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Sections induced from weakly sequentially complete spaces*

by

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Abstract. It is shown that function algebras are never weakly sequentially complete (unless finite dimensional) and then sections induced from maps from weakly sequentially complete spaces onto function algebras are studied. As a result, it is shown that for an infinite Helson set E the restriction map ϱ of the Fourier algebra $A(G)$ (that is, $L^2(G)*L^2(G)$) of a locally compact (not necessarily abelian) group onto the space $C(E)$ of continuous functions on E never admits a section π, (that is, a continuous linear map $\pi\colon C(E)\to A(G)$ with $\varrho\circ\pi = \mathrm{id}$). A set $E\subset G$ is called a Helson set provided $A(G)|E = C(E)$. A similar application to Sidon sets in the dual of a compact group is also given.

THEOREM 1. *Let A be a weakly sequentially complete commutative Banach algebra. If A is isomorphic to a closed subalgebra $\tilde{A}$ of $C_0(S)$, the continuous complex-valued functions vanishing at infinity on a locally compact Hausdorff space, then A is finite-dimensional.*

Proof. If A is infinite-dimensional, then there exists an infinite-dimensional separable subalgebra which is weakly sequentially complete. Thus we may assume that A is separable.

If $\tilde{A}$ does not separate the points of S, we embed A instead into $C_0(S/\sim)$, where for $s,t\in S$, $s\sim t$ if and only if $\tilde{f}(s)=\tilde{f}(t)$ for all $f\in A$. Thus we may assume that $\tilde{A}$ separates the points in S and hence in the Shilov boundary ∂A (since $\partial A\subset S$). Thus $\partial A\subset S$ is a metrizable locally compact space.

Let $P\subset\partial A\subset S$ denote the set of peak points of A. The set P is dense in ∂A (Bishop's theorem ([6], p. 56) since A is metrizable. It will thus suffice to show that P is finite: for then ∂A will be finite (and equal to P), and A is isomorphic to $\tilde{A}|\partial A$.

By the Lebesgue dominated convergence theorem, given a sequence $\{f_n\}\subset A$ with $\|\tilde{f}_n\|_\infty\leqslant 1$ and $\tilde{f}_n\overset{n}{\to}\chi_p$ (the characteristic function of the set $\{p\}$, $p\in P$) pointwise on S, it follows that $\{\tilde{f}_n\}$ is weakly Cauchy in $\tilde{A}$ ($\cong A$). Hence, by the weak sequential completeness of A, $\chi_p\in\tilde{A}$. Thus P consists of isolated points.

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Once again by the weak sequential completeness of A and the Lebesgue dominated convergence theorem, if P is not finite, we would have a countable subset $P' \subset P$ with $\chi_{P'} \in \tilde{A} \subset C_0(S)$. But P' would then be a compact infinite discrete set, a contradiction. ■

Remark. This result was previously announced by the authors in [3].

The fact that $C_0(S)$ is weakly sequentially complete if and only if S is finite is used by R. Edwards [4] to show that the Fourier–Stieltjes transform of the measure algebra $M(G)$ of a locally compact abelian group is not onto unless G is finite (see also [1], p. 30).

Examples of weakly sequentially complete spaces include convolution measure algebras, reflexive spaces, and the predual of a W^*-algebra, (Sakai, [8]); thus the Fourier algebra $A(G)$ (that is, $L^2(G) * L^2(G)$) of a locally compact group is weakly sequentially complete (Eymard, [5]). For G compact, a direct argument can be given to show $A(G)$ is weakly sequentially complete [2].

THEOREM 2. *Let ϱ be a continuous linear map of a weakly sequentially complete space A onto an infinite dimensional function algebra B. There does not exist a section $\pi: B \to A$; that is, a continuous linear map π for which $\varrho \circ \pi = \mathrm{id}$.*

Proof. By way of contradiction, suppose that π exists. Let $\pi^*: A^* \to B^*$ be the adjoint of π. If $\{f_n\} \subset B$ is a weak Cauchy sequence, then $\{\pi f_n\}$ is weak Cauchy in A: for $\varphi \in A^*$, note that $\langle \pi f_n, \varphi \rangle = \langle f_n, \pi^* \varphi \rangle$. Since A is weakly sequentially complete, there exists $g \in A$ for which $\pi f_n \xrightarrow{n} g$ weakly in A. Now $\varrho: A \to B$ is strongly continuous, and hence weakly continuous. Thus $f_n = \varrho(\pi f_n) \xrightarrow{n} \varrho g$ weakly in B. Hence B is also weakly sequentially complete, a contradiction by Theorem 1. ■

COROLLARY 3. *For G a locally compact group, let ϱ denote the restriction map from the Fourier algebra $A(G)$ onto the function algebra $C(E)$, where E is an infinite Helson set in G. There does not exist a continuous linear map $\pi: C(E) \to A(G)$ such that $\pi f | E = \varrho \circ \pi f = f$.*

Remark. Corollary 3, for locally compact abelian groups, appears in Graham, [7].

In the sequel, G will be a compact group and $\hat{G}$ its dual, (we use the notation of our book [1]). A subset $E \subset \hat{G}$ is a Sidon set provided $L^1(G)^\wedge | E = \mathscr{C}_0(E)$, the subset of $\mathscr{L}^\infty(\hat{G})$ consisting of those φ for which the set $\{\alpha \in E: \|\varphi_\alpha\|_\infty \geqslant \varepsilon\}$ is finite for $\varepsilon > 0$ and $\varphi_\alpha = 0$ for $\alpha \notin E$.

COROLLARY 4. *For $E \subset \hat{G}$ an infinite Sidon set, there does not exist a (bounded, linear) section π from $\mathscr{C}_0(E) \to L^1(G)$ for which $(\pi\varphi)\hat{}_a = \hat{\varphi}_a$, $a \in E$. Similarly, there does not exist a section π from $\mathscr{L}^\infty(E) \to M(G)$ for which $(\pi\varphi)\hat{}_a = \varphi_a$, $a \in E$.*

Let $E \subset \hat{G}$. We say that E is a central Sidon set provided given any $\varphi \in \mathscr{Z}\mathscr{L}^\infty(\hat{G})$, (the center of $\mathscr{L}^\infty(\hat{G})$) there exists $\mu \in M(G)$ such that $\varphi_a = \hat{\mu}_a$, $a \in E$, [2].

COROLLARY 5. *For $E \subset \hat{G}$ an infinite central Sidon set, there does not exist a section π from $\mathscr{Z}\mathscr{C}_0(E) \to L^1(G)$ for which $(\pi\varphi)\hat{}_a = \hat{\varphi}_a$, $a \in E$. Similarly, there does not exist a section π from $\mathscr{Z}\mathscr{L}^\infty(E) \to M(G)$ for which $(\pi\varphi)\hat{}_a = \hat{\varphi}_a$, $a \in E$.*

Remark. The space $\mathscr{L}^\infty(\hat{G})$, ($G$ infinite) is an infinite-dimensional C^*-algebra, and is thus not weakly sequentially complete (Sakai, [8]). One, however, can get this result quickly for $\mathscr{L}^\infty(\hat{G})$, since its center $\mathscr{Z}\mathscr{L}^\infty(\hat{G}) \cong l^\infty(\hat{G})$ is an infinite-dimensional function algebra.

References

[1] C. Dunkl and D. Ramirez, *Topics in harmonic analysis*, Appleton, New York, 1971.

[2] — *Sidon sets on compact groups*, Monatsh. für Math. 75 (1971), pp. 111–117.

[3] — *Weakly sequentially complete function algebras*, Notices Amer. Math. Soc. 19 (1972).

[4] R. Edwards, *On functions which are Fourier transforms*, Proc. Amer. Math. Soc. 5 (1954), pp. 71–78.

[5] P. Eymard, *L'algèbre de Fourier d'un groupe localement compact*, Bull. Soc. Math. France 92 (1964), pp. 181–236.

[6] T. Gamelin, *Uniform algebras*, New Jersey, 1969.

[7] C. Graham, *Helson sets and simultaneous extensions to Fourier transforms*, Studia Math. 43 (1972), pp. 57–60.

[8] S. Sakai, *On topological properties of W*-algebras*, Proc. Japan. Acad. 33 (1957), pp. 439–444.

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