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*72T-B96. CHARLES F. DUNKL and DONALD E. RAMIREZ, University of Virginia, Charlottesville. Virginia 22903. Characterizations of Sidon sets and $\Lambda_p$ sets on compact groups.

For a compact group G with dual $\hat{G}$, a set $E \subset \hat{G}$ is a Sidon set if and only if any unitary operator valued function on E can be interpolated almost surely by a Fourier-Stieltjes transform. Further, E is a $\Lambda_p$



set for $p > 2$ if and only if any unitary operator valued function on E can be interpolated by an element of the multiplier algebra of $L^p(G)$. (Received January 10, 1972.)

Characterizations of Sidon Sets

and $\Lambda_p$ Sets on Compact Groups

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Abstract

For a compact group $G$ with dual $\hat{G}$, a set $E \subset \hat{G}$ is a Sidon set if and only if any unitary operator valued function on E can be interpolated almost surely by a Fourier-Stieltjes transform. Further, E is a $\Lambda_p$ set for $p > 2$ if and only if any unitary operator valued function on E can be interpolated by an element of the multiplier algebra of $L^p(G)$.

Characterizations of Sidon Sets and $\Lambda_p$ Sets on Compact Groups

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Let G be an infinite compact group and $\hat{G}$ its dual (we will use the notation of [1, Chapters 7 and 8]). For $\alpha \in \hat{G}$, let $T_\alpha \in \alpha$. Then $T_\alpha$ is a continuous homomorphism of G into $\underset{\sim}{U}(n_\alpha)$, the group of $n_\alpha \times n_\alpha$ unitary matrices. We use $T_\alpha(x)_{ij}$ to denote the matrix entries of $T_\alpha(x)$, $1 \leq i, j \leq n_\alpha$. Let $\chi_\alpha(x) = \mathrm{Tr}(T_\alpha(x))$ (Tr = trace). We call $\chi_\alpha$ the character of $\alpha$. Let $\phi$ be a set $\{\phi_\alpha : \alpha \in \hat{G}$ where $\phi_\alpha \in \mathcal{B}(\underset{\sim}{C}^{n_\alpha})\}$ be such that $\sup\{||\phi_\alpha||_\infty : \alpha \in \hat{G}\} < \infty$ where $||\cdot||_\infty$ denotes the operator norm. The set of all such $\phi$ is denoted by $\mathcal{L}^\infty(\hat{G})$. It is a Banach algebra under the norm $||\phi||_\infty = \sup\{||\phi_\alpha||_\infty : \alpha \in \hat{G}\}$ and coordinatewise operations. For $\mu \in M(G)$, the Fourier-Stieltjes transform of $\mu$, $\hat{\mu}$, is a matrix-valued function defined for $\alpha \in \hat{G}$ by

$$\alpha \longmapsto \hat{\mu}_\alpha = \int_G T_\alpha(x^{-1})\, d\mu(x).$$

Note that $\hat{\mu} \in \mathcal{L}^\infty(\hat{G})$. We denote the center of M(G) by $\mathfrak{Z}M(G)$. Note that for $\mu \in M(G)$, $\mu \in \mathfrak{Z}M(G)$ if and only if $\hat{\mu}_\alpha = c_\alpha I_{n_\alpha}$, when $I_{n_\alpha}$ denotes the identity operator on $\underset{\sim}{C}^{n_\alpha}$.

Let $E \subset \hat{G}$. We say that E is a Sidon set if and only if given any $\phi \in \mathcal{L}^\infty(\hat{G})$, there is $\mu \in M(G)$ such that $\hat{\mu}_\alpha = \phi_\alpha$ for $\alpha \in E$. We say that E is a central Sidon set if and only if given any $\phi \in \mathfrak{Z}\mathcal{L}^\infty(\hat{G})$, (the center of $\mathcal{L}^\infty(\hat{G})$) there is $\mu \in \mathfrak{Z}M(G)$ such that $\hat{\mu}_\alpha = \phi_\alpha$ for $\alpha \in E$.

Following Rudin [6, p. 121] one can show that $E \subset \hat{G}$ is a Sidon set if any only if given any $\phi \in \mathcal{K}^{\infty}(\hat{G})$ where $\phi_\alpha$ is unitary and hermitian for $\alpha \in E$, there is $\mu \in M(G)$ such that $\sup\{||\phi_\alpha - \hat{\mu}_\alpha||_\infty : \alpha \in E\} < 1$. Also $E \subset G$ is a central Sidon set if and only if given any $\phi \in \mathcal{Z}\mathcal{K}^{\infty}(\hat{G})$ where $\phi_\alpha$ is unitary and hermitian (and thus $\phi_\alpha = \pm I_{n_\alpha}$), there is $\mu \in M(G)$ such that $\sup\{||\phi_\alpha - \hat{\mu}_\alpha||_\infty : \alpha \in E\} < 1$ (for example, see [4, p. 448] or [5]).

For $E \subset \hat{G}$, we let $\Omega(E) = \Pi_{\alpha \in E} \underset{\sim}{U}(n_\alpha)$ with its natural topology and probability measure. We now are able to state precisely our first result.

Theorem 1: Let $G$ be an infinite compact group and $E \subset \hat{G}$. For $E$ to be a Sidon set it is necessary and sufficient that there is a Borel set of positive measure $S \subset \Omega(E)$ such that for $\omega \in S$, there is a $\mu \in M(G)$ such that $\hat{\mu}|E = \omega$.

Proof: If $E$ is a Sidon set, then we can interpolate always and so the condition is satisfied. Now suppose we can interpolate with a positive probability.

Let $B$ be the unit ball in $M(G)$. Now $B$ is weak-$*$ compact. The topology we need to consider on $M(G)$ is the topology $\mathcal{T}$ of pointwise convergence on $\hat{G}$ of the Fourier-Stieltjes transforms. Now since $G$ is compact, the weak-$*$ topology is equivalent to $\mathcal{T}$ on $B$. It follows that $M(G)$ is $\sigma$-compact in $\mathcal{T}$ and that $\Lambda = (M(G)^\wedge|E) \cap \Omega$ is a measurable subset of $\Omega$. Since $\Lambda$ is a subgroup with positive measure it is open (by the Steinhaus theorem). But open subgroups are closed and since $\Lambda$ is dense in $\Omega$ it is all of $\Omega$. We are done now by the previous quoted result of Rudin. □

Remark: The zero-one law guarantees us that $(M(G)\hat{}\,|E) \cap \Omega$ has measure either zero or one.

We can use the method of proof to arrive at other similar results. We state one now as an example.

Corollary 2: Let G be an infinite compact group and $E \subset \hat{G}$. For E to be a central Sidon set it is necessary and sufficient that there is a Borel set of positive measure $S \subset \Omega = \Pi_{\alpha \varepsilon E} \{-1,1\}_\alpha$ such that for $\omega \varepsilon S$, there is a $\mu \varepsilon M(G)$ such that $\hat{\mu}|E = \omega$.

Let $E \subset \hat{G}$. We say that E is a $\Lambda_p$ set $(1 \leq p < \infty)$ if and only if $M_E(G) = \{\mu \varepsilon M(G) : \hat{\mu}_\alpha = 0 \text{ for } \alpha \notin E\}$ is setwise equal to $L_E^p(G) = \{f \varepsilon L^p(G) : \hat{f}_\alpha = 0 \text{ for all } \alpha \notin E\}$, (see [4, p. 420]). Let $M^p$ denote the subset of $\mathscr{L}^\infty(\hat{G})$ such that $\hat{\phi f} \varepsilon L^p(G)$ for each $f \varepsilon L^p(G)$ where $\hat{\phi f}$ is defined by the Fourier series $\sum_{\alpha \varepsilon \hat{G}} n_\alpha \operatorname{Tr}(\phi_\alpha \hat{f}_\alpha T_\alpha(x))$. The set $M^p$ is a subalgebra of $\mathscr{L}^\infty(\hat{G})$ and is called the $L^p$ multiplier algebra (see [2]). We denote the set $\{\phi \varepsilon M^p : \phi_\alpha = 0 \text{ for } \alpha \notin E\}$ by $M_E^p$.

We now observe that $\Lambda_p$ sets have a characterization similar to Sidon sets.

Theorem 3: For $p > 2$, E $\hat{G}$ is a $\Lambda_p$ set if and only if for each $\omega \varepsilon \Omega(E) = \Pi_{\alpha \varepsilon E} \underset{\sim}{U}(n_\alpha)$ there is a $\phi \varepsilon M_E^p$ such that $\phi|E = \omega$.

Proof: Let E be a $\Lambda_p$ set $(p > 2)$, and let $\omega \varepsilon \Omega(E)$. Given $f \varepsilon L^p(G)$, we need to show that $\hat{\omega} f \varepsilon L_E^p(G) = L_E^2(G)$ (see [4, p. 420]). But $||\hat{\omega f}||_2 = ||\omega \hat{f}||_2 \leq ||\omega||_\infty \, ||\hat{f}||_2 = ||\omega||_\infty \, ||f||_2 < \infty$.

Now let $\Omega(E) = M_E^p$. To show that E is a $\Lambda_p$ set, it will suffice to show that $L_E^2(G) \subset L_E^p(G)$. Let $f \varepsilon L_E^2(E)$. There is an $\omega \varepsilon \Omega(E)$

such that $\hat{\omega} f \in L^p_E(G)$ (see [3]); and thus since $\omega^*$ is also in $\Omega(E) = M^p_E$, we have that $f = \hat{\omega}^* (\hat{\omega} f) \in L^p_E(G)$. □

We now prove a partial result like our previous probabilistic ones.

Theorem 4: Let G be a compact abelian group and let $E \subset \hat{G}$. For E to be a $\Lambda_p$ set $(2 < p < \infty)$ it is necessary and sufficient that there is a Borel set of positive measure $S \subset \Omega(E)$ such that for $\omega \in S$, there is a $\phi \in M^p$ such that $\phi | E = \omega$.

Proof: We first observe that $M^p_E \cap \Omega(E)$ is measurable. This follows as before since $M^p$ is a dual space (see [2]). Now repeat the proof of Theorem 1. □

We are not able to handle the nonabelian compact case since it does not seem to be known whether $M^p$ is closed under the adjoint operation of $\mathcal{L}^\infty(\hat{G})$. However, we could define central $\Lambda_p$ sets (see [5]) for which the result analogous to Corollary 2 holds.

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## Footnotes

AMS 1970 subject classification:

Primary 42A18, 42A36, 43A40.

Key words and phrases:

Sidon set, $\Lambda_p$ set, multiplier algebra.

This research was supported in part by NSF contract number GP 8981.