Proc. Camb. Phil. Soc. (1968), **64**, 615
PCPS 64–77
Printed in Great Britain

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Uniform approximation by Fourier–Stieltjes coefficients

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(*Received* 19 *February* 1967)

Introduction. In chapter I, $E = \{n_k\} \subset Z$ is shown to be a Sidon set if and only if (**). For each $x \in T$,

$$\sup_{N \in Z^+} \left| \sum_{k=1}^{N} c_{n_k} e^{-in_k x} \right| \leqslant M_x < \infty \quad \text{implies that} \quad \sum_{k=1}^{\infty} |c_{n_k}| < \infty.$$

Let $E \subset Z^+$ be a lacunary sequence. In chapter II, it is constructively shown that the characteristic function of E is uniformly approximable by Fourier–Stieltjes coefficients; i.e. $\varphi_E \in M(T)^{\wedge -}$.

In Chapter III, it is shown via the construction in Chapter II that there exists $F \subset Z^+$ such that $\varphi_{F \cup -F} \in M(T)^{\wedge -}$, the von Neumann mean of $\varphi_{F \cup -F}$ is 0, but $\varphi_F \notin M(T)^{\wedge -}$; also there exists $G \subset Z^+$ such that φ_G is weakly almost periodic but $\varphi_G \notin M(T)^{\wedge -}$.

Preliminaries. Let T denote the unit circle; Z the integers; $C^B(Z)$ the algebra of (continuous) bounded functions on Z; $M(T)$ the algebra of (bounded) Borel measures on T; $M(T)^{\wedge}$ the algebra of Fourier–Stieltjes coefficients; and $M(T)^{\wedge -}$ the completion of $M(T)^{\wedge} \subset C^B(Z)$ in the sup-norm topology on Z. Let $E \subset Z$. If $M(T)^{\wedge}|E = C^B(E)$, then E is said to be a *Sidon set*. For $y \in Z$, let δ_y denote the unit point measure at y.

The object of this paper is to study the characteristic functions on Z which are uniformly approximable by Fourier–Stieltjes coefficients; i.e. the idempotents in $M(T)^{\wedge -}$.

CHAPTER I

Stechkin ((1); (2), vol. II, p. 249) has proposed the following question: For what class of subsets, E, of the integers, Z, does the following conditions hold?

(*) Let $E = \{n_k\} \subset Z$. If $\sum_{k=1}^{\infty} c_{n_k} e^{-in_k x}$ is the Fourier series of a continuous function, then $\sum_{k=1}^{\infty} |c_{n_k}| < \infty$.

Condition (*) characterizes Sidon sets ((3), p. 207). This gives a relationship between Sidon sets and the absolute convergence of *Fourier* series (see also (5)). In this chapter, the relationship between Sidon sets and the absolute convergence of *trigonometric* series is investigated. We show that $E = \{n_k\} \subset Z$ is a Sidon set if and only if

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† The author is a research associate of the Office of Naval Research, contract number N00014-66-CO269.

(**) For each x,

$$\sup_{N\in Z^+}\left|\sum_{k=1}^{N} c_{n_k}e^{-in_kx}\right| \leqslant M_x < \infty \quad \text{implies that} \quad \sum_{k=1}^{\infty}|c_{n_k}| < \infty.$$

Let $E = \{n_k\} \subset Z$ be a Sidon set. The trigonometric series $\sum_{k=1}^{\infty} c_{n_k}e^{-in_kx}$ is called a *Sidon series*. Throughout this paper, we use the convention that $|n_k| \leqslant |n_{k+1}|$.

PROPOSITION 1. *Let* $\sum_{k=1}^{\infty} c_{n_k}e^{-in_kx}$ *be a Sidon series such that for each* x,

$$\sup_{N\in Z^+}\left|\sum_{k=1}^{N} c_{n_k}e^{-in_kx}\right| \leqslant M_x < \infty. \quad \textit{Then} \quad \sum_{k=1}^{\infty}|c_{n_k}| < \infty.$$

Proof. Suppose that $\sum_{k=1}^{\infty}|c_{n_k}| = \infty$. Let $\lambda_N \in M(Z)$ be defined by

$$\sum_{k=1}^{N} c_{n_k}\delta_{n_k}\Big/ 1+\sum_{k=1}^{N}|c_{n_k}|.$$

Then $\|\lambda_N\| \leqslant 1$ and

$$|\lambda_N^{\wedge}(x)| = \left|\sum_{k=1}^{N} c_{n_k}e^{-in_kx}\right|\Big/ 1+\sum_{k=1}^{N}|c_{n_k}| \leqslant M_x\Big/ 1+\sum_{k=1}^{N}|c_{n_k}| \overset{N}{\to} 0.$$

Let f be defined on E by $f(n_k).c_{n_k} = |c_{n_k}|$. By a characterization of Sidon sets ((4), Theorem 3·2),

$$\int_Z f(x).d\lambda_N(x) = \sum_{k=1}^{N} f(n_k).d\lambda_N(n_k) \overset{N}{\to} 0.$$

But

$$\sum_{k=1}^{N} f(n_k).d\lambda_N(n_k) = \sum_{k=1}^{N}|c_{n_k}|\Big/ 1+\sum_{k=1}^{N}|c_{n_k}| \overset{N}{\to} 1.$$

This is a contradiction and thus $\sum_{k=1}^{\infty}|c_{n_k}| < \infty$. ▮

PROPOSITION 2. *Suppose* $E = \{n_k\} \subset Z$ *is not a Sidon set. Then* E *does not satisfy condition* (**).

Proof. If E is not a Sidon set, then as in Theorem 3·2 (4) we may find $(\lambda_N) \subset M(E)$ such that $\operatorname{supp}\lambda_N$ are finite and pair-wise disjoint, $\|\lambda_N\| = 1$, $\|\lambda_N^{\wedge}\|_\infty \leqslant (\tfrac{1}{2})^N$, and such that if $n \in \operatorname{supp}\lambda_N$ and $m \in \operatorname{supp}\lambda_{N+1}$, then $|n| < |m|$. Let

$$\lambda_1 = \sum_{k=1}^{i_1} c_{n_k}\delta_{n_k} \quad \text{and} \quad \lambda_N = \sum_{k=i_{N-1}+1}^{i_N} c_{n_k}\delta_{n_k}, \; N > 1.$$

Now

$$\sum_{k=1}^{M} c_{n_k}e^{-in_kx} \quad (M \in Z^+),$$

is (pointwise) bounded; but $\sum_{k=1}^{\infty}|c_{n_k}| = \infty$. ▮

THEOREM 3. *Let* $E = \{n_k\} \subset Z$. *E is a Sidon set if and only if E satisfies condition* (**):
(**) *For each* x,

$$\sup_{N\in Z^+}\left|\sum_{k=1}^{N} c_{n_k}e^{-in_kx}\right| \leqslant M_x < \infty \quad \textit{implies} \quad \sum_{k=1}^{\infty}|c_{n_k}| < \infty.$$

Remark 4. It is immediate that $E = \{n_k\} \subset Z$ is a Sidon set if and only if for all x,

$$\sup_{N\in Z^+}\left|\sum_{k=1}^{N} c_{n_k} e^{-in_k x}\right| \leqslant M < \infty \quad \text{implies that} \quad \sum_{k=1}^{\infty} |c_{n_k}| < \infty.$$

Note that Theorem 3 is an assertion about *pointwise* boundedness and not about uniform boundedness of the partial sums.

Remark 5. Zygmund ((2) vol. II, p. 249) has shown that condition (**) is satisfied by lacunary series.

Remark 6. One can replace Z in (**) with any discrete Abelian group and Z^+ with an infinite subset.

CHAPTER II

Let $E \subset Z$ and $A \subset C^B(Z)$. We say that E is a *strong peak set* for A if there exists $f \in A$ such that $f(x) \equiv 1$ on E and $|f(x)| < c < 1$ off E. If $c = 1$, E is called a *peak set.* If E is a Sidon set and a strong peak set for $M(T)^\wedge$, we say that E is a *strong peak-interpolating set.* In this chapter, it is constructively shown that for $E \subset Z^+$, a lacunary sequence, $\varphi_E \in M(T)^{\wedge -}$ and E is a strong peak set for $M(T)^\wedge$.

Let $E \subset Z$. A function $f \in L^1(T)$ such that $\hat{f} \equiv 0$ off E is called an *E-function.* If f is a trigonometric polynomial, f is called an *E-polynomial.* If $\hat{f}$ is real-valued, f is called a *real E-function.*

Let $E \subset Z$ be such that there exists a constant B such that if f is a real E-polynomial, then $\Sigma |\hat{f}(n)| \leqslant B\|f\|_\infty$. E is called a *real-Sidon* set.

PROPOSITION 1. *Let $E \subset Z$. The following are equivalent:*

(A) *E is a real-Sidon set.*

(B) *Every bounded real E-function has $\Sigma|\hat{f}(n)| < \infty$.*

(C) *To every bounded, real-valued function ϕ on E there corresponds $\mu \in M(T)$ such that $\mu^\wedge(n) = \phi(n)$ for all $n \in E$.*

(D) *E is a Sidon set.*

Proof. That (A) implies (B) and (B) implies (C) follow by modifying a theorem of Rudin ((6), p. 121).

That (C) implies (D) follows by ((6), p. 123).

Clearly (D) implies (A). ▌

LEMMA 2. *Let $E \subset Z$. E is a strong peak set for $M(T)^{\wedge -}$ if and only if $\varphi_E \in M(T)^{\wedge -}$.*

Proof. Let E be a strong set for $M(T)^{\wedge -}$. Then there exists $f \in M(T)^{\wedge -}$ such that $f \equiv 1$ on E and $|f| < c < 1$ off E.

Clearly $f^n \to \varphi_E$. ▌

PROPOSITION 3. *Let $E \subset Z$ be a Sidon set. E is a strong peak set for $M(T)^{\wedge -}$ if and only if E is a strong peak set for $M(T)^\wedge$.*

Proof. Since E is a Sidon set, there exists a constant $B \geqslant 1$ such that if $\phi \in C^B(E)$, there exists $\mu \in M(T)$ such that $\mu^\wedge \equiv \phi$ on E and $\|\mu\| \leqslant B\|\phi\|_E$, ((3), p. 207).

Let $\nu \in M(T)$ be such that $\|\nu^\wedge - \varphi_E\|_\infty < 1/4B$.

Let $\mu \in M(T)$ be such that $\mu^\wedge(n) = 1/\nu^\wedge(n)$ for $n \in E$ and $\|\mu^\wedge\|_\infty \leqslant \|\mu\| \leqslant 2B$.

Let $\lambda = \mu * \nu \in M(T)$.

$\lambda^\wedge \equiv 1$ on E and for $n \notin E$, $|\lambda^\wedge(n)| = |\mu^\wedge(n) . \nu^\wedge(n)| \leqslant 2B.(1/4B) = \frac{1}{2}$. ▮

PROPOSITION 4. *There exist strong peak sets for* $M(T)^\wedge$.

Proof. Let $E = \{n_k\} \subset Z^+$ be such that $n_{k+1}/n_k \geqslant 3$. Consider the Riesz product

$$\prod_{k=1}^{\infty} (1+\alpha_k \cos n_k x) \quad \text{represented by} \quad 1+\sum_{k=1}^{\infty} \gamma_k \cos kx;$$

i.e.

$$\prod_{k=1}^{m} (1+\alpha_k \cos n_k x) = 1+\sum_{k=1}^{\nu_m} \gamma_k \cos kx.$$

$\gamma_k = 0$ unless

$$k \in S = \{n_{i_0} \pm n_{i_1} \pm n_{i_2} \pm \ldots : i_0 > i_1 > i_2 > \ldots\} \quad ((7), \text{p. } 208).$$

Let $\alpha_k = 1$. Substituting

$$\tfrac{1}{2}\cos(n_i+n_j)x + \tfrac{1}{2}\cos(n_i-n_j)x \quad \text{for} \quad \cos n_i x \cos n_j x,$$

one has for

$$k = n_{i_0} \pm n_{i_1} \pm n_{i_2} \pm \ldots \pm n_{i_m}, \quad \text{with} \quad i_0 > i_1 > i_2 > \ldots > i_m, \quad \text{that} \quad \gamma_k = 1/2^m.$$

Thus $\gamma_k = 1$ when $k \in E$ and $0 \leqslant \gamma_k \leqslant \frac{1}{2}$ otherwise. Let $\gamma_0 = 1$ and $\gamma_{-k} = \gamma_k$. Thus $\{\frac{1}{2}(\gamma_k)\}$ are the Fourier–Stieltjes coefficients of a measure $\mu \in M(T)$ ((7), p. 209). Thus $E \cup -E \cup \{0\}$ and $E \cup -E$ are strong peak sets for $M(T)^\wedge$. ▮

THEOREM 5. *Let* $E = \{n_k\}$ *be a lacunary sequence; i.e.* $n_{k+1}/n_k \geqslant q > 1$. *Then* $E \cup -E$ *is a strong peak set for* $M(T)^\wedge$ *and* $\varphi_{E \cup -E} \in M(T)^{\wedge -}$.

Proof. Every lacunary sequence, E, is a finite union of lacunary sequences, $E_1, E_2, \ldots, E_n$ with $q_i \geqslant 3$ ((6), p. 127). By the proof of Proposition 4, $E_i \cup -E_i$ is a strong peak set for $M(T)^\wedge$. Thus $E \cup -E$ is a strong peak set for $M(T)^\wedge$, so by Lemma 2, $\varphi_{E \cup -E} \in M(T)^{\wedge -}$. ▮

PROPOSITION 6. *There exist strong peak-interpolating sets.*

Proof. Let E be as in Theorem 5. $E \cup -E$ is a strong peak set. But $E \cup -E$ is a Sidon set since every bounded function on E can be matched by a function $\mu^\wedge(n)$ where μ is a *real* measure ((3), p. 210; (5), p. 8). ▮

It is well known ((14), p. 134) that if $\mu^\wedge \in M(T)^\wedge$ is such that $\mu^\wedge(n) \to 0$ as $n \to +\infty$, then $\mu^\wedge(n) \to 0$ as $n \to -\infty$. We prove this result for completeness. It is natural to ask whether a similar result holds for $M(T)^{\wedge -}$. We show that this conjecture is false by proving that for a lacunary sequence $E \subset Z^+$, $\varphi_E \in M(T)^{\wedge -}$.

PROPOSITION 7. *Let* $\mu^\wedge \in M(T)^\wedge$ *be such that* $\mu^\wedge(n) \to 0$ *as* $n \to +\infty$. *Then* $\mu^\wedge(n) \to 0$ *as* $n \to -\infty$.

Proof. Let $\mu \geqslant 0$. Then $\mu^\wedge(-n) = \overline{\mu^\wedge(n)}$. So the proposition is clearly true in this case.

Let $\mu^\wedge \in M(T)^\wedge$ be such that $\mu^\wedge(n) \to 0$ as $n \to +\infty$. By the Radon–Nikodym theorem ((9), p. 181) there exists $f \in L^1(|\mu|)$ such that $|\mu| = f d\mu$. Let $g_\epsilon(x)$ be a trigonometric polynomial such that $\|g_\epsilon d\mu - f d\mu\| < \epsilon$. Since $(g_\epsilon d\mu)^\wedge(n) \to 0$ as $n \to +\infty$, the same is true for $(f d\mu)^\wedge$, and so for $|\mu|^\wedge$. Since $|\mu| \geqslant 0$, $|\mu|^\wedge(n) \to 0$ as $n \to -\infty$. Let $\mu = h d|\mu|$ for $h \in L^1(|\mu|)$. Let k_ϵ be a trigonometric polynomial such that $\|k_\epsilon d|\mu| - h d|\mu|\| < \epsilon$. Since $(k_\epsilon d|\mu|)^\wedge(n) \to 0$ as $n \to -\infty$, the same is true for $(h d|\mu|)^\wedge$, and so for $\mu^\wedge$. ▮

Theorem 8. *Let $E = \{n_k\} \subset Z^+$ be a lacunary sequence. Then $\varphi_E \in M(T)^{\wedge -}$ and E is a strong peak set for $M(T)^\wedge$.*

Proof. We may assume that $n_{k+1}/n_k \geqslant 3$. It will suffice to show that $\varphi_{-E} \in M(T)^{\wedge -}$. As in Proposition 4, we consider the Riesz product $\prod_{k=1}^{\infty} (1+\alpha_k \cos n_k x)$ represented by $1+\sum_{k=1}^{\infty} \gamma_k \cos kx$. Let $\alpha_k = \frac{1}{3}$. For

$$k = n_{i_0} \pm n_{i_1} \pm n_{i_2} \pm \ldots \pm n_{i_m}, \quad \text{with} \quad i_0 > i_1 > i_2 > \ldots i_m, \quad \gamma_k = \frac{1}{2^m}\frac{1}{3^{m+1}}. \text{ And } \gamma_k = 0,$$

otherwise. Thus there exists $\mu \in M(T)$ such that $\mu^\wedge(n) = \frac{1}{2}(\frac{1}{3})$ on $E \cup -E, \mu^\wedge(0) = 0$, and $0 \leqslant \mu^\wedge(n) \leqslant \frac{1}{2}(\frac{1}{18})$ otherwise.

Now consider the Riesz product

$$\prod_{k=1}^{\infty} (1+\beta_k \sin n_k x) \quad \text{represented by} \quad 1+\sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx.$$

Let $\beta_k = \frac{1}{3}$. Let

$$k = n_{i_0} \pm n_{i_1} \pm n_{i_3} \pm \ldots \pm n_{i_m}, \quad \text{with} \quad i_0 > i_1 > i_2 > \ldots > i_m,$$

then

(i) for $m = 0$, $a_k = 0$ and $b_k = \frac{1}{3}$,

(ii) for $m \geqslant 1$, either $\quad a_k = 0 \quad$ and $\quad b_k = \pm\frac{1}{2^m}\frac{1}{3^{m+1}}$,

$$\text{or} \quad b_k = 0 \quad \text{and} \quad a_k = \pm\frac{1}{2^m}\frac{1}{3^{m+1}}.$$

And $a_k = 0 = b_k$ otherwise. Let $\lambda \in M(T)$ be a measure with Fourier–Stieltjes coefficients c_n such that $c_0 = 0$ and for $n \neq 0$

$$c_{\mp n} = \tfrac{1}{2}(a \pm ib_n); \quad \text{i.e.} \quad \lambda^\wedge(-n_k) = c_{n_k} = \tfrac{1}{2}(-i\tfrac{1}{3}) = -i\tfrac{1}{6},$$
$$\lambda^\wedge(n_k) = c_{-n_k} = \tfrac{1}{2}(i\tfrac{1}{3}) = i\tfrac{1}{6},$$

and $|\lambda^\wedge(n)| \leqslant \frac{1}{2}(\frac{1}{18})$ otherwise.

Let $\nu = i\mu - \lambda$, so $\nu^\wedge = i\mu^\wedge - \lambda^\wedge$.

Thus $\quad \nu^\wedge(-n_k) = i\frac{1}{6} - (-i\frac{1}{6}) = i\frac{1}{3}, \quad \nu^\wedge(n_k) = i\frac{1}{6} - (i\frac{1}{6}) = 0, \quad \nu^\wedge(0) = 0,$

and $\quad |\nu^\wedge(n)| \leqslant |\mu^\wedge(n)| + |\lambda^\wedge(n)| \leqslant \frac{1}{36} + \frac{1}{36} = \frac{1}{18}$

otherwise. Thus $-E$ is a strong peak set for $M(T)^\wedge$ and so $\varphi_{-E} \in M(T)^{\wedge -}$. ▌

Corollary 9. *Let $E \subset Z^+$ be a lacunary sequence. Then E is a strong peak-interpolating set.*

Proof. E is a Sidon set ((6), p. 127). ▌

We now generalize Theorem 8.

Proposition 10. *Let $E = \{n_k\} \subset Z$ be a Sidon set such that $\varphi_E \in M(T)^{\wedge -}$. Then for $F \subset E, \varphi_F \in M(T)^{\wedge -}$.*

Proof. Let $\phi \in C^B(E)$ be such that $\phi(n) = 1$ on F and 0 off. Let $\mu \in M(T)$ be such that $\mu^\wedge(n) \equiv \phi(n)$ on E. Then $\varphi_F = \varphi_E \mu^\wedge \in M(T)^{\wedge-}$. ▌

Remark 11. Let $E \subset Z^+$ be a lacunary sequence. Theorem 5 shows that

$$\varphi_{E \cup -E} \in M(T)^{\wedge-}.$$

Proposition 10 implies that $\varphi_E \in M(T)^{\wedge-}$. Theorem 8 gives this result constructively.

PROPOSITION 12. *There is $\mu \in M(T)$ such that if $f = \mu^\wedge|Z^-$, then there exists no $\nu \in M(T)$ such that $\nu^\wedge|Z^- = f$ and $\|\nu^\wedge\|_\infty = \|f\|_\infty$.*

Proof. Let $E \subset Z^+$ be an infinite set such that $\varphi_E \in M(T)^{\wedge-}$. Thus there exists $\mu \in M(T)$ such that $\|\mu^\wedge - \varphi_E\|_\infty \leqslant \frac{1}{17}$. Suppose $\nu \in M(T)$ was such that $\nu^\wedge \equiv \mu^\wedge$ on Z^- and $\|\nu^\wedge\|_\infty = \|\nu^\wedge|Z^-\|_\infty$. Then $\mu - \lambda$ would be such that $(\mu-\lambda)^\wedge = 0$ on Z^- and thus absolutely continuous ((6), p. 168). But $(\mu-\lambda)^\wedge(n) \not\to 0$ as $n \to \infty$. ▌

Let $E \subset Z$. We call E an *extension set* if for all $\mu \in M(T)$ and $\epsilon > 0$, there exists $\nu \in M(T)$ such that $\nu^\wedge|E = \mu^\wedge|E$ and $\|\nu^\wedge\|_\infty < \|\mu^\wedge|E\|_\infty + \epsilon$.

LEMMA 13. *Let $E \subset Z$. Define $I(E) = \{\mu^\wedge \in M(T)^\wedge : \mu^\wedge \equiv 0$ on $E\}$. The following are equivalent*:

(A) *E is an extension set*

(B) $\|\mu^\wedge|E\|_\infty = \inf\{\|\mu^\wedge + \lambda^\wedge\|_\infty : \lambda^\wedge \in I(E)\}$.

Proof. Immediate from the definition of an extension set. ▌

PROPOSITION 14. *If E is a strong peak set for $M(T)^\wedge$, then E is an extension set.*

Proof. We follow a proof in ((12), p. 417). Clearly $\|\mu^\wedge|E\|_\infty \leqslant \inf\{\|\mu^\wedge + \lambda^\wedge\|_\infty : \lambda^\wedge \in I(E)\}$. Let $\nu^\wedge \in M(T)^\wedge$ be such that $\nu^\wedge \equiv 1$ on E and $|\nu^\wedge| < c < 1$ off E.
Let $\lambda_n^\wedge = (\nu^\wedge)^n . \mu^\wedge - \mu^\wedge$. Thus $\lambda_n^\wedge \in I(E)$ and

$$\|\mu^\wedge + \lambda_n^\wedge\|_\infty \leqslant \|(\mu^\wedge + \lambda_n^\wedge)|E\|_\infty + \|(\mu^\wedge + \lambda_n^\wedge)|Z\backslash E\|_\infty \leqslant \|\mu^\wedge|E\|_\infty + \|\mu^\wedge\|_\infty c^n.$$

Thus E is an extension set. ▌

COROLLARY 15. *If E is a Sidon set and $\varphi_E \in M(T)^{\wedge-}$, then E is an extension set; in particular lacunary sets are extension sets.*

Proof. Lemma 2 and Proposition 3. ▌

COROLLARY 16. *If E is a Sidon set and $\varphi_E \in M(T)^{\wedge-}$, then for $\epsilon > 0$ and $f \in C^B(E)$, there exists $\nu^\wedge \in M(T)^\wedge$ such that $\nu^\wedge|E = f$ and $\|\nu^\wedge\|_\infty < \|\nu^\wedge|E\|_\infty + \epsilon$.*

Proof. Since E is a Sidon set, there exists $\mu^\wedge \in M(T)^\wedge$ such that $\mu^\wedge|E = f$. Now use Corollary 15. ▌

PROPOSITION 17. *Z^- is not an extension set.*

Proof. The proof of Proposition 12. ▌

Remark 18.† Let $E = \{n_k\} \subset Z^+$. Denote by $R_s(E, n)$ the number of representations of $n \in Z$ in the form $n = \pm n_{i_1} \pm n_{i_2} \pm \ldots \pm n_{i_s}$, $i_1 < i_2 < \ldots < i_s$. Suppose that there

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† After this paper had been prepared, the author learned from Dr Robin Chaney that D. Rider [Gap series on groups and spheres, *Canad. J. of Math.* 18, (1966), 389–398] has shown that if $E \subset Z$ is such that $R_s(E, 0) \leqslant B^s$, then $\varphi_E \in M(T)^{\wedge-}$.

exists a constant B such that $R_s(E, n) \leqslant B^s$, $s \in Z^+$, $n \in \{0\} \cup E$. Then $\varphi_E \in M(T)^{\wedge-}$. This follows as in Theorem 8 and ((6), p. 124).

Remark 19. Let E_1, $E_2 \subset Z$ be Sidon sets. If $\varphi_{E_1} \in M(T)^{\wedge-}$, then $E_1 \cup E_2$ is also a Sidon set. Thus it is natural to ask if $E \subset Z$ is a Sidon set, then is $\varphi_E \in M(T)^{\wedge-}$?

CHAPTER III

Let $f \in C^B(Z)$. For $y \in Z$, we define $f_y(x) = f(x-y)$. Let βZ be the Stone–Čech compactification of Z. f is said to be *weakly almost periodic* if for all sequences $\{y_n\} \subset Z$ there exists a subsequence $\{y_i\}$ such that $\{f_{y_i}\}$ converges weakly in $C^B(Z)$; i.e. there exists $f' \in C^B(Z)$ such that for all $\mu^\beta \in M(\beta Z)$, $\mu^\beta(f_{y_i}) \rightarrow \mu^\beta(f')$.

Let $WAP(Z)$ denote the weakly almost periodic functions on Z. For $f \in WAP(Z)$, let $\mathscr{M}(f)$ be the von Neumann mean of f (10). Since $M(T)^{\wedge-} \subset WAP(Z)$, ((10), p. 233), it is natural to ask whether there exists $f \in M(T)^{\wedge-}$ with $\mathscr{M}(f) = 0$ and such that $f . \varphi_{Z^+} \notin M(T)^{\wedge-}$.†

THEOREM 1. *There exists* $\varphi_{F \cup -F} \in M(T)^{\wedge-}$ *with* $\mathscr{M}(\varphi_{F \cup -F}) = 0$ *such that* $F \subset Z^+$ *and* $\varphi_F \notin M(T)^{\wedge-}$.

Proof. Let $E = \{n_k\} \subset Z^+$ be such that $n_{k+1}/n_k \geqslant 3$. Let $F = \{n_i \pm n_j : i > j\}$. By the proof of Proposition 4, Chapter II, and since $\varphi_{E \cup -E} \in M(T)^{\wedge-}$, it follows that $\varphi_{F \cup -F} \in M(T)^{\wedge-}$. Direct computation yields $\mathscr{M}(\varphi_{F \cup -F}) = 0$. That $\varphi_F \notin M(T)^{\wedge-}$ will follow by the subsequent Lemma. ▮

Let $\{f_n\} \subset C^B(Z)$. $f_n \rightarrow 0$ *quasi-uniformly* on Z if for all $\epsilon > 0$ and $N > 0$ there exist $n_1, n_2, \ldots, n_m > N$ such that

$$\sup \min \{|f_{n_i}(y)|,\ 1 \leqslant i \leqslant m\} < \epsilon \quad (y \in Z).$$

LEMMA 2. *With the notation of Theorem* 1, $\varphi_F \notin WAP(Z)$.

Proof. Consider $(\varphi_F)_{n_k}$, $n_k \in E$. $(\varphi_F)_{n_k} \rightarrow \varphi_{E \cup -E}$ pointwise on Z. If $\varphi_F \in WAP(Z)$, then there would exist a subsequence $\{n_i\} \subset \{n_k\}$ such that $(\varphi_F)_{n_i} \rightarrow \varphi_{E \cup -E}$ weakly, and hence quasi-uniformly ((9), p. 281). But $(\varphi_F)_{n_i} \nrightarrow \varphi_{E \cup -E}$ quasi-uniformly. ▮

COROLLARY 3. *With the notation of Theorem* 1, $F \cup -F$ *is not a Sidon set.*

Proof. $\varphi_{F \cup -F} \in M(T)^{\wedge-}$ by the proof of Theorem 1. Now use Proposition 10, Chapter II, and Lemma 2. ▮

Remark 4. Professor Irving Glicksberg has shown [unpublished] that there exists $\mu \in M(T)$, $\mathscr{M}(\mu^\wedge) = 0$, such that $\mu^\wedge . \varphi_{Z^+} \notin WAP(Z)$. The author is indebted to Professor Glicksberg for communicating to him this earlier result: Let $\mu \in M(T)$ be a non-trivial continuous measure on a perfect Kronecker set, P. Thus, $\mu^\wedge(n) \nrightarrow 0$ as $n \rightarrow \infty$ ((6), p. 119). Thus there exist $\{n_k\} \subset Z^+$ such that $|\mu^\wedge(n_k)| \geqslant \epsilon > 0$, and $\nu \in M(P)$ a weak* limit of $\{e^{-in_k x} d\mu(x)\}$. So $\nu \not\equiv 0$ and $\nu^\wedge(-m) \nrightarrow 0$ as $m \rightarrow +\infty$ (see Proposition 7, Chapter II). Thus there exist $\{m_j\} \subset Z^+$ such that $|\nu^\wedge(-m_j)| \geqslant \delta > 0$. Now

$$\lim_j \lim_k (\mu^\wedge . \varphi_{Z^+})(n_k - m_j) = \lim_j \nu^\wedge(-m_j) \nrightarrow 0 \quad \text{and} \quad \lim_k \lim_j (\mu^\wedge . \varphi_{Z^+})(n_k - m_j) = 0.$$

Thus $\mu^\wedge . \varphi_{Z^+} \notin WAP(Z)$ ((11), p. 91).

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† See Berglund and Hofmann, *Compact Semitopological Semigroups and Weakly Almost Periodic Functions* (Springer–Verlag, Berlin, 1967), p. 148 for related questions.

Let $E \subset Z$ be such that $(E+y_1) \cap (E+y_2)$ is finite for all $y_1 \neq y_2$ in Z. E is called a *T-set*. Let $W_0(Z) = \{f \in WAP(Z) : \mathscr{M}(f) = 0\}$. We show that there exists $E \subset Z$ such that $\varphi_E \in WAP(Z)$ but $\varphi_E \notin M(T)^{\wedge -}$.

PROPOSITION 5. *Let $E \subset Z$ be a T-set. If $f \in C^B(Z)$ is such that the support of f is E, then $f \in W_0(Z)$.*

Proof. This result is proved in ((13), p. 218). We give a proof using quasi-uniform convergence.

Let $\{n_i\} \subset Z$ be an infinite sequence of distinct integers. Let N be any given integer and $\epsilon > 0$. To show $f \in WAP(Z)$ it suffices to show that there exists $g \in C^B(Z)$ such that $f_{n_j} \to g$ weakly for some subsequence $\{n_j\} \subset \{n_i\}$.

Suppose there exist $p, q \in Z$ such that $f_{n_i}(p) \neq 0 \neq f_{n_i}(q)$ for infinitely many i, then $(\operatorname{supp} f - p) \cap (\operatorname{supp} f - q)$ is infinite. So $p = q$.

Case 1. $\lim_i f_{n_i}(k) = 0$ for all $k \in Z$: $\operatorname{supp} f_N \cap \operatorname{supp} f_{N+1}$ is finite, say $\{k_3, \ldots, k_m\}$. By the case assumption we can find $n_3, \ldots, n_m > N$ such that $|f_{n_i}(k_i)| < \epsilon$, $3 \leqslant i \leqslant m$. Let $n_1 = N$ and $n_2 = N+1$. Then $\sup_{k \in Z} \min\{|f_{n_i}(k)| : 1 \leqslant i \leqslant m\} < \epsilon$. Any subsequence of $\{f_{n_i}\}$ has the same properties. In this case, $f \in W_0(Z)$ since $f_{n_i} \to 0$ pointwise ((9), p. 281).

Case 2. Suppose there exists a unique $\bar{k} \in Z$ such that $\limsup f_{n_i}(\bar{k}) \neq 0$:

Taking a subsequence if necessary, we may assume that $\lim_i f_{n_i}(\bar{k}) \neq 0$. As in Case 1, we may find $n_1, \ldots, n_m \geqslant N$ such that

$$\sup_{k \in Z} \min\{|f_{n_i}(k) - g(k)| : 1 \leqslant i \leqslant m\} < \epsilon,$$

where $g(n) = 0$ for $n \neq \bar{k}$ and $g(\bar{k}) = \lim_i f_{n_i}(\bar{k})$. $f_{n_i}(k) \to g(k)$ pointwise, and for $\{n_j\} \subset \{n_i\}$, $f_{n_j} \to g$ quasi-uniformly, thus $f_{n_i} \to g$ weakly.

By the remark before Case 1, we have covered all possible cases. Hence $f \in WAP(Z)$.

Finally, we note that there always exists $\{n_i\} \subset Z$ satisfying Case 1. Hence $f \in W_0(Z)$. ▌

COROLLARY 6. *Let $E = \{nk!,\ 1 \leqslant n \leqslant k,\ k = 1, 2, 3, \ldots\}$. $\varphi_E \in WAP(Z)$.*

Proof. E is a T-set ((13), p. 217.) ▌

THEOREM 7. Let $E = \{nk!,\ 1 \leqslant n \leqslant k,\ k = 1, 2, 3, \ldots\}$. Then $\varphi_E \notin M(T)^{\wedge -}$.

Proof. Recall ((2), vol. I, p. 91) that there exists $c > 0$ such that

$$\left|\sum_{n=1}^{k} \frac{\sin nx}{n}\right| \leqslant c$$

for any x and any k.

Let $S_k = \sum_{n=1}^{k} \frac{1}{n}$ and

$$\lambda_k = \frac{1}{2S_k}\left(-\frac{1}{k}\delta_{-k.k!} - \ldots - \delta_{k!} + \delta_{k!} + \tfrac{1}{2}\delta_{2k!} + \ldots + \frac{1}{k}\delta_{k.k!}\right).$$

Thus

$$\|\lambda_k\| = 1 \quad \text{and} \quad |\lambda_k^{\wedge}(x)| = \frac{1}{2S_k}\left|\sum_{n=1}^{k} \frac{1}{n}(e^{-ink!x} - e^{ink!x})\right|$$

$$= \frac{1}{2S_k}\left|-2i \sum_{n=1}^{k} \frac{\sin nk!x}{n}\right| \leqslant \frac{1}{S_k} c \to 0.$$

Thus by the characterization of $M(T)^{\wedge -}$, ((4), Theorem 1·9), if $\varphi_E \in M(T)^{\wedge -}$, then

$$\int_Z \varphi_E \, d\lambda_k \to 0. \quad \text{But} \quad \int_Z \varphi_E \, d\lambda_k = \tfrac{1}{2}.$$

So $\varphi_E \notin M(T)^{\wedge -}$. ▮

COROLLARY 8†. $M(T)^{\wedge -} \neq WAP(Z)$.

Remark 9. Corollary 8 was proved in ((13), p. 216) using a deep trigonometric inequality. The proof in this paper uses only elementary inequalities.

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† Let Γ be an infinite, non-compact, Abelian group with dual group G. Then

$$M(G)^{\wedge -} \neq WAP(\Gamma),$$

[D. Ramirez, Weakly almost periodic functions and Fourier–Stieltjes transforms. To appear. *Proc. Amer. Math. Soc.*].