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LOCALLY COMPACT SUBGROUPS OF THE SPECTRUM OF THE MEASURE ALGEBRA III

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The maximal ideal space of the measure algebra of a locally compact abelian (LCA) group has the structure of a compact commutative semitopological semigroup (separately continuous multiplication). Idempotents in the semigroup correspond to certain algebraic projections on the measure algebra. In this paper we study the maximal groups about certain idempotents.

NOTATION: For an infinite LCA group G, let M(G) denote the Banach algebra of finite regular Borel measures on G, and let Δ_G denote the maximal ideal space of M(G). We will use our previously established notation from [2] and [3].

We sketch the results from [2]. Let ε be an idempotent in $c\ell\,\hat{G}$ and $H(\varepsilon)$ the maximal subgroup of $c\ell\,\hat{G}$ containing ε . If ε is

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associated with an R-projection P_1 of M(G) where P_1 is the projection of M(G) determined by a finer LC-topology $\mathcal{T}_1$ for G then $H(\varepsilon) \simeq \hat{G}_1$ (algebraically and topologically), and $H(\varepsilon)$ is the maximal subgroup of Δ_G containing ε. Conversely, if ε is an idempotent in $c\ell\,\hat{G}$ with $H(\varepsilon)$ locally compact (in the induced topology from $c\ell\,\hat{G}$), then there is a finer LC-topology $\mathcal{T}_1$ for the group G, with the induced projection P_1 corresponding to ε. Further $(G, \mathcal{T}_1)\hat{} \simeq H(\varepsilon)$. Such an idempotent will be called an LC-idempotent. Thus the maximal subgroups of $c\ell\,\hat{G}$ which are locally compact are determined by the finer LC-topologies for the group G.

In [3] we showed that a maximal subgroup of $c\ell\,\hat{G}$ is not necessarily locally compact (in the induced topology from $c\ell\,\hat{G}$). Indeed, we showed this for the group $G = T^{\omega}$ the countable product of the unit circle T with the product topology. Recently, this result has been shown in general by G. Brown [1].

We now state some conventions which will hold throughout. The group G will denote a nondiscrete LCA group with topology $\mathcal{T}$. A net will always be unbounded above. Idempotents in Δ_G will be denoted by $\varepsilon, \varepsilon_\alpha, \cdots$, and given the natural ordering : $\varepsilon_\alpha \leq \varepsilon_\beta$ if and only if $\varepsilon_\alpha \times \varepsilon_\beta = \varepsilon_\alpha$ (where $\times$ is the multiplication in Δ_G). We will use additive notation for G, and multiplicative notation for groups of characters, such as $\hat{G}$.

DEFINITION: Let $\{\mathfrak{F}_\alpha : \alpha \in A\}$ be an increasing net of Raikov families (see [2, p.96]) for G ($\alpha \leq \beta$ if and only if $\mathfrak{F}_\alpha \subset \mathfrak{F}_\beta$). We assume $\mathfrak{F}_\alpha \neq \mathfrak{F}_\beta$ for $\alpha \neq \beta$. We let $\mathfrak{F}^{\#}$ be the Raikov family generated by $\{\mathfrak{F}_\alpha : \alpha \in A\}$. Each element of the family $\mathfrak{F}^{\#}$ has the form

$F = \bigcup_{n=1}^{\infty} F_n, F_n \in \mathfrak{F}_{\alpha_n}$ ($\{\alpha_n\}$ a sequence in A).

We call $\mathfrak{F}^{\#}$ the increasing limit of $\{\mathfrak{F}_\alpha : \alpha \in A\}$.

THEOREM 1: *Let $\{\mathfrak{F}_\alpha : \alpha \in A\}$ be an increasing net of Raikov families, and let $\mathfrak{F}^{\#}$ be the increasing limit of $\{\mathfrak{F}_\alpha : \alpha \in A\}$. Let the rings R_α, $R^{\#}$; the ideals I_α, $I^{\#}$; and the R-projections P_α, $P^{\#}$ be associated with $\mathfrak{F}_\alpha$, $\mathfrak{F}^{\#}$ respectively. Then $P^{\#} = \lim_{\alpha \in A} P_\alpha$ in the SO-topology, $I^{\#} = \bigcap_{\alpha \in A} I_\alpha$, and $R^{\#} = c\ell\{\bigcup_{\alpha \in A} R_\alpha\}$.*

Proof. For $\mu \in M(G), \mu \in I^{\#}$ if and only if $|\mu|(S) = 0$ for all $S \in \mathfrak{F}^{\#}$ if and only if $|\mu|(S) = 0$ for all $S \in \mathfrak{F}_\alpha$, ($\alpha \in A$) by the monotone convergence theorem, if and only if $\mu \in I_\alpha$, all $\alpha \in A$. Thus

$$I^{\#} = \bigcap_{\alpha \in A} I_\alpha .$$

Let $\mu \in R^{\#}$ and $\varepsilon > 0$. Choose $S \in \mathfrak{F}^{\#}$ such that $|\mu|$ is concentrated on S. By the monotone convergence theorem, there exists $B \in \mathfrak{F}_\alpha$ (some α) such that $|\mu|(S \setminus B) < \varepsilon$. Thus

$\mu|B \in R_\alpha$ and $||\mu|B-\mu|| < \varepsilon$. It follows that $R^{\#} \subset c\ell(\bigcup_{\alpha \in A} R_\alpha)$. The other inclusion is trivial.

Let $\mu \in M(G)$ and write $\mu = \mu_R + \mu_I$ with $\mu_R \in R^{\#}$ and $\mu_I \in I^{\#}$. For $\varepsilon > 0$, let $S \in \mathfrak{F}_\alpha$ (some α) such that $||\mu_R - \mu|S|| < \varepsilon/2$. For $\beta > \alpha$, $S \in \mathfrak{F}_\beta$. Since $\mu_I \in I^{\#} = \bigcap_{\alpha \in A} I_\alpha$, $P_\beta(\mu_I) = 0$ for $\beta > \alpha$. Thus

$$P_\beta(\mu|S + \mu_I) = \mu|S, \quad \text{for } \beta > \alpha \quad .$$

Thus

$$||P^{\#}(\mu) - P_\beta(\mu)|| = ||\mu_R - \mu|S + \mu|S - P_\beta(\mu)||$$

$$= ||\mu_R - \mu|S + P_\beta(\mu|S + \mu_I - \mu)||$$

$$\leq ||\mu_R - \mu|S|| + ||\mu|S - \mu_R|| < \varepsilon,$$

and so $P^{\#} = \lim_{\alpha \in A} P_\alpha$ (in the SO-topology).□

<u>DEFINITION</u>: Let $\{\mathfrak{F}_\alpha : \alpha \in A\}$ be a decreasing net of Raikov families for G ($\alpha \leq \beta$ if and only if $\mathfrak{F}_\beta \subset \mathfrak{F}_\alpha$). We assume $\mathfrak{F}_\alpha \neq \mathfrak{F}_\beta$ for $\alpha \neq \beta$. Let $\mathfrak{F}^\flat$ be the Raikov family defined by

$$\mathfrak{F}^\flat = \bigcap \{\mathfrak{F}_\alpha : \alpha \in A\}.$$

We call $\mathfrak{F}^\flat$ the decreasing limit of $\{\mathfrak{F}_\alpha : \alpha \in A\}$.

<u>THEOREM 2</u>. <u>Let A be a net with a cofinal sequence. Suppose $\{\mathfrak{F}_\alpha : \alpha \in A\}$ is a decreasing net of Raikov families for G, and let $\mathfrak{F}^\flat$ be the decreasing limit of $\{\mathfrak{F}_\alpha : \alpha \in A\}$. Let R_α, $R^\flat$; I_α, $I^\flat$; P_α, $P^\flat$ be associated with $\mathfrak{F}_\alpha$, $\mathfrak{F}^\flat$ respectively.</u>

Then $P^{\flat} = \lim_{\alpha \in A} P_\alpha$ *in the strong operator topology,* $I^{\flat} = c\ell(\bigcup_{\alpha \in A} I_\alpha)$, *and* $R = \bigcap_{\alpha \in A} R_\alpha$.

Proof. Let $\mu \in R^{\flat}$, then μ is concentrated on some $B \in \mathfrak{F}^{\flat}$, and so $\mu \in \bigcap_{\alpha \in A} R_\alpha$. Conversely suppose $\mu \in \bigcap_{\alpha \in A} R_\alpha$ and $\varepsilon > 0$. Choose $\alpha_1, \alpha_2, \cdots$ a cofinal sequence in A. Pick $K_1 \in \mathfrak{F}_{\alpha_1}$ to be a compact subset of G such that $||\mu|K_1 - \mu|| < \varepsilon/2$. Suppose $K_1, K_2, \cdots, K_n$ have been chosen. Now pick $K_{n+1} \in \mathfrak{F}_{\alpha_{n+1}}$ to be a compact subset of G such that $K_{n+1} \subset K_n$ and $||\mu|K_{n+1} - \mu|| < \varepsilon/2 + \varepsilon/4 + \cdots + \varepsilon/2^{n+1}$. Let $K = \bigcap_{n=1}^{\infty} K_n$, a compact subset of G. Then $K \in \mathfrak{F}^{\flat}$ and $||\mu|K - \mu|| \leq \varepsilon$. Since $R^{\flat}$ is closed it follows that $\mu \in R^{\flat}$. Thus $R^{\flat} = \bigcap_{\alpha \in A} R_\alpha$.

Let $\mu \in M(G), \mu \geq 0$ and $\varepsilon > 0$. We wish to find $\alpha \in A$ such that $\beta \geq \alpha$ implies $||P_\beta(\mu) - P^{\flat}(\mu)|| < \varepsilon$. By induction there exists a monotone decreasing sequence of compact subsets $K_n \subset G$ such that $K_n \in \mathfrak{F}_{\alpha_n}$ and $||\mu|K_n - P_{\alpha_n}\mu|| < \varepsilon(1/2^2 + \cdots + 1/2^{n+1})$. Let $K = \bigcap_{n=1}^{\infty} K_n \in \mathfrak{F}^{\flat}$, then $\mu|K = P^{\flat}\mu|K \leq P^{\flat}\mu$. By the monotone convergence theorem choose n such that $||\mu|K_n - \mu|K|| < \varepsilon/2$. Thus for $\beta \geq \alpha_n$,

$$P^{\flat}(\mu) \leq P_\beta(\mu) \leq P_{\alpha_n}(\mu) = \mu|K_n + \nu$$

$= \mu|K + \nu' + \nu \leq P^{b}(\mu) + \nu' + \nu$

(where $\nu',\nu \in M(G)$ with $||\nu'||,||\nu|| < \varepsilon/2$), and so $||P^{b}(\mu) - P_{\beta}(\mu)|| < \varepsilon$. Hence $P_{\alpha} \overset{\alpha}{\to} P^{b}$ in the SO-topology.

Clearly $I^{b} \supset c\ell(\bigcup_{\alpha \in A} I_{\alpha})$, so let $\mu \in I^{b}$. Write $\mu = P_{\alpha}\mu + \nu_{\alpha}$ where $\nu_{\alpha} \in I_{\alpha}$. Since $\mu \in I^{b}$, $P^{b}(\mu) = 0$. Since $P_{\alpha}\mu \overset{\alpha}{\to} P^{b}\mu = 0$, $\nu_{\alpha} \overset{\alpha}{\to} \mu$ and so $I^{b} = c\ell(\bigcup_{\alpha \in A} I_{\alpha})$. $\square$

<u>THEOREM 3</u>. <u>Let</u> ε <u>be an idempotent in</u> $c\ell(\hat{G})$ <u>and suppose</u> ϕ <u>is a continuous function on</u> $\overline{H(\varepsilon)}$ $(\subset c\ell(\hat{G}))$. <u>Then</u> ϕ <u>is a restriction to</u> $\overline{H(\varepsilon)}$ <u>of the Gel'fand transform</u> $\tilde{\mu}$ <u>of a measure</u> $\mu \in E_{\varepsilon}(M(G))$ <u>if and only if there exists</u> $B < \infty$ <u>such that for</u>

$a_1,\cdots,a_n \in \mathbb{C}$ and $\gamma_1,\cdots,\gamma_n \in \hat{G}$,

$$(*) \quad |\textstyle\sum_{i=1}^{n} a_i \phi(\gamma_i \times \varepsilon)| \leq B||\textstyle\sum_{i=1}^{n} a_i\gamma_i||_{\infty}.$$

Proof. Let $\mu \in M(G)$, then for $a_1,\cdots,a_n \in \mathbb{C}$ and $\gamma_1,\cdots,\gamma_n \in \hat{G}$,

$$|\textstyle\sum_{i=1}^{n} a_i \tilde{\mu}(\gamma_i \times \varepsilon)| = |\int_G \textstyle\sum_{i=1}^{n} a_i\gamma_i dE_{\varepsilon}\mu|$$

$$\leq ||\textstyle\sum_{i=1}^{n} a_i\gamma_i||_{\infty}||E_{\varepsilon}\mu|| \quad .$$

Conversely, suppose ϕ satisfies (*). We define the continuous function $\psi: c\ell(\hat{G}) \to \mathbb{C}$ by $\psi(\tau) = \phi(\tau \times \varepsilon), \tau \in c\ell(\hat{G})$. Now for $a_1,\cdots,a_n \in \mathbb{C}$ and $\gamma_1,\cdots,\gamma_n \in \hat{G}$, we have

$$|\sum_{i=1}^{n} a_i \psi(\gamma_i)| = |\sum_{i=1}^{n} a_i \phi(\gamma_i \times \varepsilon)| \le B || \sum_{i=1}^{n} a_i \gamma_i ||_\infty$$

and so there exists by Eberlein's theorem [5,p.32] a measure $\mu \in M(G)$ with $\hat{\mu}(\gamma) = \psi(\gamma)$, $\gamma \in \hat{G}$. But then $\tilde{\mu} = \psi$ on $c\ell(\hat{G})$. Hence for $\gamma \in \hat{G}, \tilde{\mu}(\gamma\times\varepsilon) = \psi(\gamma\times\varepsilon) = \phi(\gamma\times\varepsilon\times\varepsilon) = \phi(\gamma\times\varepsilon)$, and so $\tilde{\mu} = \phi$ on $\overline{H(\varepsilon)}$. We now observe that $\mu \in E_\varepsilon(M(G))$ since for $\gamma \in \hat{G}$,

$$\hat{\mu}(\gamma) = \psi(\gamma) = \phi(\varepsilon\times\gamma) = \phi(\varepsilon\times\varepsilon\times\gamma) = \psi(\varepsilon\times\gamma)$$
$$= \tilde{\mu}(\varepsilon\times\gamma) = (E_\varepsilon \mu)\hat{}\,(\gamma). \square$$

DEFINITION: Let $\{\pi_\alpha\} \subset \Delta_G$ be a net and $\pi \in \Delta_G$. We say that $\{\pi_\alpha\}$ converges strongly to π if and only if $E_{\pi_\alpha} \overset{\alpha}{\to} E_\pi$ in the SO-topology. Note that strong convergence clearly implies convergence in Δ_G. Indeed strong convergence is equivalent to convergence in Δ_G for idempotents. (Let μ be a probability measure on G and $\varepsilon_\alpha \overset{\alpha}{\to} \varepsilon$ in Δ_G. Then $||E_{\varepsilon_\alpha}\mu - E_\varepsilon \mu|| = \int_G |f^\mu_{\varepsilon_\alpha} - f^\mu_\varepsilon| d\mu = \int_G (f^\mu_{\varepsilon_\alpha} - f^\mu_\varepsilon)^2 d\mu$ $= \tilde{\mu}(\varepsilon_\alpha) + \tilde{\mu}(\varepsilon) - 2\tilde{\mu}(\varepsilon_\alpha \times \varepsilon) \overset{\alpha}{\to} 0$, where $f^\mu_\pi \in L^\infty(\mu)$ is the generalized character corresponding $\pi \in \Delta_G$, [2].)

THEOREM 4. <u>Let</u> $\{\varepsilon_\alpha\}_{\alpha\in A} \subset c\ell(\hat{G})$ <u>be a net of LC-idempotents, such that</u> $\varepsilon_\alpha \ge \varepsilon_\beta$ <u>for</u> $\alpha,\beta \in A$ <u>with</u> $\alpha<\beta$. <u>Let</u> ε <u>be the limit point of</u> $\{\varepsilon_\alpha\}$, <u>and so</u> $\varepsilon \le \varepsilon_\alpha, \alpha \in A$. <u>Then</u>

$$E_\varepsilon(M(G)) = \bigcap_{\alpha \in A} E_{\varepsilon_\alpha}(M(G)).$$

Proof. Suppose $\mu \in E_\varepsilon(M(G))$ and so

$E_\varepsilon(\mu) = \mu$. Thus for each $\alpha \in A$,

$$E_{\varepsilon_\alpha}(\mu) = E_{\varepsilon_\alpha}(E_\varepsilon(\mu)) = E_\varepsilon(\mu) = \mu,$$

and so $\mu \in E_{\varepsilon_\alpha}(M(G))$.

Conversely, suppose $\mu \in \bigcap_{\alpha \in A} E_{\varepsilon_\alpha}(M(G))$, and so $E_{\varepsilon_\alpha}(\mu) = \mu, \alpha \in A$. Now since $\varepsilon_\alpha \overset{\alpha}{\to} \varepsilon$ in Δ_G, $\varepsilon_\alpha \overset{\alpha}{\to} \varepsilon$ strongly, and so $\mu = E_{\varepsilon_\alpha}(\mu) \overset{\alpha}{\to} E_\varepsilon(\mu)$. Thus $\mu = E_\varepsilon(\mu)$. □

Let G_d denote G with the discrete topology, and let $\varepsilon_o \in \Delta_G$ be the idempotent corresponding to the sup-norm bounded projection $P_o : M(G) \to M(G_d)$. (The idempotent ε_o is the identity in the Bohr group of $\hat{G}$, $\beta\hat{G}$, in $c\ell\hat{G}$.) Note $\beta\hat{G}$ is just the set of all characters on G. We sometimes identify $\beta\hat{G}$ with $H(\varepsilon_o)$.

For $\pi \in \Delta_G$, let $j\pi \in \beta\hat{G}$ be defined by $(j\pi)(x) = (\delta_x)^\sim(\pi)$ $(x \in G)$. The map $j : \Delta_G \to \beta\hat{G}$ is a semigroup morphism. Further for ε an idempotent in Δ_G the map j takes $H(\varepsilon)$ onto a dense subgroup of $\beta\hat{G}$ (see [2, Proposition 7]).

DEFINITION: Let ε_∞ be the idempotent in $c\ell\hat{G}$ which is the limit of the idempotents $\{\varepsilon_\alpha\}$ where $\{\varepsilon_\alpha\}$ corresponds to a decreasing net of LC-topologies for the group G. Then we call ε_∞ an inductive limit idempotent.

PROPOSITION 5. _Let_ ε_∞ _be an inductive limit idempotent in_ $c\ell\hat{G}$. _Then the map_ $j : H(\varepsilon_\infty) \to \beta\hat{G}$ _is one-to-one._

Proof. Let $\sigma, \sigma' \in H(\varepsilon_\infty)$ and $j\sigma = j\sigma'$. We will show that $\sigma = \sigma'$. Now $j(\varepsilon_\alpha \times \sigma) = j(\varepsilon_\alpha)j(\sigma) = j(\varepsilon_\alpha)j(\sigma') = j(\varepsilon_\alpha \times \sigma')$. Since j is one-to-one on $H(\varepsilon_\alpha)$, $\varepsilon_\alpha \times \sigma = \varepsilon_\alpha \times \sigma'$. Now as $\varepsilon_\alpha \overset{\alpha}{\to} \varepsilon_\infty$, $\varepsilon_\alpha \times \sigma \overset{\alpha}{\to} \varepsilon_\infty \times \sigma$ and $\varepsilon_\alpha \times \sigma' \to \varepsilon_\infty \times \sigma'$. Thus $\sigma = \varepsilon_\infty \times \sigma = \varepsilon_\infty \times \sigma' = \sigma'$. □

THEOREM 6. Let ε_∞ be an inductive limit idempotent in $c\ell\hat{G}$, then $\varepsilon_o \times H(\varepsilon_\infty) = \bigcap_{\alpha \in A} \varepsilon_o \times H(\varepsilon_\alpha)$.

Proof. Let $\pi \in H(\varepsilon_\infty)$. Put $\pi_\alpha = \pi \times \varepsilon_\alpha$, then $\pi_\alpha \in H(\varepsilon_\alpha)$ (recall that $\varepsilon_\alpha \leq \varepsilon_\infty$ for all α). Now $(j\pi_\alpha)(x) = (j\pi)(x)(j\varepsilon_\alpha)(x) = (j\pi)(x)$ $(x \in G)$ since $j\varepsilon_\alpha = 1$. Thus $\varepsilon_o \times \pi_\alpha = \varepsilon_o \times (\pi \times \varepsilon_\alpha) = (\varepsilon_o \times \varepsilon_\alpha) \times \pi = \varepsilon_o \times \pi$. Therefore $\varepsilon_o \times H(\varepsilon_\infty) \subset \bigcap_{\alpha \in A} \varepsilon_o \times H(\varepsilon_\alpha)$.

Conversely, let $\pi \in \bigcap_{\alpha \in A} \varepsilon_o \times H(\varepsilon_\alpha)$.. For each $\alpha \in A$, there exists $\gamma_\alpha \in H(\varepsilon_\alpha)$ such that $\gamma_\alpha \times \varepsilon_o = \pi$. Indeed γ_α is unique in $H(\varepsilon_\alpha)$, for if $\gamma'_\alpha \in H(\varepsilon_\alpha)$ such that $\gamma'_\alpha \times \varepsilon_o = \pi$, then $j(\gamma_\alpha) = \gamma_\alpha \times \varepsilon_o = \gamma'_\alpha \times \varepsilon_o = j(\gamma'_\alpha)$, and so $\gamma_\alpha = \gamma'_\alpha$ (as a function on G) and so $\gamma_\alpha = \gamma'_\alpha$ in Δ_G. Thus for $\alpha < \beta, \gamma_\beta \times \varepsilon_\alpha = \gamma_\alpha$ since $\gamma_\beta \times \varepsilon_\alpha \in H(\varepsilon_\alpha)$ and $\gamma_\beta \times \varepsilon_\alpha \times \varepsilon_o = \gamma_\beta \times \varepsilon_o = \pi$. Let σ be a cluster point of $\{\gamma_\alpha\}$. Now $\gamma_\alpha = \gamma_\alpha \times \varepsilon_\alpha = \gamma_\alpha \times (\varepsilon_\alpha \times \varepsilon_\infty) = (\gamma_\alpha \times \varepsilon_\alpha) \times \varepsilon_\infty = \gamma_\alpha \times \varepsilon_\infty$. Thus $\sigma = \sigma \times \varepsilon_\infty$. Fix $\alpha \in A$. For $\beta > \alpha$, $\overline{\gamma}_\alpha \times \gamma_\beta = (\overline{\gamma}_\alpha \times \varepsilon_\alpha) \times \gamma_\beta = \overline{\gamma}_\alpha \times (\varepsilon_\alpha \times \gamma_\beta) = \overline{\gamma}_\alpha \times \gamma_\alpha = \varepsilon_\alpha$. Thus $\overline{\gamma}_\alpha \times \sigma = \varepsilon_\alpha$. Let τ be a cluster point of $\{\overline{\gamma}_\alpha\}$. Then $\tau \times \sigma = \varepsilon_\infty$, and so $\sigma \in H(\varepsilon_\infty)$.

Finally, recall that $\gamma_\alpha \times \varepsilon_o = \pi$, and so $\sigma \times \varepsilon_o = \pi$. Thus $\pi \in \varepsilon_o \times H(\varepsilon_\infty)$. □

REMARK. Let $X: G^{\#} \to T$ be a (not necessarily continuous) character on $G^{\#}$ where $G^{\#}$ is the inductive limit of the LC-topologies corresponding to $\{\varepsilon_\alpha\}$. Then X is $\mathcal{T}^{\#}$-continuous if and only if X is continuous for each LC-topology from the inductive limit. Thus $(G^{\#})^{\wedge} = \bigcap_{\alpha \in A} (G_\alpha)^{\wedge} = H(\varepsilon_\infty)$, as a group of characters.

THEOREM 7. <u>Let</u> ε_∞ <u>be an inductive limit idempotent in</u> $c\ell\hat{G}$, <u>then the set</u> $H(\varepsilon_\infty)^{\wedge}$ <u>of continuous characters on</u> $H(\varepsilon_\infty)$ <u>is algebraically isomorphic to the group</u> G.

Proof. Let $\rho_\pi : \Delta_G \to \Delta_G$ $(\pi \in \Delta_G)$ be defined by $\rho_\pi(\tau) = \pi \times \tau$, $\tau \in \Delta_G$. The morphism ρ_{ε_o} takes $H(\varepsilon_\infty)$ onto a dense subgroup of $\beta\hat{G}$ and so $G \subset H(\varepsilon_\infty)^{\wedge}$.

Conversely, the morphism $\rho_{\varepsilon_\infty}$ takes $H(1) = \hat{G}$ onto a dense subgroup of $H(\varepsilon_\infty)$. Thus $H(\varepsilon_\infty)^{\wedge} \subset G^{\wedge\wedge} = G$. □

THEOREM 8. <u>Let</u> ε_∞ <u>be an inductive limit idempotent in</u> $c\ell\hat{G}$, <u>then on</u> $H(\varepsilon_\infty)$ <u>there exists three equivalent group topologies</u>:

(1) $\mathcal{T}_w$, <u>the relative topology from</u> Δ_G,

(2) $\mathcal{T}^b$, <u>the projective limit topology on</u> $H(\varepsilon_\infty)$ <u>given by the system</u> $\{H(\varepsilon_\alpha) : \alpha \in A\}$, <u>and</u> (3) $\mathcal{T}_K$, <u>the topology of uniform convergence on the</u> $\mathcal{T}_\alpha$ <u>-compact subsets of</u> G

viewing $H(\varepsilon_\infty)$ *as a subgroup of characters on* G *under the map* j.

Proof. Since $\varepsilon_\alpha \le \varepsilon_\infty$, $\rho_{\varepsilon_\alpha}$ takes $H(\varepsilon_\infty)$ onto a subgroup of $H(\varepsilon_\alpha)$. The topology is the weakest (group) topology such that the maps $\rho_{\varepsilon_\alpha}: H(\varepsilon_\infty) \to H(\varepsilon_\alpha)$ is continuous (where $H(\varepsilon_\alpha)$ has the relative topology from Δ_G, that is, $H(\varepsilon_\alpha) \simeq \hat{G}_\alpha$). Let $\{\sigma_\iota\}$ be a net from $H(\varepsilon_\infty)$ and let $\sigma \in H(\varepsilon_\infty)$. The net $\sigma_\iota \xrightarrow{\mathfrak{I}} \sigma$ in Δ_G if and only if $\sigma_\iota \times \varepsilon_\alpha \xrightarrow{\mathfrak{I}} \sigma \times \varepsilon_\alpha$ for each α; since $|\tilde{\mu}(\sigma_\iota) - \tilde{\mu}(\sigma)| \le 2||E_{\varepsilon_\alpha}\mu - E_{\varepsilon_\infty}\mu|| + |\tilde{\mu}(\varepsilon_\alpha \times \sigma_\iota) - \tilde{\mu}(\varepsilon_\alpha \times \sigma)|$, $(\mu \in M(G))$. Thus (1) and (2) are equivalent.

That $\mathfrak{T}_\kappa \supset \mathfrak{T}^b$ is clear since $\mathfrak{T}_\kappa$ is a group topology for which $H(\varepsilon_\infty) \to H(\varepsilon_\alpha)$ is continuous. Conversely, let $\{\sigma_\iota\}$ be a net in Δ_G which converges in $\mathfrak{T}^b$ to σ. Thus $\varepsilon_\alpha \times \sigma_\iota \xrightarrow{\mathfrak{I}} \varepsilon_\alpha \times \sigma$ in $H(\varepsilon_\alpha)$ for each α, and so in $\mathfrak{T}_\kappa$. $\square$

DEFINITION: Let $L^*(G)$ denote the closed subalgebra of $M(G)$ generated by the set of $L^1(G, \mathfrak{T}_1)$ where $\mathfrak{T}_1$ is an LC-topology finer than $\mathfrak{T}$ (the given topology on G). We call $L^*(G)$ the maximal group algebra.

REMARK. The algebra $L^*(G)$ has the form $\sum \oplus L^1(G, \mathfrak{T}_1)$, where $\mathfrak{T}_1$ is an LC-topology finer than the given topology on G (see Varopoulos [6, p. 503]). The maximal ideal space Δ^* of $L^*(G)$ has been characterized by Varopoulos [6,p.507], and using this characterization we have that Δ^* is equal to

$$\bigcup\{H(\varepsilon):\varepsilon \text{ is an LC-idempotent}\} \cup \bigcup\{H(\varepsilon_\infty):\varepsilon_\infty \text{ is an inductive limit idempotent}\}.$$

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