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FOURIER-STIELTJES TRANSFORMS AND WEAKLY ALMOST PERIODIC FUNCTIONALS FOR COMPACT GROUPS

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Let G be a compact group and H a closed subgroup. A function in the Fourier algebra of H can be extended to a function in the Fourier algebra of G without increase in norm and with an arbitrarily small increase in sup-norm. For G a compact Lie group, the space of Fourier-Stieltjes transforms is not dense in the space of weakly almost periodic functionals on the Fourier algebra of G.

We let G denote an infinite compact group and $\hat{G}$ its dual. We use the notation of [1, Chapters 7 and 8], [2], and [3]. Recall $A(G)$ denotes the Fourier algebra of G (an algebra of continuous functions on G), and $\mathscr{L}^\infty(\hat{G})$ denotes its dual space under the pairing $\langle f, \phi \rangle$ ($f \in A(G)$, $\phi \in \mathscr{L}^\infty(\hat{G})$). Further, note $\mathscr{L}^\infty(\hat{G})$ is identified with the C^*-algebra of bounded operators on $L^2(G)$ commuting with right translation. The module action of $A(G)$ on $\mathscr{L}^\infty(\hat{G})$ is defined by the following: for $f \in A(G)$, $\phi \in \mathscr{L}^\infty(\hat{G})$, $f \cdot \phi \in \mathscr{L}^\infty(\hat{G})$ by $\langle g, f \cdot \phi \rangle = \langle fg, \phi \rangle$, $g \in A(G)$. Also $\|f \cdot \phi\|_\infty \leqq \|f\|_A \|\phi\|_\infty$.

Let $\phi \in \mathscr{L}^\infty(\hat{G})$. We call ϕ a weakly almost periodic functional if and only if the map $f \mapsto f \cdot \phi$ from $A(G)$ to $\mathscr{L}^\infty(\hat{G})$ is a weakly compact operator. The space of all such is denoted by $W(\hat{G})$.

Let $M(G)$ denote the measure algebra of G. For $\mu \in M(G)$, the Fourier-Stieltjes transform of μ, $\mathscr{F}\mu$, is a matrix-valued function in $\mathscr{L}^\infty(\hat{G})$ defined for $\alpha \in \hat{G}$ by

$$\alpha \mapsto (\mathscr{F}\mu)_\alpha = \int_G T_\alpha(x^{-1})\, d\mu(x) \ \ (T_\alpha \in \alpha) .$$

We denote the closure of $\mathscr{F}M(G)$ in $\mathscr{L}^\infty(\hat{G})$ by $\mathscr{M}(\hat{G})$. In [2], we showed that $W(\hat{G})$ is a closed subspace of $\mathscr{L}^\infty(\hat{G})$, and that $\mathscr{M}(\hat{G}) \subset W(\hat{G})$ with the inclusion proper when G is a direct product of an infinite collection of nontrivial compact groups. In this paper, we show the inclusion is proper for all compact Lie groups.

We first state a standard lemma.

LEMMA 1. *Let A, B be compact subsets of a topological group G. Suppose $AB \subset U$, U an open subset of G. Then there is an open neighborhood V of the identity e of G such that $AVB \subset U$.*

PROPOSITION 2. *Let G be a compact group and H a closed subgroup. Let W be an open subset of G with $H \cap \bar{W} = \emptyset$. Then there is a continuous positive definite function p on G with $p(x) = 1$, $x \in H$, and $p(x) = 0$, $x \in W$.* (Note $p \in A(G)$ and $\|p\|_A = 1$.)

Proof. Let U be an open subset of G with $H \subset U$, and $U \cap W = \emptyset$. Choose V_1 an open neighborhood of e with $HV_1H \subset U$. Now let V be an open neighborhood of e with $VV \subset V_1$ and $V = V^{-1}$. Thus $HVVH \subset HV_1H \subset U$.

Let $p = (m_G(HV))^{-1} \chi_{HV} * \chi_{VH}$ (m_G is normalized Haar measure on G and χ_A denotes the characteristic function of A). Then $p(x) = (m_G(HV))^{-1} m_G(xHV \cap HV)$, $x \in G$. Thus for $x \in H$, $p(x) = 1$. If $p(x) \neq 0$, then $x\ HV \cap HV \neq \emptyset$, and so $x \in HVVH \subset U$.

THEOREM 3. *Let G be a compact group and H a closed subgroup. Let $f \in A(H)$ and $\varepsilon > 0$. Then there exists $g \in A(G)$, $\|g\|_A = \|f\|_A$, $g|H = f$, and $\|g\|_\infty \leqq \|f\|_\infty + \varepsilon$.*

Proof. Let h be an extension of f to G with $\|h\|_A = \|f\|_A$ (see [1, Chapter 8]). Let $V = \{x \in G\colon |h(x)| > \|f\|_\infty + \varepsilon\}$. Now let p be as in Proposition 2, and let $g = ph$.

We now state a characterization of $\mathscr{M}(\hat{G})$. The proof for abelian groups is in [1, Chapter 3]. The proof for nonabelian groups is analogous.

THEOREM 4. *Let G be a compact group and $\phi \in \mathscr{L}^\infty(\hat{G})$. For $\phi \in \mathscr{M}(\hat{G})$ it is necessary and sufficient that whenever $\{f_n\}$ is a sequence from $A(G)$ with $\|f_n\|_A \leqq 1$ and $\|f_n\|_\infty \xrightarrow{n} 0$ we have $\langle f_n, \phi\rangle \xrightarrow{n} 0$.*

THEOREM 5. *Let G be a compact Lie group. Then $\mathscr{M}(\hat{G}) \neq W(\hat{G})$.*

Proof. Let H be a total subgroup of G; that is, H is the circle group. Now $\mathscr{M}(\hat{H}) \neq W(\hat{H})$, (see [1, Chapter 4]).

Let π_1 denote the restriction map of $A(G)$ onto $A(H)$ and let $\hat{\pi}$ denote the adjoint map of $\mathscr{L}^\infty(\hat{H})$ into $\mathscr{L}^\infty(\hat{G})$. In [3], we showed that

$$\hat{\pi}\mathscr{M}(\hat{H}) \subset \mathscr{M}(\hat{G}) \text{ and } \hat{\pi}W(\hat{H}) \subset W(\hat{G})\,.$$

Let $\phi \in W(\hat{H}) \backslash \mathscr{M}(\hat{H})$. Now $\hat{\pi}\phi \in W(\hat{G})$ so we need only show that $\hat{\pi}\phi \notin \mathscr{M}(\hat{G})$. Since $\phi \notin \mathscr{M}(\hat{H})$, there is a sequence $\{f_n\} \subset A(H)$, $\|f_n\|_A \leqq 1$, $\|f_n\|_\infty \xrightarrow{n} 0$ with $|\langle f_n, \phi\rangle| \geqq \varepsilon$ (some $\varepsilon > 0$). Extend f_n to $g_n \in$

$A(G)$ by Theorem 3 with $\|g_n\|_A \leqq 1$ and $\|g_n\|_\infty \xrightarrow{n} 0$. But $\langle g_n, \tilde{\pi}\phi \rangle = \langle \pi_1 g_n, \phi \rangle = \langle f_n, \phi \rangle$, and so $\tilde{\pi}\phi \notin \mathscr{M}(\hat{G})$.

REMARK. If a compact group G has a closed subgroup H with $\mathscr{M}(\hat{H}) \neq W(\hat{H})$, then $\mathscr{M}(\hat{G}) \neq W(\hat{G})$, (in particular, if G contains an infinite abelian subgroup). Indeed, it is an open question whether an infinite compact group always contains an infinite abelian subgroup.

COROLLARY 6. *Let G be a compact group with H a closed subgroup. Then*

$$\tilde{\pi}(W(\hat{H}) \backslash \mathscr{M}(\hat{H})) \subset W(\hat{G}) \backslash \mathscr{M}(\hat{G}) .$$

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