & 7\bullet \end{array}.$$

Figure 5

For each possible pair of rows $I_2 = \{i_1, i_2\}$, we compute $ACE(\Sigma[I_2])$. There are $\binom{9}{2} + 9 = 45$ possible pairs. However, many pairs have the same eigenvalues for $\Sigma[I_2]$. Using equality of the eigenvalues of $\Sigma[I_2]$ breaks the 45 pairs into 11 equivalence classes. These are shown in Table 1 and the accompanying diagrams in Figure 6.

Table 1: 3^2 factorial design

Equivalence Classes for $I_2 = [i_1, i_2]$

	1	2	3	4	5	6	7	8	9
1	I								
2	A	J							
3	B	A	I						
4	A	F	D	J					
5	C	G	C	G	K				
6	D	F	A	H	G	J			
7	B	D	E	A	C	D	I		
8	D	H	D	F	G	F	A	J	
9	E	D	B	D	C	A	B	A	I

Table 2: Values of $ACE(I_2)$ for the 3^2 Factorial Design

A	B	C	D	E	F	G	H	I	J	K^*
.5463	.5516	.5237	.5479	.5522	.5436	.5210	.5460	.5612	.5533	.5156

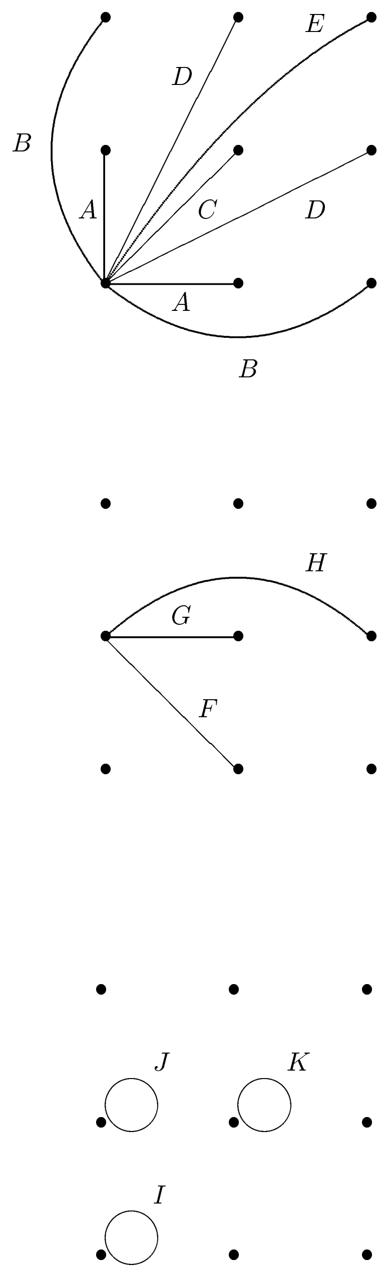

Figure 6

Table 2 reports the values of $ACE(\Sigma(I_2))$ for the different equivalence classes for the 3^2 factorial design. The minimum $ACE(I_2)$ occurs with the augmented design that has replicated the center run. The equivalence class B has the minimum C_D value. This augmented design has replicated two adjacent corners.

The ACE covariance ratio is

$$ACECR(I_2 = \{5, 5\}) = \frac{0.5156}{0.5829} = 0.8845 \tag{6.6}$$

showing that the expected radius of the confidence ellipsoid has been reduced 12% by replicating the center run. We omit the rotated 3^2 factorial design Z since the results are identical to those with the 3^2 factorial design X. For the equiradial design Δ, we label the design according to Figure 7.

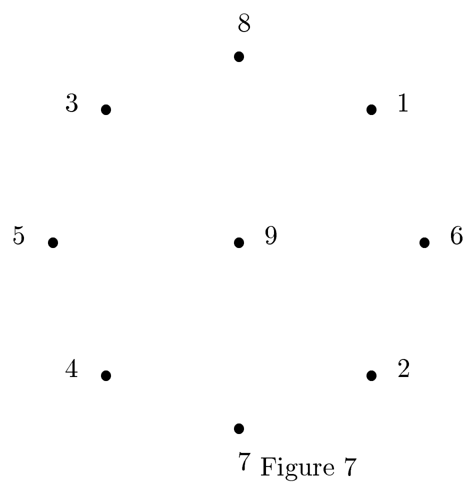

Figure 7

The equivalence classes for the equiradial design Δ are given in Table 3. Figure 8 gives a geometric interpretation of these equivalence classes.

Table 3: Equiradial design

Equivalence Classes for $I_2 = [i_1, i_2]$

	1	2	3	4	5	6	7	8	9
1	I								
2	A	I							
3	A	B	I						
4	B	A	A	I					
5	C	C	D	D	J				
6	D	D	C	C	F	J			
7	C	D	C	D	G	G	J		
8	D	C	D	C	G	G	F	J	
9	E	E	E	E	H	H	H	H	K

Table 4: Values of $ACE(I_2)$ for the Equiradial Design

A	B	C	D	E	F	G	H	I	J	K^*
.6463	.6461	.6497	.6503	.5551	.6542	.6542	.5597	.6532	.6594	.5297

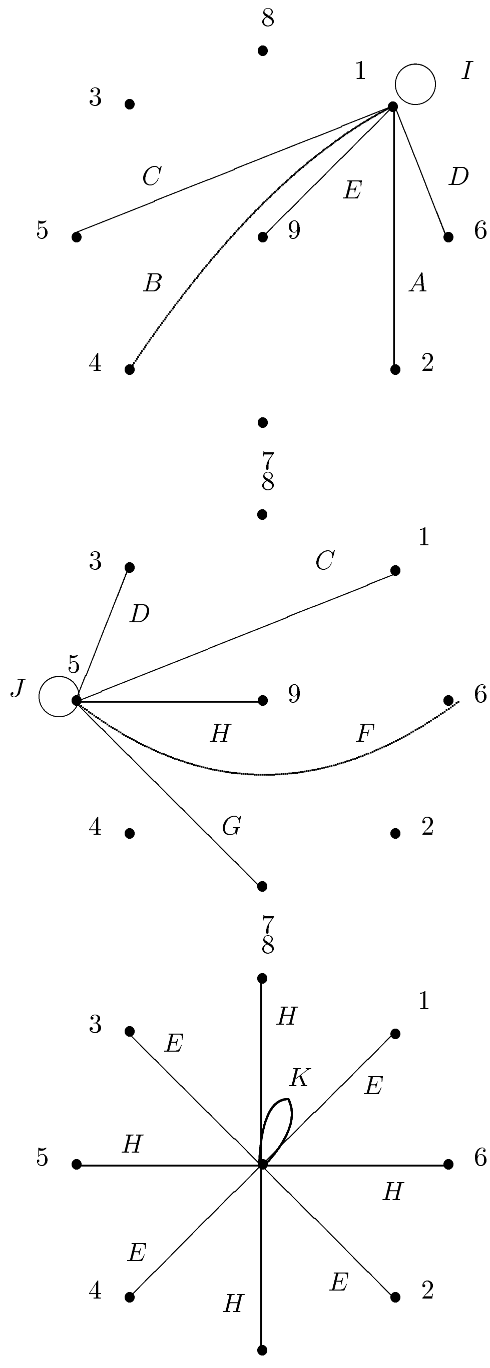

Figure 8

Table 4 reports the values of $ACE(I_2)$ for the different equivalence classes for the equiradial design. The minimum $ACE(I_2)$ occurs with the augmented design that has replicated the center run. The equivalence classes E and H have the minimum C_D value. This augmented design has replicated the center run and a perimeter run.

The ACE covariance ratio is

$$ACECR(I_2 = \{9, 9\}) = \frac{0.5297}{0.6739} = 0.7860 \tag{6.7}$$

showing that the expected radius of the confidence ellipsoid has been reduced 21% by replicating the center run.

7. Summary

For a design matrix Σ, we have introduced the design criterion $ACE(\Sigma|\Xi)$ (Equation 5.1) which is a scaling of the integrated average of the standard errors for tests $H_0 : u^{'}\beta = 0$, where the distribution of u satisfies $\mathcal{L}(U) = N_p(0, \Xi)$. When $\Xi = I_p$, $ACE(\Sigma|\Xi)$ reduces to $ACE(\Sigma)$ (Equation 1.4) the integrated average of the standard errors for tests $H_0 : u^{'}\beta = 0$, where the distribution of u is uniform over the sphere Ω in $\mathbf{R}^p$. We have given the basic properties of ACE, and the relationship between ACE and the expected radius of an ellipsoid $\mathcal{E}(r)$.

We have used ACE to find the optimal design in the family Φ consisting of nine points which are symmetric, reflexive, and have balanced variances for the main effects. We have used ACE to determine points of influence for the 3^2 factorial design X and the equiradial design Δ.

A. Appendix

For a design $X(a) \in \Phi$, the moment matrix has

$$X'(a)X(a) = \begin{bmatrix} 9 & 0 & 0 & 2(a^2+b^2) & 2(a^2+b^2) & 0 \\ 0 & 2(a^2+b^2) & 0 & 0 & 0 & 0 \\ 0 & 0 & 2(a^2+b^2) & 0 & 0 & 0 \\ 2(a^2+b^2) & 0 & 0 & b^4+2a^4 & b^4 & 0 \\ 2(a^2+b^2) & 0 & 0 & b^4 & b^4+2a^4 & 0 \\ 0 & 0 & 0 & 0 & 0 & b^4 \end{bmatrix}. \tag{A.1}$$

The covariance matrix $\Sigma(a) = (X'X)^{-1/2}$ has its $(2, 2)$ and $(3, 3)$ terms equal to $1/(2(a^2 + b^2))$ the variance for the main effects. Thus, if we require $r^2 = a^2 + b^2$ to be a constant, then the variances for the main effects are equal. This is our motivation for the choice of the normalization use for the family of designs in Φ.

B. Appendix

The expectation $E_\omega((u'\Sigma u)^{1/2})$ can be represented as a Carlson function $R_{1/2}(1/2, \ldots, 1/2; y_1, \ldots, y_p)$. The applications to statistics for Carlson functions are noted in Exton (1976). The extension to expectations with respect to the general distribution $N_p(0, \Xi)$ is given in Carlson (1972). This result has been used by Dickey (1983) to compute moments of Dirichlet distributions. The result we need can be shown by an application of the double expectation theorem.

$$\begin{aligned}
E((u^{'}\Sigma u)^{1/2}|N_p(0,\Xi)) &= E_\rho(E_\omega((u'\Sigma u)^{1/2} \mid u'\Xi^{-1}u = \rho^2)) \qquad \text{(B.1)} \\
&= E_\rho(E_\omega((v'\Xi^{1/2}\Sigma\Xi^{1/2}v)^{1/2} \mid v'v = \rho^2)) \text{ (with } v = \Xi^{-1/2}u) \\
&= E_\rho(E_\omega(\left(\frac{v'\Xi^{1/2}\Sigma\Xi^{1/2}v}{\rho^2}\right)^{1/2} \rho) \mid v'v = \rho^2) \\
&= ACE(\Xi^{1/2}\Sigma\Xi^{1/2})E(\sqrt{\chi_p^2}) = \frac{2^{1/2}\Gamma(\frac{p+1}{2})}{\Gamma(\frac{p}{2})} ACE(\Xi^{1/2}\Sigma\Xi^{1/2}),
\end{aligned}$$

since $\rho^2 = v'v = u'\Xi^{-1}u$ is distributed as χ_p^2. Here we use the fact that

$$\mu_r^{'}(\chi_p) = 2^{r/2}\Gamma\left(\frac{p+r}{2}\right)/\Gamma\left(\frac{p}{2}\right) \qquad \text{(B.2)}$$

correcting an error in Johnson and Kotz (1970, page 197).

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