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Sidon Sets on Compact Groups

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Let G be a compact group and $\hat{G}$ its dual (we will use our notation from [3]). For $\alpha \in \hat{G}$, let $T_\alpha \in \alpha$. Then T_α is a continuous homomorphism of G into $U(n_\alpha)$, the group of $n_\alpha \times n_\alpha$ unitary matrices. We use $T_\alpha(x)_{ij}$ to denote the matrix entries of $T_\alpha(x)$, $1 \leq i, j \leq n_\alpha$. Let $\chi_\alpha(x) = Tr(T_\alpha(x))$ (Tr = trace).

We call χ_α the character of α. Let φ be a set $\{\varphi_\alpha : \alpha \in \hat{G}\}$ where $\varphi_\alpha \in \mathfrak{B}(C^{n_\alpha})$ be such that $\sup\{\|\varphi_\alpha\|_\infty : \alpha \in \hat{G}\} < \infty$ where $\|\cdot\|_\infty$ denotes the operator norm. The set of all such φ is denoted by $\mathfrak{L}^\infty(\hat{G})$. It is a Banach algebra under the norm $\|\varphi\|_\infty = \sup\{\|\varphi_\alpha\|_\infty : \alpha \in \hat{G}\}$ and coordinatewise operations. For $\mu \in M(G)$, the Fourier transform of μ, $\hat{\mu}$, is a matrix-valued function defined for $\alpha \in \hat{G}$ by

$$\alpha \longrightarrow \hat{\mu}_\alpha = \int_G T_\alpha(x^{-1})\, d\mu(x).$$

Note that $\hat{\mu} \in \mathfrak{L}^\infty(\hat{G})$. For A, subalgebra of $M(G)$, we denote its center by $\mathfrak{Z}A$. Note that for $\mu \in M(G)$, $\mu \in \mathfrak{Z}M(G)$ if and only if $\hat{\mu}_\alpha = c_\alpha I_{n_\alpha}$, where I_{n_α} denotes the identity operator on C^{n_α}.

Let $E \subset \hat{G}$. We say that E is a Sidon set if given any $\varphi \in \mathfrak{L}^\infty(G)$, there is $\mu \in M(G)$ such that $\hat{\mu}_\alpha = \varphi_\alpha$ for $\alpha \in E$. We say that E is a central Sidon set if given any $\varphi \in \mathfrak{Z}\mathfrak{L}^\infty(\hat{G})$ (the center of $\mathfrak{L}^\infty(\hat{G})$), there is $\mu \in \mathfrak{Z}M(G)$ such that $\hat{\mu}_\alpha = \varphi_\alpha$ for $\alpha \in E$.

Following Rudin [10, p. 121] one can show that $E \subset \hat{G}$ is a Sidon set if given any $\varphi \in \mathfrak{L}^\infty(\hat{G})$ where φ_α is unitary and hermitian for $\alpha \in E$, there is $\mu \in M(G)$ such that $\sup\{\|\varphi_\alpha - \hat{\mu}_\alpha\|_\infty : \alpha \in E\} < 1$. Also $E \subset \hat{G}$ is a central Sidon set if given any $\varphi \in \mathfrak{Z}\mathfrak{L}^\infty(\hat{G})$ where φ_α is unitary and hermitian (and thus $\varphi_\alpha = \pm I_{n_\alpha}$), there is $\mu \in M(G)$ such that

$$\sup \{\| \varphi_\alpha - \hat{\mu}_\alpha \|_\infty : \alpha \in E\} < 1,$$

(see for example [5, p. 448]).

For $\varphi \in \mathfrak{L}^\infty(\hat{G})$, we put $\| \varphi \|_1 = \Sigma_{\alpha \in \hat{G}} n_\alpha \| \varphi_\alpha \|_1$ where $\|\varphi_\alpha\|_1 = Tr(|\varphi_\alpha|)$. Let

$$\mathfrak{L}^1(\hat{G}) = \{\varphi \in \mathfrak{L}^\infty(\hat{G}) : \| \varphi \|_1 = \sum_{\alpha \in \hat{G}} n_\alpha \| \varphi_\alpha \|_1 < \infty\}.$$

Now $\mathfrak{L}^1(\hat{G})$ is a Banach space under this norm. For $\varphi \in \mathfrak{L}^1(\hat{G})$, let $Tr(\varphi) = \Sigma_{\alpha \in \hat{G}} n_\alpha Tr(\varphi_\alpha)$. Let $A(G)$ be the set of $f \in C(G)$ (the continuous functions on G) for which $\hat{f} \in \mathfrak{L}^1(\hat{G})$. Note that $A(G)$ is isomorphic to $\mathfrak{L}^1(\hat{G})$ and is a Banach algebra of continuous functions under the pointwise operations and the norm $\| f \|_A = \| \hat{f} \|_1$ (see, for example, [5, p. 260].) It is called the Fourier algebra of G. Note that

$$\mathfrak{Z}A(G) = \{f \in A(G) : f(xy) = f(yx) \text{ for all } x, y, \in G\}.$$

1. Proposition. *Let G be compact and $\varphi \in \mathfrak{L}^\infty(\hat{G})$. The following are equivalent:*

(A) $\varphi \in M(G)^{\wedge -}$ *(closure in $\mathfrak{L}^\infty(\hat{G})$).*

(B) *If $f_n \in A(G)$, $\| f_n \|_A \leq 1$, and $f_n \xrightarrow{n} 0$ pointwise on G, then*

$$Tr(\varphi \hat{f}_n) = \Sigma_{\alpha \in \hat{G}} n_\alpha Tr(\varphi_\alpha (f_n)^\wedge_\alpha) \xrightarrow{n} 0.$$

Proof. The proof follows the method given in [9]. □

Similarly, we have:

2. Proposition. *Let G be compact and $\varphi \in \mathfrak{Z}\mathfrak{L}^\infty(\hat{G})$. The following are equivalent:*

(A) $\varphi \in \mathfrak{Z}M(G)^{\wedge -}$ *(closure in $\mathfrak{L}^\infty(\hat{G})$).*

(B) *If $f_n \in \mathfrak{Z}A(G)$, $\| f_n \|_A \leq 1$, and $f_n \xrightarrow{n} 0$ pointwise on G, then*

$$Tr(\varphi \hat{f}_n) \xrightarrow{n} 0.$$

We will later show for G an infinite compact group that $\mathfrak{Z}M(G)^{\wedge -}$ is proper in $\mathfrak{Z}\mathfrak{L}^\infty(\hat{G})$.

Let $E \subset \hat{G}$. We define $\varphi_E \in \mathfrak{L}^\infty(\hat{G})$ by $(\varphi_E)_\alpha = I_{n_\alpha}$ if $\alpha \in E$ and $(\varphi_E)_\alpha = 0$ if $\alpha \notin E$. We call E a peak set if $\varphi_E \in M(G)^{\wedge -}$ (closure in $\mathfrak{L}^\infty(\hat{G})$). If $E \subset \hat{G}$ is both a peak set and a Sidon set, then we call it a peak-Sidon set. If for $\varepsilon > 0$, there is $\mu \in M(G)$ such that $\hat{\mu}_\alpha = I_{n_\alpha}$ for $\alpha \in E$ and $\| \mu_\alpha \|_\infty < \varepsilon$ for $\alpha \notin E$, then we call E a strong peak set.

3. Proposition. *Let $E \subset \hat{G}$ be a Sidon set. If E is a peak set then E is a strong peak set.*

Proof. Since E is a Sidon set, there is a constant $C \geq 1$ such that if $\varphi \in \mathfrak{L}^\infty(\hat{G})$, there is $\mu \in M(G)$ such that $\hat{\mu}_\alpha = \varphi_\alpha$ for $\alpha \in E$ and

$$\| \mu \| \leq C \sup \{ \| \varphi_\alpha \|_\infty : \alpha \in E \},$$

(see for example, [5, p. 416]). Now let $\nu \in M(G)$ be such that

$$\| \hat{\nu}_\alpha - I_{n_\alpha} \|_\infty < 1/4C$$

for $\alpha \in E$ and $\| \hat{\nu}_\alpha \|_\infty < 1/4C$ for $\alpha \notin E$. Let $\mu \in M(G)$ be such that $\hat{\mu}_\alpha = (\hat{\nu}_\alpha)^{-1}$ for $\alpha \in E$ and

$$\| \hat{\mu} \|_\infty \leq \| \mu \| \leq C \sup \{ \| (\hat{\nu}_\alpha)^{-1} \|_\infty : \alpha \in E \} \leq 2\, C$$

since

$$\| (\hat{\nu}_\alpha)^{-1} - I_{n_\alpha} \|_\infty \leq \| I_{n_\alpha} - \hat{\nu}_\alpha \|_\infty \, / \, (1 - \| I_{n_\alpha} - \hat{\nu}_\alpha \|_\infty).$$

Let $\lambda = \mu * \nu$. Then $\hat{\lambda}_\alpha = I_{n_\alpha}$ for $\alpha \in E$ and

$$\| \hat{\lambda}_\alpha \|_\infty \leq \| \hat{\nu}_\alpha \|_\infty \| \hat{\mu}_\alpha \|_\infty < (1/4C)\,(2C) = 1/2 \text{ for } \alpha \notin E.$$

Now $(\hat{\lambda})_n \xrightarrow{n} \varphi_E$ in $M(G)^{\widehat{\ }-}$. □

We now characterize peak-Sidon sets.

4. Proposition. *Let G be compact. The following are equivalent:*

(A) *E is a peak-Sidon set.*

(B) *For $\varepsilon > 0$ and $\varphi \in \mathfrak{L}^\infty(\hat{G})$, there is $\mu \in M(G)$ with $\hat{\mu}_\alpha = \varphi_\alpha$ for $\alpha \in E$ and $\| \hat{\mu}_\alpha \|_\infty < \varepsilon$ for $\alpha \notin E$.*

(C) *If $\varphi \in \mathfrak{L}^\infty(\hat{G})$ with $\varphi_\alpha = 0$ for $\alpha \notin E$, $f_n \in A(G)$, $\| f_n \|_A \leq 1$, and $f_n \xrightarrow{n} 0$ pointwise on G, then $Tr\,(\varphi \hat{f}_n) \xrightarrow{n} 0$.*

Proof. That (A) and (B) are equivalent follows from Proposition 3. That (A) and (C) are equivalent follows from Proposition 1 and the equivalent characterization of a Sidon set stated before Proposition 1. □

Similarly, we have:

5. Proposition. *Let $E \subset \hat{G}$ be a central Sidon set. If E is a peak set, then E is a strong peak set.*

Proof. For $\varepsilon > 0$, let $\mu \in M(G)$ be such that $\| \hat{\mu} - \varphi_E \|_\infty < \varepsilon$. Define $\nu \in M(G)$ by $\int_G f d\nu = \int_G \int_G f\,(yxy^{-1})\, d\mu(x)\, dm_G(y)$ where $f \in C(G)$ and m_G denotes the Haar measure on G. Then $\nu \in \mathfrak{Z}M(G)$ since

$$\int_G R(t)\, f d\nu = \int_G L(t^{-1})\, f d\nu \text{ where } R(t)\, f(x) = f(xt)$$

and $L(t)\,f(x) = f(t^{-1}x)$. Also $\| \hat{\mu} \|_\infty \geq \| \hat{\nu} \|_\infty$. It follows that

$$\varphi_E \in \mathfrak{Z}M(G)^{\wedge -}.$$

The proof now follows that of Proposition 3. □

6. Proposition. *Let G be compact. The following are equivalent:*

(A) *E is a peak-central Sidon set.*

(B) *For* $\varepsilon > 0$ *and* $\varphi \in \mathfrak{Z}\mathfrak{L}^\infty(\hat{G})$, *there is* $\mu \in \mathfrak{Z}M(G)$ *with* $\hat{\mu}_\alpha = \varphi_\alpha$ *for* $\alpha \in E$ *and* $\| \hat{\mu}_\alpha \|_\infty < \varepsilon$ *for* $\alpha \notin E$.

(C) *If* $\varphi \in \mathfrak{Z}\mathfrak{L}^\infty(\hat{G})$ *with* $\varphi_\alpha = 0$ *for* $\alpha \notin E$, $f_n \in \mathfrak{Z}A(G)$, $\| f_n \|_A \leq 1$, *and* $f_n \xrightarrow{n} 0$ *pointwise on G, then* $Tr\,(\varphi \hat{f}_n) \xrightarrow{n} 0$.

We now show for G an infinite compact group that $\hat{G}$ is never a central Sidon set. This will answer a question of Parker [7].

7. Proposition. *Let G be an infinite compact group. Then* $\hat{G}$ *is not a central Sidon set.*

Proof. If $\hat{G}$ is a central Sidon set, then there is a constant C such that for each $\varphi \in \mathfrak{Z}\mathfrak{L}^\infty(\hat{G})$, there is $\mu \in \mathfrak{Z}M(G)$ with $\| \mu \| \leq C \| \varphi \|_\infty$ and $\hat{\mu} = \varphi$. The map $\varphi \longrightarrow \mu$ is one-to-one, continuous, and in fact topological ($\| \hat{\mu} \|_\infty \leq \| \mu \|$). Thus $\mathfrak{Z}M(G)$ contains an infinite dimensional commutative B^*-subalgebra, $\mathfrak{A}$, which we may assume separable. Thus $\mathfrak{A}$ is sequentially weakly complete which is impossible for infinite dimensional commutative B^*-algebras. □

The proof of Proposition 7 is a modification of the method introduced by Edwards [4] for the abelian case. Observe that we have shown for G an infinite compact group that $\mathfrak{Z}M(G)^{\wedge -}$ is proper in $\mathfrak{Z}\mathfrak{L}^\infty(\hat{G})$. Hence $M(G)^{\wedge -}$ is proper in $\mathfrak{L}^\infty(\hat{G})$.

8. Proposition. *Let E be a central Sidon set and suppose* $\varphi_E \in M(G)^\wedge$. *Then E is finite.*

Proof. Let $\varphi_E \in M(G)^\wedge$. Then for each $\varphi \in \mathfrak{Z}\mathfrak{L}^\infty(\hat{G})$ with $\varphi_\alpha = 0$ for $\alpha \notin E$, there is $\mu \in M(G)$ with $\hat{\mu}_\alpha = \varphi_\alpha$ for $\alpha \in E$ and $\hat{\mu}_\alpha = 0$ for $\alpha \notin E$. Now repeat the proof of Proposition 7.□

Let $\mathfrak{C}_0(\hat{G}) = \{ \varphi \in \mathfrak{L}^\infty(\hat{G}) :$ given $\varepsilon > 0$, $\| \varphi_\alpha \|_\infty \geq \varepsilon$ for only finitely many $\alpha \in \hat{G} \}$. For G infinite, $\mathfrak{Z}L^1(G)^\wedge$ is dense in $\mathfrak{Z}\mathfrak{C}_0(\hat{G})$, but proper (for otherwise $\hat{G}$ would be a central Sidon set). In particular, $L^1(G)^\wedge$ is proper in $\mathfrak{C}_0(\hat{G})$. Alternatively, to show that $\mathfrak{Z}L^1(G)^\wedge$ is proper in $\mathfrak{Z}\mathfrak{C}_0(\hat{G})$

it suffices to show that $\mathfrak{Z}A(G)$ is sequentially weakly complete. For then, $\mathfrak{Z}A(G) \neq \mathfrak{Z}C(G)$ and so $\mathfrak{Z}A(G)^* \cong \mathfrak{Z}\mathfrak{L}^1(\hat{G})^* = \mathfrak{Z}\mathfrak{L}^\infty(\hat{G}) \neq \mathfrak{Z}M(G)^\wedge$ and hence G is not a central Sidon set (see for example [7]).

9. Proposition. *$A(G)$ is sequentially weakly complete.*

Proof. Now $A(G) \cong \mathfrak{L}^1(\hat{G})$. So let $\varphi^{(n)} \in \mathfrak{L}^1(\hat{G})$ and suppose $\{\varphi^{(n)}\}$ is weak Cauchy (i. e. in $\sigma(\mathfrak{L}^1(\hat{G}), \mathfrak{L}^\infty(\hat{G}))$). Now since $\mathfrak{C}_0(\hat{G})^* \cong \mathfrak{L}^1(\hat{G})$ (see, for example, [5, p. 74]), $\{\varphi^{(n)}\}$ is weak-* Cauchy. Thus there is $\varphi \in \mathfrak{L}^1(\hat{G})$ such that $\varphi^{(n)} \xrightarrow{n} \varphi$ in the weak-* topology (see, for example, [2, p. 55]). Now let $\psi^{(n)} = \varphi^{(n)} - \varphi$. Then $\psi^{(n)} \xrightarrow{n} 0$ in the weak-* topology and $\{\psi^{(n)}\}$ is weak Cauchy. It will suffice to show that $\psi^{(n)} \xrightarrow{n} 0$ strongly. If not then we may assume that for all n, $\| \psi^{(n)} \|_1 > 1$. There is a finite set $F_1 \subset \hat{G}$ such that $\Sigma_{\alpha \in F_1} n_\alpha \| \psi_\alpha^{(1)} \|_1 > \frac{1}{2}$ and $\Sigma_{\alpha \notin F_1} n_\alpha \| \psi_\alpha^{(1)} \|_1 < \frac{1}{5}$.

Let $n_1 = 1$. Define $n_2 > n_1$ such that

$$\Sigma_{\alpha \in F_1} n_\alpha \| \psi_\alpha^{(n_2)} \|_1 < \frac{1}{5}.$$

Choose a finite set $F_2 \subset \hat{G}$ disjoint from F_1 such that

$$\Sigma_{\alpha \in F_2} n_\alpha \| \psi_\alpha^{(n_2)} \|_1 > \frac{1}{2} \text{ and}$$

$$\Sigma_{\alpha \notin F_1 \cup F_2} n_\alpha \| \psi_\alpha^{(n_2)} \|_1 < \frac{1}{5}.$$

In general, given n_{k-1} define $n_k > n_{k-1}$ such that

$$\Sigma_{\alpha \in F_1 \cup \ldots \cup F_{k-1}} n_\alpha \| \psi_\alpha^{(n_k)} \|_1 < \frac{1}{5}.$$

Choose a finite set $F_k \subset \hat{G}$ disjoint from $F_1 \cup \ldots \cup F_{k-1}$ such that $\Sigma_{\alpha \in F_k} n_\alpha \| \psi_\alpha^{(n_k)} \|_1 > \frac{1}{2}$ and $\Sigma_{\alpha \notin F_1 \cup \ldots \cup F_k} n_\alpha \| \psi_\alpha^{(n_k)} \|_1 < \frac{1}{5}$.

Define $\gamma \in \mathfrak{L}^\infty(\hat{G})$ with $\| \gamma \|_\infty \leq 1$ such that γ is 0 off $\bigcup_{k=1}^{\infty} F_k$ and

$$\Sigma_{\alpha \in F_k} n_\alpha \| \psi_\alpha^{(n_k)} \|_1 = \Sigma_{\alpha \in F_k} n_\alpha \, Tr\,(\gamma_\alpha \psi_\alpha^{(n_k)}).$$

$$\text{Now let } (\theta)_\alpha = \begin{cases} (\gamma)_\alpha & \text{on } F_{2k} \\ -(\gamma)_\alpha & \text{on } F_{2k+1} \end{cases}$$

Now since $\{ \psi^{(n_k)} \}_{k=1}^{\infty}$ is weak Cauchy, $\{ Tr\,(\theta \psi^{(n_k)}) \}_{k=1}^{\infty}$ is Cauchy. But

$$
\begin{aligned}
&\left| Tr\,(\theta\psi^{(2k)} - \theta\psi^{(2k+1)}) \right| \\
&= \left| \Sigma_{\alpha\in F_{2k}}\, n_\alpha\, Tr\,(\gamma_\alpha\psi_\alpha^{(2k)}) + \Sigma_{\alpha\notin F_{2k}}\, n_\alpha\, Tr\,(\gamma_\alpha\psi_\alpha^{(2k)}) \right. \\
&\left. + \Sigma_{\alpha\in F_{2k+1}}\, n_\alpha\, Tr\,(\gamma_\alpha\psi_\alpha^{(2k+1)}) + \Sigma_{\alpha\notin F_{2k+1}}\, n_\alpha\, Tr\,(\gamma_\alpha\psi_\alpha^{(2k+1)}) \right| \\
&\geq \Sigma_{\alpha\notin F_{2k}}\, n_\alpha \| \psi_\alpha^{(2k)} \|_1 - \Sigma_{\alpha\in F_1\cup\ldots\cup F_{2k-1}}\, n_\alpha \| \psi_\alpha^{(2k)} \|_1 \\
&- \Sigma_{\alpha\notin F_1\cup\ldots\cup F_{2k}}\, n_\alpha \| \psi^{(2k)} \|_1 \\
&+ \Sigma_{\alpha\in F_{2k+1}}\, n_\alpha \| \psi_\alpha^{(2k+1)} \|_1 - \Sigma_{\alpha\in F_1\cup\ldots\cup F_{2k}}\, n_\alpha \| \psi_\alpha^{(2k+1)} \|_1 \\
&- \Sigma_{\alpha\notin F_1\cup\ldots\cup F_{2k+1}}\, n_\alpha \| \psi_\alpha^{(2k+1)} \|_1 \\
&> \frac{1}{2} - \frac{1}{5} - \frac{1}{5} + \frac{1}{2} - \frac{1}{5} - \frac{1}{5} = 1 - \frac{4}{5} = \frac{1}{5} > 0. \ \square
\end{aligned}
$$

The proof of this proposition is modelled after the corresponding result for l_1 ([1, p. 143]). This result could be extended to more general settings (see [6, p. 359]). That B^*-algebras cannot be sequentially weakly complete unless finite dimensional was shown by RAJAGOPALAN [8]. We now state some corollaries.

10. Corollary. *$\mathfrak{Z}A(G)$ is sequentially weakly complete.*

11. Corollary. *$A(G)$ has no infinite dimensional B^*-subalgebras.* In particular, if H is a Helson set in G, then H has no interior.

For H a closed subgroup of G (a compact group) we define

$$A_H(G) = \{f \in A(G) : f(hx) = f(x) \text{ for all } x \in G, h \in H\} \text{ and}$$

$$A_{HH}(G) = \{f \in A(G) : f(h_1 x h_2) = f(x) \text{ for all } x \in G, h_1, h_2 \in H\}.$$

Similarly, one defines $C_H(G)$ and $C_{HH}(G)$. These algebras are related to the study of absolutely convergent series of ultraspherical polynomials.

12. Corollary. *If $\{Hx : x \in G\}$ is an infinite set, then $A_H(G)$ is proper in $C_H(G)$.*

13. Corollary. *If $\{HxH : x \in G\}$ is an infinite set, then $A_{HH}(G)$ is proper in $C_{HH}(G)$.*

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