

RESEARCH ARTICLE

LOCALLY COMPACT SUBGROUPS
OF THE SPECTRUM OF THE MEASURE ALGEBRA

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The maximal ideal space Δ_G of the measure algebra $M(G)$ of a locally compact abelian group G is a compact commutative semitopological semigroup. In this paper a class of locally compact subgroups of the closure of G , the dual group of G , in Δ_G is characterized. Each such group is the dual of the abstract group G with some stronger locally compact topology than that of G . There is no more than one such group about any idempotent in the closure of \hat{G} . In a previous paper the authors showed that every stronger locally compact topology on G determines an idempotent in the closure of \hat{G} . In other words, an exact description is given of those idempotents in the closure of \hat{G} which are contained in locally compact maximal subgroups of Δ_G .

NOTATION. Denote the dual of an LCA (locally compact abelian) group G by \hat{G} . For LCA groups G and H , and a morphism (continuous homomorphism) $\phi: G \rightarrow H$, there exists the dual morphism $\hat{\phi}: \hat{H} \rightarrow \hat{G}$, defined by $\hat{\phi}\gamma(x) = \gamma(\phi x)$ for $x \in G$, $\gamma \in \hat{H}$. For G an LCA group, $M(G)$ denotes the measure algebra, and Δ_G denotes the maximal ideal space of $M(G)$. As

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usual, we identify Δ_G with the space of nonzero multiplicative linear functionals on $M(G)$ furnished with the Gelfand topology. Let $M_d(G)$ be the closed subalgebra of discrete measures and $L^1(G)$ the ideal of absolutely continuous measures. For $\mu \in M(G)$, $\hat{\mu}$ is the Fourier-Stieltjes transform given by $\hat{\mu}(\gamma) = \int_G \gamma d\mu$, and $\tilde{\mu}$ is the Gelfand transform, a continuous function on Δ_G . We will identify \hat{G} with a subset of Δ_G , so that $\tilde{\mu}(\gamma) = \hat{\mu}(\gamma)$ for $\mu \in M(G)$, $\gamma \in \hat{G}$. For $x \in G$, let δ_x be the unit mass at x . Generally we will use additive notation for G , and multiplicative notation for \hat{G} .

Henceforth G denotes a fixed LCA group.

We now recall some facts about Raikov systems.

DEFINITION 1. A nontrivial class \mathcal{F} of σ -compact subsets of G is called a Raikov system if

- 1) $A \in \mathcal{F}$, B a σ -compact subset of G and $B \subset A$ imply $B \in \mathcal{F}$,
- 2) if $\{A_n\}_{n=1}^{\infty} \subset \mathcal{F}$ then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$,
- 3) $A, B \in \mathcal{F}$ implies $A + B \in \mathcal{F}$, and
- 4) $A \in \mathcal{F}$ and $x \in G$ imply $A + x \in \mathcal{F}$.

THEOREM [5]. Given a Raikov system \mathcal{F} there exists a unique bounded algebraic projection P on $M(G)$ such that $PM(G) = \{\mu \in M(G): \mu \text{ is carried by some } A \in \mathcal{F}\}$ and the kernel of P , an ideal, is the set $\{\mu \in M(G): |\mu|_A = 0 \text{ for all } A \in \mathcal{F}\}$.

We will call such a projection a Raikov projection.

DEFINITION 2. Let $\phi: H \rightarrow G$ be a monomorphism. Let G_H be the group G topologized so that ϕH

has the H -topology and is open in G_H . Note G_H is LCA and the identity: $G_H \rightarrow G$ is continuous. Let P be the Raikov projection associated with the Raikov system generated by the G_H -compact sets. We say P is the Raikov projection induced by ϕ .

DEFINITION 3. Let $\phi: H \rightarrow G$ be a morphism. Define $\phi^*: M(H) \rightarrow M(G)$, a homomorphism, by $\phi^*\mu(E) = \mu(\phi^{-1}E)$ for each Borel set $E \subset G$, $\mu \in M(H)$. Equivalently $(\phi^*\mu)^\wedge(\gamma) = \hat{\mu}(\hat{\phi}\gamma)$, for $\gamma \in \hat{G}$. If ϕ is a monomorphism then ϕ^* is an isometric isomorphism of $M(H)$ into $M(G)$ (see [4] and [6]).

PROPOSITION 1. With the hypotheses of the two prior definitions, the range of P may be identified with $M(G_H)$. If further, ϕ maps H onto G , then ϕ^* is an isometric isomorphism of $M(H)$ onto $M(G_H)$. We now single out an important set of operators on $M(G)$.

DEFINITION 4. Say T is an R-homomorphism of $M(G)$ if T is a bounded nonzero homomorphism of $M(G)$ into itself, such that $(T\mu)|E = T(\mu|E)$ for $\mu \in M(G)$, E Borel $\subset G$. That is, T commutes with restriction to Borel sets. If T is a projection and an R-homomorphism, then T is called an R-projection.

PROPOSITION 2. If T is an R-homomorphism, and $x \in G$, then $T\delta_x = \chi_T(x)\delta_x$, where χ_T is some character (not necessarily continuous) of G . If further T is an R-projection, then $T\delta_x = \delta_x$.

Proof. For any $\mu \in M(G)$, $T\mu = T(\mu*\delta_0) = T\mu*T\delta_0$, but $T \neq 0$, so $T\delta_0 \neq 0$. For any $x \in G$, $T\delta_x| \{x\} = T\delta_x$, so there exists a complex number $\chi_T(x)$

such that $T\delta_x = \chi_T(x) \delta_x$. But since $T\delta_{x+y} = T(\delta_x * \delta_y) = T\delta_x * T\delta_y$, we have $\chi_T(x+y) = \chi_T(x) \chi_T(y)$, for all $x, y \in G$, and $\chi_T(0) \neq 0$ so χ_T is a character of G . If T is an R -projection, then $(\chi_T)^2 = \chi_T$, so $T\delta_x = \delta_x$, all $x \in G$. \square

COROLLARY 3. The set of R -homomorphisms is closed under composition, and is a semigroup with identity I (where I is the identity map on $M(G)$).

Proof. It suffices to observe, that $S \circ T(\delta_0) = S(\delta_0) = \delta_0$ so $S \circ T \neq 0$, for R -homomorphisms S and T . \square

We briefly sketch the details of the isomorphism between the semigroup $\hat{\mathcal{E}}$ of R -homomorphisms and the compact space Δ_G (see [2, Chapter 1]). For each $\tau \in \Delta_G$ there exists a unique generalized character $\{f_\tau^\mu\}$, where $f_\tau^\mu \in L^\infty(\mu)$ for each μ of the form $\mu = \exp|\nu|$, $\nu \in M(G)$; and for each $\lambda \in L^1(\mu)$ (that is, the space of elements of $M(G)$ which are absolutely continuous with respect to μ) we have the relation $\hat{\lambda}(\tau) = \int_G f_\tau^\mu d\lambda$. Observe that for each such μ , that $\tau|L^1(\mu)$ is a bounded linear functional and thus determines an element f_τ^μ of $L^\infty(\mu)$ with the required property. Note further that if μ_1 and μ_2 are both exponentials, and $\mu_1 \in L^1(\mu_2)$ then $f_\tau^{\mu_2} = f_\tau^{\mu_1} \mu_1$ -almost everywhere. One can also prove that for $\mu = \exp|\nu|$, $\nu \in M(G)$, that $f_\tau^\mu(x+y) = f_\tau^\mu(x) f_\tau^\mu(y)$ for $(x,y) \in G \times G$, $\mu \times \mu$ -almost everywhere.

Now to each $\tau \in \Delta_G$ we associate the R -homomorphism E_τ defined by $E_\tau \lambda = f_\tau^{\mu\lambda}$ (where $\lambda \in M(G)$),

and $\mu = \exp|\lambda|$.

Conversely, if E is an R -homomorphism, define $\tau \in \Delta_G$ by $\hat{\mu}(\tau) = (E\mu)^\wedge(1)$, ($\mu \in M(G)$), then it can be shown that $E = E_\tau$ (see [2, Chapter 1]). Now we give Δ_G the structure of a semigroup so that for $\sigma, \tau \in \Delta_G$, $\sigma \times \tau$ is described by the generalized character $f_\sigma^\mu f_\tau^\mu$; or equivalently, that $E_{\sigma \times \tau} = E_\sigma E_\tau$. Further it can be shown that multiplication (\times) is separately continuous on Δ_G , so Δ_G becomes a commutative compact semitopological semigroup with identity 1, (for a general reference see [1]). The R -projections correspond to idempotents in Δ_G . Note that \hat{G} is embedded homeomorphically into Δ_G , in fact, as a subgroup, since for $\gamma_1, \gamma_2 \in \hat{G}$ we have $\gamma_1 \times \gamma_2 = \gamma_1 \gamma_2$; the generalized character f_γ^μ for γ is exactly γ (as a function on G).

The space that will hold our interest is the closure of \hat{G} , $c\ell\hat{G}$, a compact subsemigroup of Δ_G . For B a set of symmetric maximal ideals of Δ_G , \bar{B} consists of all $\tau \in \Delta_G$ for which $|\hat{\mu}(\tau)| \leq \sup\{|\hat{\mu}(\pi)| : \pi \in B\}$; for if the inequality holds, τ can be extended to a multiplicative linear functional on $C(\bar{B})$ (since $M(G)^\wedge | \bar{B}$ is sup-norm dense in $C(\bar{B})$ by the Stone-Weierstrass theorem), and so $\tau \in \bar{B}$. Thus for $\tau \in \Delta_G$, $\tau \in c\ell\hat{G}$ if and only if $|\mu(\tau)| \leq \|\hat{\mu}\|_\infty$, for all $\mu \in M(G)$. An equivalent formulation is that $\|(E_\tau \mu)^\wedge\|_\infty \leq \|\hat{\mu}\|_\infty$, $\mu \in M(G)$; and to see this, observe that $(E_\tau \mu)^\wedge(\gamma) = \int \gamma f_\tau^\lambda d\mu = \hat{\mu}(\gamma \times \tau)$, for $\gamma \in \hat{G}$, $\mu \in M(G)$, $\lambda = \exp|\mu|$. Clearly $\sup\{|\hat{\mu}(\sigma)| : \sigma \in c\ell\hat{G}\} = \|\hat{\mu}\|_\infty$ and $\gamma \times \tau \in c\ell\hat{G}$ for all

$\gamma \in \hat{G}$ if and only if $\tau \in \text{cl}\hat{G}$.

Observe that any Raikov projection is an R-projection. However, Yu. Šreider [8] has constructed an R-projection which is not a Raikov projection. In our paper [3] we proved the following:

THEOREM 4. Let $\phi: H \rightarrow G$ be a monomorphism, and let P be the Raikov projection induced by ϕ . Then $\|(P\mu)^\wedge\|_\infty \leq \|\hat{\mu}\|_\infty$. Thus $P = E_\varepsilon$, for $\varepsilon \in \text{cl}\hat{G}$.

We will describe how Theorem 4 gives the existence of LCA subgroups of $\text{cl}\hat{G}$, and then characterize the LCA subgroups of $\text{cl}\hat{G}$ which arise this way.

For further illustrations of the theory of R-projections we give the following:

PROPOSITION 5. If P is an R-projection and $PL^1(G) \neq \{0\}$, then $P = I$.

Proof. Write $P = E_\varepsilon$, ε an idempotent in Δ_G . Then for any $\mu \in M(G)$, $(P\mu)^\wedge(\gamma) = \hat{\mu}(\gamma \times \varepsilon)$, but if $\mu \in L^1(G)$, then $\hat{\mu} = 0$ off \hat{G} . (Note that \hat{G} is the spectrum of $L^1(G)$; see [7, p. 7].) Hence if $PL^1(G) \neq \{0\}$, there exists $\gamma_1, \gamma_2 \in \hat{G}$ such that $\gamma_1 \times \varepsilon = \gamma_2$, but then $\varepsilon = \overline{\gamma_1 \gamma_2} \in \hat{G}$, so $\varepsilon = 1$. \square

Recall from Proposition 2 that if P is an R-projection then P is the identity on $M_d(G)$.

EXAMPLE. There exists an LCA group G , and a bounded algebraic projection P on $M(G)$, such that $PL^1(G) = \{0\}$, P is the identity on $M_d(G)$,

$\|(P\mu)^\wedge\|_\infty \leq \|\hat{\mu}\|_\infty$, all $\mu \in M(G)$, but P is not an R-projection.

Proof. Let G be an LCA group having subgroups H and $K \neq \{0\}$ such that H has its own LCA

topology, and K is compact, $K \subset H$ and such that the injection maps $K \rightarrow H \rightarrow G$ are continuous, and not open. Let P_K be the Raikov projection $M(G) \rightarrow M(G_K)$ induced by $K \rightarrow G$, and let P_H be the Raikov projection $M(G) \rightarrow M(G_H)$ induced by $H \rightarrow G$. Let $T: M(G) \rightarrow M(G)$ be defined by

$$T\mu = (P_H\mu) * m_K + (\delta_o - m_K) * P_K\mu, \text{ for } \mu \in M(G),$$

where m_K is the normalized Haar measure on K . A direct computation shows that T is a projection on $M(G)$ (note that $P_H m_K = P_K m_K = m_K$).

The Fourier-Stieltjes transform of $T\mu$ is given by

$$(T\mu)^\wedge(\gamma) = \begin{cases} (P_H\mu)^\wedge(\gamma), & \gamma \in K^\perp \\ (P_K\mu)^\wedge(\gamma), & \gamma \notin K^\perp \end{cases}$$

(where K^\perp is the annihilator of K in \hat{G}). Thus $\|(T\mu)^\wedge\|_\infty \leq \|\hat{\mu}\|_\infty$, and T is an algebraic projection, which is the identity on $M_d(G)$ and $TL^1(G) = \{0\}$. To show that T is not an R -projection it suffices to show $T(\gamma\mu) \neq \gamma T\mu$, for some $\gamma \in \hat{G}$, $\mu \in M(G)$, (since R -projections commute with multiplication by bounded continuous functions). Let $\gamma \notin K^\perp$, and let $\mu \in L^1(G_H)$ such that $\hat{\mu}(1) \neq 0$, then $(T(\gamma\mu))^\wedge(\bar{\gamma}) = (P_K(\gamma\mu))^\wedge(\bar{\gamma}) = 0$ but $(\gamma T\mu)^\wedge(\bar{\gamma}) = (T\mu)^\wedge(1) = (P_H\mu)^\wedge(1) = \hat{\mu}(1) \neq 0$. \square

DEFINITION 5. Let ε be an idempotent in $c\ell\hat{G}$. Then the maximal subgroup of $c\ell\hat{G}$ containing ε , denoted by $H(\varepsilon)$, is the set $\{\sigma \in c\ell\hat{G}: \sigma \times \varepsilon = \sigma, \text{ and there exists } \tau \in c\ell\hat{G} \text{ such that } \sigma \times \tau = \varepsilon\}$. It can be shown that $\sigma \in H(\varepsilon)$ implies $|f_\sigma^\mu| = f_\varepsilon^\mu$, for

all $\mu = \exp|\lambda|$, $\lambda \in M(G)$; and for $\{\sigma_\alpha\}$ a net in $H(\epsilon)$, $\sigma_\alpha \xrightarrow{\alpha} \sigma \in H(\epsilon)$ (in the Δ_G topology) if and only if $\int_G |f_{\sigma_\alpha}^\mu - f_\sigma^\mu| d\mu \xrightarrow{\alpha} 0$ for all $\mu = \exp|\lambda|$, $\lambda \in M(G)$. Thus $H(\epsilon)$ has jointly continuous multiplication, and is thus a topological group.

DEFINITION 6. For ϵ an idempotent in $c\ell\hat{G}$, let ρ_ϵ be the continuous map $\gamma \mapsto \gamma \times \epsilon$ for $\gamma \in \hat{G}$.

PROPOSITION 6. For ϵ an idempotent in $c\ell\hat{G}$, ρ_ϵ is a continuous monomorphism of \hat{G} with dense range in $H(\epsilon)$. Further $c\ell H(\epsilon) = \epsilon \times c\ell\hat{G} = \{\sigma \in c\ell\hat{G}: \sigma \times \epsilon = \sigma\}$.

Proof. Since ϵ is an idempotent, ρ_ϵ is a continuous homomorphism. Now suppose for some $\gamma \in \hat{G}$ that $\epsilon \times \gamma = \epsilon$, then for each $x \in G$, $(\delta_x)^\wedge(\epsilon \times \gamma) = (E_\epsilon \delta_x)^\wedge(\gamma) = (\delta_x)^\wedge(\gamma) = \gamma(x)$ and $(\delta_x)^\wedge(\epsilon \times \gamma) = (\delta_x)^\wedge(\epsilon) = 1$; hence $\gamma = 1$ and ρ_ϵ is a monomorphism. Finally $\epsilon \times \hat{G} \subset H(\epsilon) \subset \epsilon \times c\ell\hat{G}$, and $c\ell(\epsilon \times \hat{G}) = \epsilon \times c\ell\hat{G}$. \square

PROPOSITION 7. Let ϵ be an idempotent in $c\ell\hat{G}$, then there exists a morphism $j: H(\epsilon) \rightarrow \beta\hat{G}$, the Bohr compactification of \hat{G} , which is identified with $(G_d)^\wedge$, (G_d is G with the discrete topology). Further $j \circ \rho_\epsilon = i$, the canonical injection of \hat{G} into $\beta\hat{G}$; so j has dense range.

Proof. For each $\sigma \in H(\epsilon)$, define a character on G (hence an element of $\beta\hat{G}$) by $j\sigma(x) = (\delta_x)^\wedge(\sigma)$, $x \in G$; ($j\sigma$ is a character by the argument of Proposition 2). Further if a net $\{\sigma_\alpha\} \subset H(\epsilon)$ converges to $\sigma \in H(\epsilon)$, then for each x , $j\sigma_\alpha(x) \xrightarrow{\alpha} j\sigma(x)$,

since $j\sigma_\alpha(x) = (\delta_x)^\wedge(\sigma_\alpha) \rightarrow (\delta_x)^\wedge(\sigma)$. Let $\gamma \in \hat{G}$, $x \in G$, then $j\rho_\varepsilon\gamma(x) = (\delta_x)^\wedge(\gamma \times \varepsilon) = (E_\varepsilon \delta_x)^\wedge(\gamma) = (\delta_x)^\wedge(\gamma) = \gamma(x)$. \square

DEFINITION 7. Let ε be an idempotent in $c\ell\hat{G}$, then we say L is a good group containing ε if L is a locally compact (in the Δ_G -topology) subgroup of $H(\varepsilon)$ and $L \supset \rho_\varepsilon \hat{G}$.

We will show that the good groups arise exactly as the dual groups of the G_H 's, the abstract group G furnished with LCA topologies so that $\text{id}: G_H \rightarrow G$ is continuous. Further there is at most one good group containing an idempotent.

PROPOSITION 8. Let ε be an idempotent in $c\ell\hat{G}$ and let L be a good group containing ε , then $\hat{\rho}_\varepsilon: \hat{L} \rightarrow G$ is a monomorphism onto G .

Proof. Since $\rho_\varepsilon: \hat{G} \rightarrow L$ has dense range, we see that $\hat{\rho}_\varepsilon$ is a monomorphism. Consider the map j defined in Proposition 7 restricted to L . Then $j \circ \rho_\varepsilon = i$, where $i: \hat{G} \rightarrow \mathcal{B}\hat{G}$. Passing to the duals, we obtain $\hat{i} = \hat{\rho}_\varepsilon \circ \hat{j}$, and $\hat{i}: G_d \rightarrow G$. Hence $\hat{\rho}_\varepsilon$ is onto. \square

For notational convenience put $\phi = \hat{\rho}_\varepsilon^*$. Applying Proposition 1 to the above situation we obtain that ϕ is an isometric isomorphism of $M(L)$ onto $M(G_L)$. We now show that the Raikov projection induced by $\hat{\rho}_\varepsilon$ is actually E_ε .

THEOREM 9. Let ε be an idempotent in $c\ell\hat{G}$, and let L be a good group containing ε . Then E_ε is the Raikov projection induced by $\hat{\rho}_\varepsilon$, and \hat{L} is isomorphic under $\hat{\rho}_\varepsilon$ to G_L , G with a stronger

LCA topology.

Proof. The action of ϕ can be described as follows: for $\mu \in M(\hat{L})$, $\gamma \in \hat{G}$, $(\phi\mu)^\wedge(\gamma) = \hat{\mu}(\gamma \times \varepsilon)$. First we show that $\phi M(\hat{L}) \subset E_\varepsilon M(G)$. Since $\varepsilon \in \text{cl}\hat{G}$, there exists a net $\{\gamma_\alpha\} \subset \hat{G}$ such that $\gamma_\alpha \xrightarrow{\alpha} \varepsilon$. Let $\mu \in M(\hat{L})$, $\gamma \in \hat{G}$ then

$$(\phi\mu)^\wedge(\gamma_\alpha \gamma) = \hat{\mu}(\gamma_\alpha \times \gamma \times \varepsilon) \xrightarrow{\alpha} \hat{\mu}(\varepsilon \times \gamma \times \varepsilon) = (\phi\mu)^\wedge(\gamma)$$

since $\hat{\mu}$ is continuous on L . But

$$(\phi\mu)^\vee(\gamma_\alpha \gamma) \xrightarrow{\alpha} (\phi\mu)^\vee(\varepsilon \times \gamma) \text{ hence } (\phi\mu)^\vee(\varepsilon \times \gamma) = (\phi\mu)^\wedge(\gamma), \text{ and so } E_\varepsilon \phi\mu = \phi\mu. \text{ To show that } E_\varepsilon M(G) \subset \phi M(\hat{L}), \text{ let } \mu \text{ be a positive measure in } E_\varepsilon M(G). \text{ To see that these positive measures span } E_\varepsilon M(G), \text{ observe that } E_\varepsilon \mu = f_\varepsilon^\lambda \mu, \text{ for } \mu \in M(G), \lambda = \exp|\mu|, \text{ and } f_\varepsilon^\lambda \geq 0; \text{ hence } \mu \geq 0 \text{ implies } E_\varepsilon \mu \geq 0. \text{ Then the function } \tilde{\mu}|_L \text{ is positive definite on the dense subgroup } \rho_\varepsilon \hat{G} \text{ of } L \text{ and is continuous and bounded on } L; \text{ note for } \gamma \in \hat{G}, \tilde{\mu}(\gamma \times \varepsilon) = (E_\varepsilon \mu)^\vee(\gamma) = \hat{\mu}(\gamma). \text{ Hence } \tilde{\mu}|_L \text{ is positive definite on } L, \text{ (suppose } f \text{ is a continuous function on a topological group } X \text{ and is positive definite on a dense subgroup } Y; \text{ then for } n=1,2,\dots, c_i \in \mathbb{C}, x_i \in X \text{ for } i=1,\dots,n \text{ the expression } \sum_{i,j} c_i \overline{c_j} f(x_j^{-1} x_i) \text{ can be approximated by expressions of the form } \sum_{i,j} c_i \overline{c_j} f(y_j^{-1} y_i), y_i \in Y, \text{ by taking } y_1, \dots, y_n \text{ close enough to } x_1, \dots, x_n \text{ respectively, thus } f \text{ is positive definite on } X), \text{ and by Bochner's theorem (see [7, p. 19]) there exists } \nu \in M(\hat{L}) \text{ such that } \hat{\nu}(\varepsilon \times \gamma) = \tilde{\mu}(\varepsilon \times \gamma) \text{ for each } \gamma \in \hat{G}. \text{ Clearly } \mu = \phi\nu.$$

Let P be the Raikov projection induced by $\hat{\rho}_\varepsilon$, but then $P M(G) = M(\hat{G}_L) = \phi M(L) = E_\varepsilon M(G)$. Since P and E_ε commute, this shows that $P = E_\varepsilon$. \square

Conversely, let G_H be the abstract group G with an LCA topology stronger than that of G . Let $P = E_\varepsilon$ be the Raikov projection induced by $\text{id}: G_H \rightarrow G$, then ε is an idempotent in $\text{cl}\hat{G}$. (see Theorem 4). Now consider \hat{G} as a subgroup of \hat{G}_H (both groups are groups of characters of the abstract group G), then the morphism $\rho_\varepsilon: \hat{G} \rightarrow H(\varepsilon)$ extends to \hat{G}_H in the following way: let $\chi \in \hat{G}_H$, then put $\rho_\varepsilon \chi = \sigma$, where σ is defined by $f_\sigma^\mu = \chi f_\varepsilon^\mu$, for $\mu = \exp|\nu|$, $\nu \in M(G)$.

THEOREM 10. Under the preceding hypotheses,
 $\rho_\varepsilon \hat{G}_H$ is a good group containing ε . Further
 $\rho_\varepsilon \hat{G}_H = H(\varepsilon)$, and $H(\varepsilon)$ is the maximal subgroup
of Δ_G containing ε .

Proof. The topology on \hat{G}_H is induced by the Fourier-Stieltjes transforms of the elements of $M(G_H)$. The Δ_G -topology on $\rho_\varepsilon \hat{G}_H$ is induced by the functions $\tilde{\mu}$, $\mu \in M(G)$. But for $\mu \in M(G)$, $\chi \in \hat{G}_H$, we have $\tilde{\mu}(\rho_\varepsilon \chi) = \tilde{\mu}(\varepsilon \times \rho_\varepsilon \chi) = (E_\varepsilon \mu)^\sim(\rho_\varepsilon \chi) = (E_\varepsilon \mu)^\sim(\chi)$, where the last term is the G_H -Fourier-Stieltjes transform of $E_\varepsilon \mu \in M(G_H)$. Hence $\rho_\varepsilon \hat{G}_H$ in the Δ_G -topology is homeomorphic to \hat{G}_H , so $\rho_\varepsilon \hat{G}_H$ is LCA, and contains $\rho_\varepsilon \hat{G}$. Thus $\rho_\varepsilon \hat{G}_H$ is a good group.

The topologies on the maximal ideal space Δ_{G_H} of $M(G_H)$ and on $\varepsilon \times \Delta_G$ are both induced by the

is homeomorphic to $\varepsilon \times \Delta_G$. Indeed, a generalized character argument shows that Δ_{G_H} is isomorphic to $\varepsilon \times \Delta_G$ as a semitopological semigroup (see the remark immediately preceding Theorem 10).

The annihilator of $L^1(G_H)$ in $\varepsilon \times \Delta_G$ is $(\varepsilon \times \Delta_G) \cap \rho_\varepsilon \hat{G}_H$. Now let $\sigma \in \Delta_G$ with $\sigma \times \varepsilon = \sigma$ (that is, $\sigma \in \varepsilon \times \Delta_G$), and suppose there exists $\tau \in \Delta_G$ such that $\sigma \times \tau = \varepsilon$. We now show that $\sigma \in \rho_\varepsilon \hat{G}_H$. For $\mu = \exp|\lambda|$, $\lambda \in M(G)$, we have $|f_\sigma^\mu| = f_\varepsilon^\mu$. In particular, let $\lambda \in L^1(G_H)$, $\lambda \neq 0$, and $\mu = \exp|\lambda|$. Then $f_\varepsilon^\mu = 1$ ($|\mu|$ almost everywhere), so $|f_\sigma^\mu| = 1$ ($|\mu|$ almost everywhere). Thus $f_\sigma^\mu \in L^1(\lambda)$, and there exists a bounded continuous function g on G_H such that $\int_{G_H} f_\sigma^\mu g d\lambda \neq 0$. But $g\lambda \in L^1(G_H)$ and so $\sigma \in \rho_\varepsilon \hat{G}_H$. \square

COROLLARY 11. Let ε be an idempotent in $\text{cl}\hat{G}$, and let L be a good group containing ε . Then $L = H(\varepsilon)$, and $H(\varepsilon)$ is the maximal subgroup of Δ_G containing ε . Hence $H(\varepsilon)$ is the unique good group containing ε .

Proof. Apply Theorem 10 to G_L^\wedge . \square

Taylor [9] showed that the maximal subgroup of Δ_G containing 1 is \hat{G} . In a future paper the authors will present an example of an LCA group G such that $\text{cl}\hat{G}$ contains an idempotent ε with $H(\varepsilon)$ not locally compact.

REFERENCES

- [1] Berglund, J. and K. H. Hofmann, Compact Semi-topological Semigroups and Weakly Almost Periodic Functions. Springer-Verlag, Berlin, 1967.
- [2] Dunkl, C. and D. Ramirez. Topics in Harmonic Analysis. Appleton-Century-Crofts, New York, 1971.
- [3] Dunkl, C. and D. Ramirez. Bounded projections on Fourier-Stieltjes transforms. Proc. Amer. Math. Soc.
- [4] Dunkl, C. and D. Ramirez. Homomorphisms on groups and induced maps on certain algebras of measures. Trans. Amer. Math. Soc.
- [5] Hewitt, E., The asymmetry of certain algebras of Fourier-Stieltjes transforms. Michigan Math. J. 5 (1958), 149-158.
- [6] Johnson, B., Isometric isomorphisms of measure algebras. Proc. Amer. Math. Soc. 15 (1964), 186-188.
- [7] Rudin, W., Fourier Analysis on Groups. Interscience Publishers, New York, 1962.
- [8] Šreider, Yu., The structure of maximal ideals in rings of measures with convolution. Amer. Math. Soc. Transl. (1) 8 (1962), 365-391.
- [9] Taylor, J., The structure of convolution measure algebras. Trans. Amer. Math. Soc. 119 (1965), 150-166.

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