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A Linear Programming Problem in Harmonic Analysis*

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ABSTRACT

For a given subset E of the natural numbers it is desired to maximize $\Sigma_{n\in E}a_n$ subject to $a_n > 0$, $1+\Sigma_{n\in E}a_n\cos n\theta > 0$ for $\theta\in[0,\pi]$. A dual program is defined, and a duality principle is established. Extensions to other series of functions are given, and these include the motivating example of P. Delsarte [*Philips Res. Repts.* 27 (1972), 272–289].

1. INTRODUCTION

To motivate our results, we consider the following problem: for $a_1,a_2,a_3,a_4,a_5 > 0$ with $g(\theta)=1+\Sigma_{l=1}^{5}a_l\cos l\theta \geqslant 0$ for all $\theta\in[0,\pi]$, find the maximum of $A=\Sigma_{l=1}^{5}a_l$, the sum of the coefficients.

Primal problem: Given E, a finite subset of the natural numbers $\{1,2,\dots\}$, maximize $A=\Sigma_{l\in E}a_l$ subject to the constraints

$$a_l > 0 \qquad (l\in E)$$

and

$$1+\sum_{l\in E}(\cos l\theta)a_l \geqslant 0 \qquad (\theta\in[0,\pi]).$$

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To estimate the solution of the above problem, we solve the primal problem (an infinite linear program) with $\theta \in [0,\pi]$ replaced by $\theta \in \{\pi/250, 2\pi/250, 3\pi/250, \ldots, \pi\}$—a large (but finite) linear program. The solution turns out to be $A = 5.00$ with

$$\begin{aligned} a_1 &= 1.67, \\ a_2 &= 1.33, \\ a_3 &= 1.00, \\ a_4 &= 0.66, \\ a_5 &= 0.33. \end{aligned}$$

This finite approximation does indeed give the correct answer for the infinite problem with $E = \{1,2,3,4,5\}$. We show that for $E = \{1,\ldots,n\}$ the maximum value is n, and we will also consider arbitrary subsets E. The device is to establish a dual program.

Dual problem: Given E, a finite subset of the natural numbers, minimize $N = \|\mu\|$(the norm of the measure μ on the unit circle T) for μ a nonnegative measure and subject to the constraints

$$\hat{\mu}(k) \leq -1 \qquad (k \in E \cup -E).$$

Here $\hat{\mu}(k)$ is the Fourier-Stieltjes coefficient at k defined by

$$\hat{\mu}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\theta}\, d\mu(\theta).$$

THEOREM 1. *If the dual problem is feasible with solution N, then the primal problem is bounded with $A \leq N$.*

Proof. For $\varepsilon > 0$ let μ be a nonnegative measure on T with $\hat{\mu} \leq -1$ on E and $\|\mu\| < N + \varepsilon$. Let $g(\theta) = 1 + \sum_{l \in E} a_l \cos l\theta \geq 0$ $(a_l \geq 0)$; then

$$\begin{aligned} 0 &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)\, d\mu(\theta) = \sum_{n=-\infty}^{\infty} \hat{g}(n)\,\hat{\mu}(n) \\ &= \hat{\mu}(0) + \textstyle\sum_{l \in E} a_l \hat{\mu}(l) \\ &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} 1\, d\mu - \sum_{l \in E} a_l. \end{aligned}$$

Thus

$$\sum_{l \in E} a_l \leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} 1\, d\mu = \| \mu \| < N + \varepsilon.$$

It follows that the maximum of $A = \sum_{l \in E} a_l \leqslant N$. ∎

Example 1. Returning to the example with $E = \{1,2,3,4,5\}$, let us first note that

$$g(\theta) = 1 + \frac{10}{6}\cos\theta + \frac{8}{6}\cos 2\theta + \frac{6}{6}\cos 3\theta + \frac{4}{6}\cos 4\theta + \frac{2}{6}\cos 5\theta$$

is a Fejér polynomial; and so $g \geqslant 0$ and A is at least 5.

In general with $E = \{1, 2, \ldots, n\}$, let

$$g(\theta) = 1 + 2\sum_{l=1}^{n}\left(1 - \frac{l}{n+1}\right)\cos l\theta;$$

then

$$g(\theta) = \sum_{|l| < n}\left(1 - \frac{|l|}{n+1}\right)e^{il\theta}$$

$$= \frac{1}{n+1}\left[\frac{\sin\frac{1}{2}(n+1)\theta}{\sin\frac{1}{2}\theta}\right]^2 \geqslant 0.$$

Also

$$2\sum_{l=1}^{n}\left(1 - \frac{l}{n+1}\right) = n.$$

The dual program is to minimize $N = \| \mu \|$ with μ a nonnegative measure on the circle and

$$\hat{\mu}(k) \leqslant -1 \qquad \text{for} \quad k = 1,\ 2,\ 3,\ 4,\ 5.$$

Consider

$$\nu = \sum_{j=0}^{5} \delta\left(\frac{2\pi j}{6}\right).$$

(Here $\delta(p)$ denotes the unit point measure at $p \in [-\pi, \pi]$.) Then

$$\hat{\nu}(k) = 0 \qquad (k = \pm 1, \pm 2, \pm 3, \pm 4, \pm 5).$$

Let

$$\mu = \nu - \delta(0),$$

a nonnegative measure on T with

$$\hat{\mu}(k) = -1 \qquad (k = \pm 1, \pm 2, \pm 3, \pm 4, \pm 5),$$

and so

$$N \leqslant \|\mu\| = 5.$$

By Theorem 1, $A \leqslant N \leqslant 5$. But we know that A is at least 5, and thus $A = 5$.

More generally we have the following:

THEOREM 2. *Let* $E = \{1, 2, \ldots, n\}$. *Then the solution of the primal problem is* n *with*

$$a_l = 1 - \frac{l}{n+1} \qquad (l = 1, 2, \ldots, n).$$

We now extend the primal problem to infinite subsets E of $\{1, 2, \ldots,\}$ as follows:

$$\text{maximize} \quad A = \sum_{l \in E} a_l$$

subject to the same constraints and with F a finite subset of E. The dual problem is unchanged for infinite subsets. Theorem 1 is valid for infinite subsets, and the proof remains the same.

EXAMPLE 2. Let $E=\{1,3,5,\dots\}$. Then the solution of the primal problem has $A=1$:

Clearly A is at least 1, since $1+\cos\theta \geqslant 0$. To see that $A \leqslant 1$ we offer two arguments. Firstly, write $g(\theta)=1+\sum_{l=0}^{n} a_{2l+1}\cos(2l+1)\theta \geqslant 0$, and evaluate at $\theta=\pi/2$ to get

$$1-\sum_{l=0}^{n} a_{2l+1} \geqslant 0.$$

Secondly, let ν be the measure defined by

$$\nu=\delta(0)+\delta(\pi).$$

Then

$$\hat{\nu}(k)=\begin{cases} 0, & k \text{ odd},\\ 2, & k \text{ even}.\end{cases}$$

Let $\mu=\nu-\delta(0)$. Then $\hat{\mu}(k)=-1$ for k odd; thus

$$A \leqslant N \leqslant \|\mu\|=1.$$

We have thus shown that the primal problem may be bounded for infinite subsets. Of course, for $E=\{1,2,\dots\}$, the primal problem is unbounded by Theorem 2.

EXAMPLE 3. The solution of the primal problem becomes challenging even for small subsets. Let $E=\{2,3\}$. Here we can show by *ad hoc* methods that the solution of the primal problem is $A=1.2361$ with $a_2=0.7416$ and $a_3=0.4945$.

We will turn our attention to the duality principle mentioned in Theorem 1. But first we will need this lemma.

LEMMA. *Let $\{F_\alpha\}$ be a collection of closed convex subsets of the Euclidean space $\mathbf{R}^m$. Suppose $F=\cap F_\alpha$ is a nonempty (convex) compact set. Given U an open neighborhood of F, there exists a finite subset $\alpha_1,\dots,\alpha_n$ with $\cap_{i=1}^{n} F_{\alpha_i} \subset U$.*

Proof. We may suppose U is convex and has compact closure K. The sets $F_\alpha \cap K$ are compact, and $\cap(F_\alpha \cap K)=F$ is a nonempty compact subset of K. Thus there exists a finite family $F_{\alpha_1},\ldots,F_{\alpha_n}$ with

$$(F_{\alpha_1}\cap K)\cap\cdots\cap(F_{\alpha_n}\cap K)\subset U,$$

so

$$(F_{\alpha_1}\cap\cdots\cap F_{\alpha_n})\cap K\subset U.$$

We wish to show $\cap_{i=1}^{n}F_{\alpha_i}\subset U$. If not, let $q\in(\cap_{i=1}^{n}F_{\alpha_i})\setminus U$. Choose $p\in F\cap K$, so the line segment $[p,q]$ from p to q is in $\cap_{i=1}^{n}F_{\alpha_i}$. Choose $t\in[p,q]$ with $t\in K$ but $t\notin U$. Thus $t\in(F_{\alpha_1}\cap\cdots\cap F_{\alpha_n})\cap K\subset U$, a contradiction. So $\cap_{i=1}^{n}F_{\alpha_i}\subset U$, as required. ∎

Convexity as a hypothesis can not be easily removed: consider the example $F_n=\{0\}\cup[n,\infty)\subset\mathbf{R}$, $n=1,2,\ldots$.

THEOREM 3. *If the primal problem is bounded with solution A, then the dual problem is feasible and $N\leqslant A$.*

Proof. Let E be a subset of $\{1,2,\ldots\}$, and assume that the primal problem is bounded with solution A. It will suffice to show for each $\varepsilon>0$ and for each finite subset F of E that there exists a nonnegative measure μ_F on the circle with $\|\mu_F\|\leqslant A+\varepsilon$ and $\hat{\mu}_F\leqslant -1$ on $F\cap -F$. For then a weak-* cluster point μ_ε of $\{\mu_F\}$ has $\|\mu_\varepsilon\|\leqslant A+\varepsilon$, and $\hat{\mu}_\varepsilon\leqslant -1$ on $E\cap -E$. Let μ be a weak-* cluster point of $\{\mu_\varepsilon\}$, and thus $\hat{\mu}\leqslant -1$ on $E\cap -E$ and $\|\mu\|\leqslant A$.

Let $P=\{g:g(\theta)=1+\sum_{l\in F}a_l\cos l\theta$ with $a_l\geqslant 0$ $(l\in F)\}$. To each $g\in P$ associate the finite tuple $(a_l)_{l\in F}$. Now P is a locally compact space with the topology of pointwise convergence of the coefficients. (Indeed, P is isomorphic to a subset of the Euclidean space $\mathbf{R}^m$, $m=$ cardinality of F.) For each finite subset Υ of the circle, define

$$H(\Upsilon)=\{g\in P:g(\theta)\geqslant 0\ (\theta\in\Upsilon)\}.$$

Then $\cap\{H(\Upsilon):\Upsilon$ a finite subset of $T\}=\{g\in P:g\geqslant 0\}$. Denote the set $\{g\in P:g\geqslant 0\}$ by K.

Now K is a compact set, since for $g\in K$, $\sum_{l\in F}a_l\leqslant A$. Let U be the open neighborhood of K defined by

$$U=\left\{g\in P:\sum_{l\in F}a_l<A+\varepsilon\right\}.$$

Since K is a compact set and $\cap H(\Upsilon) = K$, there exists a finite subset Υ of T with $H(\Upsilon) \subset U$ (from the lemma).

Consider the primal linear program:

$$\text{maximize} \quad \sum_{l \in F} a_l$$

subject to the finite number of constraints

$$a_l \geqslant 0 \qquad (l \in F),$$

$$\sum_{l \in F} (-\cos l\theta) a_l \leqslant 1 \qquad (\theta \in \Upsilon).$$

By the above choice of Υ, the maximum is less than $A + \varepsilon$.

The finite dual program is:

$$\text{minimize} \quad \sum_{\theta \in \Upsilon} y_\theta$$

subject to the finite number of constraints

$$y_\theta \geqslant 0 \qquad (\theta \in \Upsilon),$$

$$\sum_{\theta \in \Upsilon} (\cos l\theta)\, y_\theta \leqslant -1 \qquad (l \in F).$$

Let $\mu_F = \frac{1}{2} \sum_{\theta \in \Upsilon} y_\theta [\delta(\theta) + \delta(-\theta)]$. Then $\hat{\mu}_F(l) \leqslant -1$ $(l \in F \cap -F)$ and $\|\mu_F\| = \sum_{\theta \in \Upsilon} y_\theta < A + \varepsilon$. ∎

2. EXTENSIONS

The results in Sec. 1 have been stated for cosine series for motivation. However, the reader will see that few of the properties of cosine series were needed. Here we give the general setting.

Definition 1. Let X be a compact space with a fixed base point $e \in X$. Let $\{f_\alpha\}_{\alpha \in J}$ be a set of linearly independent bounded real-valued continuous functions on X with $f_0 = 1$ (0 a distinguished element of J) and $\|f_\alpha\|_\infty = f_\alpha(e) = 1$ $(\alpha \in J)$.

The primal and dual problems for the family $\{f_\alpha\}_{\alpha\in J}$ are:

Primal problem: Let $E \subset J\setminus\{0\}$.

Maximize

$$A=\sum_{l\in F} a_l=\sum_{l\in F} a_l f_l(e) \qquad (F \text{ finite } \subset E)$$

subjects to the constraints

$$a_l \geqslant 0 \qquad (l\in E),$$

$$1+\sum_{l\in F} a_l f_l(x) \geqslant 0 \qquad (x\in X).$$

Dual problem: Let $E \subset J\setminus\{0\}$. Minimize

$$N=\|\mu\|=\int_X f_0\, d\mu$$

for μ a nonnegative measure on X subject to the constraints

$$\hat{\mu}_\alpha=\int_X f_\alpha\, d\mu \leqslant -1 \qquad (\alpha\in E).$$

The proofs from Sec. 1 yield

THEOREM 4. *Let $\{f_\alpha\}$ be as above. If the dual problem is feasible with solution N, then the primal problem is bounded with $A \leqslant N$. If the primal problem is bounded with solution A, then the dual problem is feasible with $N \leqslant A$.*

EXAMPLE 4. The Krawtchouk polynomials k_n $(0\leqslant n\leqslant N)$ are an orthogonal set of polynomials on the discrete set $X=\{0,1,2,\ldots,n\}$, where $0<p<1$, N is a positive integer, and the weight function is

$$\binom{N}{x} p^x (1-p)^{N-x} \qquad (x=0,1,\ldots,N).$$

The polynomials are given in terms of the hypergeometric functions by

$$(k_n(x;p,N)=(1-p)^n\left(\tfrac{x}{n}\right)F[-n,x-N;\,x-n+1;\,p/(p-1)]$$

$$=\frac{p^n(-N)_n}{n!}F[-n,-x;\,-N;\,1/p].$$

We normalize the Krawtchouk polynomials by

$$K_n(x;p,N)=\frac{k_n(x;p,N)}{k_n(0;p,N)}.$$

Szegö [4, p. 36] gives the explicit formula

$$K_n(x;p,N)=\sum_{j=0}^{\min(x,n)}\left(\frac{p-1}{p}\right)^j\binom{N-x}{n-j}\binom{x}{j}\Big/\binom{N}{n}.$$

The Krawtchouk polynomials appear (see [3]) as the symmetrized characters of the product of a two-point hypergroup.

In fact, Theorem 4 applied to the Krawtchouk polynomials $\{K_n\}_{n=0}^N$, producing the linear-programming bounds on codes, was the particular result suggesting this paper and is due to Delsarte [1, pp. 280–284].

EXAMPLE 5. Let $\alpha\geqslant\beta\geqslant-\frac{1}{2}$ and let $P_n^{(\alpha,\beta)}$ be the Jacobi polynomials of degree n and index (α,β) normalized to have $P_n^{(\alpha,\beta)}(1)=1$(see Szegö[5, pp. 29, 168]). Then the family $\{P_n^{(\alpha,\beta)}:n\in Z_+\}$ satisfies the hypotheses of Theorem 4 with $X=[-1,1]$ and $e=1$.

EXAMPLE 6. Let G be a compact group with identity e. Let χ be a character of G, and define

$$f_\chi=\frac{1}{2}\frac{\chi+\bar{\chi}}{\chi(e)}.$$

The set $\{f_\chi:\chi \text{ a character}\}$ satisfies the hypotheses of Theorem 4 with $X=G$.

DEFINITION 2. Let G be a compact Abelian group with character group $\hat{G}$. Let E be a subset of $\hat{G}$ such that the primal problem is bounded for the set of functions

$$f_\chi=\tfrac{1}{2}(\chi+\bar{\chi})\qquad(\chi\in\hat{G}).$$

We say that E is LP-bounded.

DEFINITION 3. Let G be a compact Abelian group with character group $\hat{G}$. Let E be a subset of $\hat{G}$ with the property that for any bounded function on E there exists $\mu\in M(G)$ with $\hat{\mu}|E=f$. Then E is called a *Sidon set*. (For

$G = T$ and $\hat{G} = Z$, one can construct Sidon sets in Z by taking finite unions of *lacunary sequences* $\{n_i\}_{i=1}^{\infty}$, that is ones with $\frac{n_{i+1}}{n_i} > q > 1$.)

THEOREM 5. *Let G be a compact Abelian group and $E \subset \hat{G}$ a symmetric Sidon set ($E = -E$); then E is LP-bounded.*

Proof. Drury [2] has shown for a bounded real function g on E with $g(x) = g(-x)$ $(x \in E)$ that there exists a nonnegative measure μ on G with $\hat{\mu} = g$ on $E \setminus \{0\}$. Thus, letting $g = -1$, one has that the dual problem is feasible, and so the primal problem is bounded. ■

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