

Uniform approximation by Fourier-Stieltjes transforms†

By DONALD E. RAMIREZ

University of Washington, University of Virginia

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Introduction. Let G be a locally compact Abelian group; Γ the dual group of G ; $C^B(\Gamma)$ the algebra of continuous, bounded functions on Γ ; $C_0(\Gamma)$ the algebra of continuous functions on Γ which vanish at infinity; $M(G)$ the algebra of bounded Borel measures on G ; $M(G)^\wedge$ the algebra of Fourier-Stieltjes transforms; and $M(G)^\wedge^-$ the completion of $M(G)^\wedge$ in the sup-norm topology on Γ . The object of this paper is to study the natural pairing between $M(G)^\wedge$ and $M(\Gamma)$.

The basic result of this research is a characterization of $M(G)^\wedge^-$. The following theorem, which is a generalization of a reported result of A. Beurling and E. Hewitt, is proved:

For f a continuous, bounded function on Γ , $f \in M(G)^\wedge^-$ if and only if $\{\lambda_n\} \subset M(\Gamma)$, $\|\lambda_n\| \leq M$, and $\lambda_n^\wedge(x) \xrightarrow{n} 0$ for all $x \in G$ implies

$$\int_{\Gamma} f d\lambda_n \xrightarrow{n} 0.$$

[In the sequel 'n' will denote a natural number.]

In Chapter I, we prove the above theorem.

In Chapter II, we characterize $M(G)^\wedge$ in a manner similar to the above theorem:

For f a continuous, bounded function on Γ , $f \in M(G)^\wedge$ if and only if $\{\lambda_n\} \subset M(\Gamma)$, $\|\lambda_n^\wedge\|_\infty \leq M$, and $\lambda_n^\wedge(x) \xrightarrow{n} 0$ for all $x \in G$ implies

$$\int_{\Gamma} f d\lambda_n \xrightarrow{n} 0.$$

In Chapter III, we characterize Sidon sets: E is a Sidon set if and only if for each bounded function, f , on E ; $\{\lambda_n\} \subset M(E)$, $\|\lambda_n^\wedge\|_\infty \leq 1$ (or $\|\lambda_n\| \leq 1$) and $\lambda_n^\wedge(x) \xrightarrow{n} 0$ implies that

$$\int_E f d\lambda_n \xrightarrow{n} 0.$$

Chapter I. Characterization of $M(G)^\wedge^-$

We prove in this chapter:

THEOREM. For $f \in C^B(\Gamma)$, the following are equivalent:

(A) $f \in M(G)^\wedge^-$.

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(B) If $\{\lambda_n\} \subset M(\Gamma)$, $\|\lambda_n\| \leq M$, and $\lambda_n^*(x) \rightarrow 0$ for all $x \in G$, then

$$\int_{\Gamma} f d\lambda_n \xrightarrow{n} 0.$$

Beurling and Hewitt ((6), p. 138) have reported this result in the case that G and Γ are σ -compact. The author is indebted to Dr Frank T. Birtel for communicating to him the suggestion of Dr George Reid that we apply Grothendieck's completion theorem to the natural pairing between the transform algebra, $M(G)^\wedge$, and the measure algebra, $M(\Gamma)$.

For $\mu^\wedge \in M(G)^\wedge$ and $\lambda \in M(\Gamma)$, define

$$\langle \mu^\wedge, \lambda \rangle = \int_{\Gamma} \mu^\wedge d\lambda.$$

1.1. LEMMA. $\langle \cdot, \cdot \rangle$ is a pairing.

Proof. If $\langle \mu^\wedge, \lambda \rangle = 0$ for all $\lambda \in M(\Gamma)$ —in particular for λ a point measure—then $\mu^\wedge = 0$. The Fourier–Stieltjes uniqueness theorem ((9), p. 17) implies that $\mu = 0$.

If $\langle \mu^\wedge, \lambda \rangle = 0$ for all $\mu^\wedge \in M(G)^\wedge$ —in particular for $\mu^\wedge \in C_0(\Gamma)$ —then $\lambda = 0$ since $M(G)^\wedge \cap C_0(\Gamma)$ is sup-norm dense in $C_0(\Gamma)$ ((9), p. 9). ■

Let w denote the weak topology on $M(\Gamma)$ from this pairing with $M(G)^\wedge$. For $\{\lambda_\alpha\} \subset M(\Gamma)$, $\lambda_\alpha \xrightarrow{\alpha} 0$ in w if and only if $\langle \mu^\wedge, \lambda_\alpha \rangle \xrightarrow{\alpha} 0$ for all $\mu^\wedge \in M(G)^\wedge$. Let B_n denote the n -ball of $M(\Gamma)$. Let $C_n = \{\lambda \in B_n : \lambda^\wedge \text{ has compact support}\}$.

1.2. LEMMA. B_n is w -closed.

Proof. Let $\{\lambda_\alpha\} \subset B_n$ be such that $\lambda_\alpha \xrightarrow{\alpha} \lambda$ in w , $\lambda \in M(\Gamma)$. We wish to show that $\|\lambda\| \leq n$. Now $\langle \mu^\wedge, \lambda_\alpha \rangle \xrightarrow{\alpha} \langle \mu^\wedge, \lambda \rangle$ for $\mu^\wedge \in M(G)^\wedge \cap C_0(\Gamma)$. If $\|\mu^\wedge\|_\infty \leq 1$, then

$$|\langle \mu^\wedge, \lambda_\alpha \rangle| \leq \|\mu^\wedge\|_\infty \|\lambda_\alpha\| \leq n,$$

so $|\langle \mu^\wedge, \lambda \rangle| \leq n$. Since $M(G)^\wedge \cap C_0(\Gamma)$ is sup-norm dense in $C_0(\Gamma)$, $\|\lambda\| \leq n$. ■

1.3. LEMMA. B_n is w -bounded.

Proof. Let $V = \{\lambda \in M(\Gamma) : |\langle \mu^\wedge, \lambda \rangle| \leq 1, \mu^\wedge \in M(G)^\wedge\}$.

V is a typical sub-basic w -neighbourhood of 0. We wish to find $\rho > 0$ such that $B_n \subset \rho V$. Since $|\langle \mu^\wedge, \lambda \rangle| \leq \|\mu^\wedge\|_\infty \|\lambda\|$, we let $\rho = n \|\mu^\wedge\|_\infty$. ■

1.4. LEMMA. On $M(G)^\wedge$, the topology, \mathcal{T}_{B_n} , of uniform convergence on the norm balls, B_n , is equivalent to the sup-norm topology on Γ .

Proof. Let $\{\mu_\alpha^\wedge\} \subset M(G)^\wedge$.

Suppose $\mu_\alpha^\wedge \xrightarrow{\alpha} 0$ in \mathcal{T}_{B_n} , then

$$\sup\{|\langle \mu_\alpha^\wedge, \lambda \rangle|, \lambda \in B_n\} \xrightarrow{\alpha} 0.$$

Hence $\|\mu_\alpha^\wedge\|_\infty = \sup\{|\langle \mu_\alpha^\wedge, \lambda \rangle|, \lambda \text{ a unit point measure}\} \xrightarrow{\alpha} 0$.

Suppose $\|\mu_\alpha^\wedge\|_\infty \xrightarrow{\alpha} 0$. Then

$$\sup\{|\langle \mu_\alpha^\wedge, \lambda \rangle|, \lambda \in B_n\} \leq \|\mu_\alpha^\wedge\|_\infty \cdot n \xrightarrow{\alpha} 0. \quad \blacksquare$$

1.5. PROPOSITION. For $f \in C^B(\Gamma)$, the following are equivalent:

(A) $f \in M(G)^\wedge$.

(C) The linear functional on $M(\Gamma)$ determined by f , i.e. $\lambda \rightarrow \int_{\Gamma} f d\lambda$, is w -continuous when restricted to the norm balls, B_n , of $M(\Gamma)$.

Proof. The norm balls, B_n , are convex, circled, w -closed, w -bounded, and

$$\cup \{B_n, n = 1, 2, \dots\} = M(\Gamma).$$

Hence Grothendieck's completion theorem ((7), p. 271) asserts that for $f \in C^B(\Gamma)$, f is in the closure of $M(G)^\wedge$ with respect to the topology of uniform convergence on the norm balls, B_n , if and only if (C) is satisfied. The proposition now follows immediately from Lemma 1.4.

Represent the functions, f , in $C^B(G)$ as operators, T_f , on $C_0(G)$ by $T_f(g) = f \cdot g, g \in C_0(G)$. Let WO be the weak operator topology on $C^B(G)$ via this representation; and SO be the strong operator topology. Let $\{f_\alpha\} \subset C^B(G)$:

(i) $f_\alpha \xrightarrow{\alpha} 0$ in WO if and only if for $g \in C_0(G)$ and $\mu \in M(G)$,

$$\int_G f_\alpha g d\mu \xrightarrow{\alpha} 0.$$

(ii) $f_\alpha \xrightarrow{\alpha} 0$ in SO if and only if for $g \in C_0(G)$, $\|f_\alpha g\|_\infty \xrightarrow{\alpha} 0$.

Note that the SO topology is Buck's 'strict' topology ((2), p. 97). Hence on sup-norm bounded sets, the SO topology is equivalent to the compact-open topology ((2), p. 98). Recall that convex sets have the same closures in WO and SO ((4), p. 477).

Viewing $M(\Gamma)$ as a subalgebra of $C^B(G)$ via the Fourier-Stieltjes transformation induces the WO and SO topologies on $M(\Gamma)$.

1.6. LEMMA. On the norm balls, B_n , the weak topology, w , is equivalent to the weak operator topology, WO . In general, $WO \subset w$.

Proof. Let $\{\lambda_\alpha\} \in M(\Gamma)$.

Suppose $\lambda_\alpha \xrightarrow{\alpha} 0$ in w , i.e.

$$\langle \mu^\wedge, \lambda_\alpha \rangle = \int_{\Gamma} \mu^\wedge d\lambda_\alpha = \int_G \lambda_\alpha^\wedge d\mu \xrightarrow{\alpha} 0$$

for each $\mu \in M(G)$. For $g \in C_0(G)$ and $\mu \in M(G)$, $g d\mu \in M(G)$. So

$$\int_G \lambda_\alpha^\wedge g d\mu \xrightarrow{\alpha} 0.$$

Hence $\lambda_\alpha \xrightarrow{\alpha} 0$ in WO .

Suppose $\lambda_\alpha \xrightarrow{\alpha} 0$ in WO in B_n , i.e.

$$\|\lambda_\alpha\| \leq n \quad \text{and} \quad \int_G \lambda_\alpha^\wedge g d\mu \xrightarrow{\alpha} 0 \quad \text{for } g \in C_0(G) \quad \text{and} \quad \mu \in M(G).$$

Let $\mu \in M(G)$. There exists a compact subset, K , of G such that $\|\mu|_K - \mu\| \leq \epsilon/3n$. Let $g \in C_0(G)$ be such that $g = 1$ on K and $\|g\|_\infty \leq 1$. Let α_0 be such that if $\alpha \geq \alpha_0$, then

$$\left| \int_G \lambda_\alpha^\wedge g d\mu \right| \leq \epsilon/3.$$

We wish to find α_1 such that if $\alpha \geq \alpha_1$, then

$$\left| \int_G \lambda_\alpha^\wedge d\mu \right| \leq \epsilon.$$

Let $\alpha_1 = \alpha_0$. Then for $\alpha \geq \alpha_1$,

$$\begin{aligned} \left| \int_G \lambda_\alpha^\wedge d\mu \right| &\leq \left| \int_K \lambda_\alpha^\wedge d\mu \right| + \left| \int_{G/K} \lambda_\alpha^\wedge d\mu \right| \leq \left| \int_K \lambda_\alpha^\wedge g d\mu \right| + \|\lambda_\alpha^\wedge\|_\infty \cdot \|\mu|_{G/K}\| \leq \left| \int_G \lambda_\alpha^\wedge g d\mu \right| \\ &+ \left| \int_{G/K} \lambda_\alpha^\wedge g d\mu \right| + n \cdot \epsilon/3n \leq \epsilon/3 + \|\lambda_\alpha^\wedge g\|_\infty \cdot \|\mu|_{G/K}\| + \epsilon/3 \leq \epsilon/3 + n \cdot \epsilon/3n + \epsilon/3 = \epsilon. \quad \blacksquare \end{aligned}$$

1.7. LEMMA. Let $\{\lambda_n\} \subset B_M$. If $\lambda_n^\wedge(x) \xrightarrow{n} 0$ for all $x \in G$, then $\lambda_n \xrightarrow{n} 0$ in w ; and conversely.

Proof. Let $\{\lambda_n\} \subset M(\Gamma)$ be such that $\|\lambda_n\| \leq M$ and $\lambda_n^\wedge(x) \xrightarrow{n} 0$ for all $x \in G$. We wish to show that $\lambda_n \xrightarrow{n} 0$ in w , i.e. let $\mu^\wedge \in M(G)^\wedge$ and show $\langle \mu^\wedge, \lambda_n \rangle \xrightarrow{n} 0$. Let K be a compact subset of G such that $\|\mu - \mu|_K\| \leq \epsilon/2M$. Since $\|\lambda_n^\wedge\|_\infty \leq M$, $\{\lambda_n^\wedge|_K\}$ are sup-norm bounded and pointwise convergent to 0. Therefore,

$$\int_K \lambda_n^\wedge d\mu|_K \xrightarrow{n} 0$$

by the Lebesgue dominated convergence theorem ((3), p. 109); hence there exists n_0 such that if $n \geq n_0$, then

$$\left| \int_K \lambda_n^\wedge d\mu|_K \right| \leq \epsilon/2.$$

Hence for $n \geq n_0$,

$$\begin{aligned} |\langle \mu^\wedge, \lambda_n \rangle| &= \left| \int_\Gamma \mu^\wedge d\lambda_n \right| = \left| \int_G \lambda_n^\wedge d\mu \right| \leq \left| \int_K \lambda_n^\wedge d\mu \right| + \left| \int_{G/K} \lambda_n^\wedge d\mu \right| \leq \epsilon/2 \\ &+ \|\lambda_n^\wedge\|_\infty \|\mu|_{G/K}\| \leq \epsilon/2 + M \cdot \epsilon/2M = \epsilon. \end{aligned}$$

We remark that the converse is clear by considering unit point measures. ▀

1.8. LEMMA. Let $f \in C^B(\Gamma)$. The following are equivalent:

(D) The linear functional on $M(\Gamma)$ determined by f , i.e. $\lambda \rightarrow \int_\Gamma f d\lambda$, is *SO*-continuous when restricted to the norm balls, B_n , of $M(\Gamma)$.

(E) The linear functional on $M(\Gamma)$ determined by f , i.e. $\lambda \rightarrow \int_\Gamma f d\lambda$, is *WO*-continuous when restricted to the norm balls, B_n , of $M(\Gamma)$.

Proof. Since $WO \subset SO$, (E) implies (D).

Let $\lambda \rightarrow \int_\Gamma f d\lambda$ be *SO*-continuous on B_n . Let K be its kernel. Let $\{\lambda_\alpha\} \subset B_n \cap K$ be such that $\lambda_\alpha \rightarrow \lambda$ in *SO*, $\lambda \in M(\Gamma)$. Since $WO \subset SO$, $\lambda_\alpha \rightarrow \lambda$ in *WO*. Since $\{\lambda_\alpha\} \subset B_n$, $\lambda_\alpha \rightarrow \lambda$ in w by Lemma 1.6. But B_n is w -closed by Lemma 1.2, so $\lambda \in B_n$. By the *SO*-continuity of the linear functional, $\lambda \in K$. Hence $\lambda \in B_n \cap K$, and so $B_n \cap K$ is *SO*-closed. Since B_n is convex, $B_n \cap K$ is also *WO*-closed. This clearly implies (E) ((1), p. 84, ex. 18a). ▀

1.9 THEOREM. Let $f \in C^B(\Gamma)$. The following are equivalent:

- (A) $f \in M(G)^{\wedge-}$.
- (B) If $\{\lambda_n\} \subset M(\Gamma)$, $\|\lambda_n\| \leq M$ and $\lambda_n^\wedge(x) \xrightarrow{n} 0$ for all $x \in G$, then $\int_\Gamma f d\lambda_n \xrightarrow{n} 0$.
- (C) $\lambda \rightarrow \int_\Gamma f d\lambda$ is w -continuous on B_n .
- (D) $\lambda \rightarrow \int_\Gamma f d\lambda$ is SO -continuous on B_n .
- (E) $\lambda \rightarrow \int_\Gamma f d\lambda$ is WO -continuous on B_n .

Proof.

- (A) is equivalent to (C) by Proposition 1.5.
- (C) is equivalent to (E) by Lemma 1.6.
- (C) implies (B) by Lemma 1.7.
- (D) is equivalent to (E) by Lemma 1.8.

It suffices to show that (B) implies (D). First, we show that $\lambda \rightarrow \int_\Gamma f d\lambda$ is SO -continuous on C_n . For suppose not. Hence there exists $\epsilon > 0$ such that for compact subsets, K , of G and $\delta > 0$ we have a $\lambda_{K,\delta}$ in $V_{K,\delta} = \{\lambda \in C_N : \|\lambda^\wedge\|_{\infty, K} \leq \delta\}$ such that

$$\left| \int_\Gamma f d\lambda_{K,\delta} \right| \geq \epsilon.$$

Let λ_1 be any measure in C_N such that

$$\left| \int_\Gamma f d\lambda_1 \right| \geq \epsilon.$$

Let $\lambda_2 = \lambda_{\text{supp } \lambda_1^\wedge, 1}$, $\lambda_3 = \lambda_{\text{supp } \lambda_1^\wedge \cup \text{supp } \lambda_2^\wedge, \frac{1}{2}}$,
and $\lambda_{n+1} = \lambda_{\text{supp } \lambda_1^\wedge \cup \dots \cup \text{supp } \lambda_n^\wedge, (1/n)}$. $\lambda_n^\wedge(x) \xrightarrow{n} 0$

for all $x \in G$ and $\{\lambda_n\} \subset C_N \subset B_N$. Hence (B) implies that

$$\int_\Gamma f d\lambda_n \xrightarrow{n} 0.$$

This is a contradiction since $\left| \int_\Gamma f d\lambda_n \right| \geq \epsilon$.

That $\lambda \rightarrow \int_\Gamma f d\lambda$

is SO -continuous on C_n relative to $\bigcup_{n=1}^\infty C_n$ implies (by the proof of 1.8) that

$$\lambda \rightarrow \int_\Gamma f d\lambda$$

is WO -continuous on C_n and hence w -continuous on C_n . Let K be the kernel of the linear functional. Thus $K \cap C_n$ is w -closed. Let \mathcal{T}^u be the finest locally convex topology on

$M(\Gamma)$ which coincides with the w -topology on the sets B_n . Thus $K \cap C_n$ is \mathcal{F}^U -closed in $\bigcup_{n=1}^{\infty} C_n$ and

$$\lambda \rightarrow \int_{\Gamma} f d\lambda$$

is \mathcal{F}^U -continuous on $\bigcup_{n=1}^{\infty} C_n$. Extend this linear functional by the Hahn-Banach Theorem to a \mathcal{F}^U -continuous linear functional on $M(\Gamma)$. By the proof of Grothendieck's completion theorem, the extended linear functional has the form

$$\lambda \rightarrow \int_{\Gamma} g d\lambda \quad \text{where } g \in M(G)^{\wedge-}.$$

It remains to show that $f = g$. This follows since the group algebra, $L^1(\Gamma)$, is Tauberian ((8), p. 92). ■

1.10. PROPOSITION. Let Z denote the integers and f a (continuous) bounded function on Z be such that $f(10^k + n) = e^{in\frac{1}{2}}$ ($k=1, 2, 3, \dots, 1 \leq n \leq k$). Then $f \notin M(T)^{\wedge-}$ where T is the unit circle, the dual group of the integers.

Proof. Let

$$\lambda_k = \frac{1}{k} \sum_{n=1}^k e^{-in\frac{1}{2}} \delta_{10^k+n},$$

where δ_{10^k+n} is the unit point measure at $10^k + n$ ($k=1, 2, 3, \dots, 1 \leq n \leq k$). Then $\|\lambda_k\| = 1$. It is known ((11), vol. I, p. 200) that for each $x \in T$, there is a $C_x > 0$ such that

$$\left| \sum_{n=1}^k e^{in\frac{1}{2}} e^{inx} \right| \leq C_x \cdot k^{\frac{1}{2}}.$$

$$\text{Hence } |\lambda_k^{\wedge}(x)| = \left| \frac{1}{k} \sum_{n=1}^k e^{-in\frac{1}{2}} e^{-i(10^k+n)x} \right| = \left| \frac{1}{k} \sum_{n=1}^k e^{in\frac{1}{2}} e^{inx} \right| \leq \frac{1}{k} \cdot C_x \cdot k^{\frac{1}{2}} \xrightarrow{k} 0.$$

By Theorem 1.9, $f \in M(T)^{\wedge-}$ implies that

$$\int_Z f d\lambda_k \xrightarrow{k} 0.$$

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$$\left| \int_{\Gamma} f d\lambda_k \right| = \frac{1}{k} \sum_{n=1}^k 1 = 1 \xrightarrow{k} 0.$$

Hence $f \notin M(T)^{\wedge-}$. ■

1.11. Remark. In Theorem 1.9, one may not replace pointwise convergence with almost everywhere convergence. Let $\lambda_n = (1/n)(\delta_1 + \dots + \delta_n) \in M(Z)$. $\|\lambda_n\| = 1$ and $\lambda_n^{\wedge}(x) \xrightarrow{n} 0$ for $x \neq 0$, ((11), vol. I, p. 142). Now

$$\int_Z 1 \cdot d\lambda_n = 1 \xrightarrow{n} 0.$$

1.12. Remark. Since for $\mu \in M(G)$, $\|\mu^{\wedge}\|_{\infty} = \sup \{|\langle \mu^{\wedge}, \lambda \rangle|, \lambda \in L^1(\Gamma) \cap B_1\}$, Theorem 1.9 is valid with $L^1(\Gamma)$ replacing $M(\Gamma)$. The proof is essentially the same.

1.13. Remark. Another proof of Proposition 1.5 has recently been published by R. E. Edwards, ((5), p. 254).

Chapter II. Characterization of $M(G)^\wedge$

2.1. PROPOSITION. Let $f \in C^B(\Gamma)$ and $B_n^\wedge = \{\lambda \in M(\Gamma) : \|\lambda^\wedge\|_\infty \leq n\}$. The following are equivalent.

(A') $f \in M(G)^\wedge$.

(C') The linear functional on $M(\Gamma)$ determined by f , i.e. $\lambda \rightarrow \int_\Gamma f d\lambda$, is w -continuous on the sup-norm balls B_n^\wedge of $M(\Gamma)$.

Proof. As in Lemma 1.1, $\langle \cdot, \cdot \rangle$ is a pairing. B_n^\wedge is w -closed since if

$$\{\lambda_\alpha\} \cup \{\lambda\} \subset M(\Gamma), \quad \|\lambda_\alpha^\wedge\|_\infty \leq n,$$

and
$$\int_G \lambda_\alpha^\wedge d\mu = \int_\Gamma \mu^\wedge d\lambda_\alpha = \langle \mu^\wedge, \lambda_\alpha \rangle \xrightarrow{\alpha} \langle \mu^\wedge, \lambda \rangle = \int_\Gamma \mu^\wedge d\lambda = \int_G \lambda^\wedge d\mu$$

for all $\mu^\wedge \in M(G)^\wedge$ —in particular for μ a unit point measure; hence $\|\lambda^\wedge\|_\infty \leq n$ and $\lambda \in B_n^\wedge$.

B_n^\wedge is w -bounded as in Lemma 1.3 since $|\langle \mu^\wedge, \lambda \rangle| \leq \|\mu\| \cdot \|\lambda^\wedge\|_\infty$.

Let $\mathcal{T}_{B_n^\wedge}$ be the topology on $M(G)^\wedge$ of uniform convergence on the sup-norm balls B_n^\wedge with respect to this pairing. $\mathcal{T}_{B_n^\wedge}$ is equivalent to the measure-norm topology on $M(G)^\wedge$ since for $\mu \in M(G)^\wedge$,

$$\|\mu\| = (1/n) \sup \{ |\langle \mu^\wedge, \lambda \rangle| : \lambda \in B_n^\wedge \} \quad \text{and} \quad |\langle \mu^\wedge, \lambda \rangle| \leq \|\mu\| \cdot \|\lambda^\wedge\|_\infty.$$

By Grothendieck's completion theorem ((7), p. 271) we may assert for $f \in C^B(\Gamma)$ that $f \in M(G)^\wedge$ if and only if $\lambda \rightarrow \int_\Gamma f d\lambda$ is w -continuous when restricted to the sets B_n^\wedge . ■

2.2. THEOREM. Let $f \in C^B(\Gamma)$. The following are equivalent:

(A') $f \in M(G)^\wedge$.

(B') If $\{\lambda_n\} \subset M(\Gamma)$, $\|\lambda_n^\wedge\|_\infty \leq M$, and $\lambda_n^\wedge(x) \xrightarrow{n} 0$ for all $x \in G$, then

$$\int_\Gamma f d\lambda_n \xrightarrow{n} 0.$$

(C') $\lambda \rightarrow \int_\Gamma f d\lambda$ is w -continuous on B_n^\wedge .

(D') $\lambda \rightarrow \int_\Gamma f d\lambda$ is SO -continuous on B_n^\wedge .

(E') $\lambda \rightarrow \int_\Gamma f d\lambda$ is WO -continuous on B_n^\wedge .

Proof. (A') is equivalent to (C') by Proposition 2.1. (C') is equivalent to (E') by the proof of Lemma 1.6. (C') implies (B') by the proof of Lemma 1.7. (D') implies (E') by the proof of Lemma 1.8. (B') implies (D') by the proof of Theorem 1.9. ■

2.3. PROPOSITION. $WO \subset w \subset SO$ on $M(\Gamma)$.

Proof. That $WO \subset w$ is part of Lemma 1.6.

Let $\lambda_\alpha \xrightarrow{\alpha} 0$ in SO , $\{\lambda_\alpha\} \subset M(\Gamma)$. To show $\lambda_\alpha \xrightarrow{\alpha} 0$ in w .

Suppose not. Then there exists $\mu \in M(G)$ such that

$$\int_G \lambda_\alpha^\wedge d\mu \not\xrightarrow{\alpha} 0.$$

Let $g \in C_0(G)$ be such that

$$\int_G \left| \frac{1}{g} \right| d|\mu| = M < \infty.$$

Such a function exists as noted in (2), p. 99). Let λ_{α_n} be such that

$$\|\lambda_{\alpha_n}^\wedge \cdot g\|_\infty \leq \frac{1}{n} \quad \text{and} \quad \left| \int_G \lambda_{\alpha_n}^\wedge d\mu \right| \geq \epsilon.$$

Now $\left| \int_G \lambda_{\alpha_n}^\wedge d\mu \right| \leq \int_G |\lambda_{\alpha_n}^\wedge| \left| \frac{1}{g} \right| d|\mu| \leq \|\lambda_{\alpha_n}^\wedge g\|_\infty \int_G \left| \frac{1}{g} \right| d|\mu| \leq \frac{1}{n} \cdot M \xrightarrow{n} 0.$

This is a contradiction and hence $w \subset SO$. ■

2.4. COROLLARY. Let $M_d(\Gamma)$ be the subalgebra of $M(\Gamma)$ consisting of all discrete measures. Let $f \in C^B(\Gamma)$. The following are equivalent:

(A') $f \in M(G)^\wedge$.

(I') $\lambda \rightarrow \int_\Gamma f d\lambda$ is *SO*-continuous on $M(\Gamma)$.

(J') $\lambda \rightarrow \int_\Gamma f d\lambda$ is *WO*-continuous on $M(\Gamma)$.

(K') $\lambda \rightarrow \int_\Gamma f d\lambda$ is *SO*-continuous on $M_d(\Gamma)$.

(L') $\lambda \rightarrow \int_\Gamma f d\lambda$ is *WO*-continuous on $M_d(\Gamma)$.

Proof. That (I') is equivalent to (J') is immediate since convex sets, in particular, the kernel of a linear functional, have the same closures in *WO* or *SO*. Similarly, (K') is equivalent to (L').

Let (A') be satisfied. Then $\lambda \rightarrow \int_\Gamma f d\lambda$ is *w*-continuous on $M(\Gamma)$, and hence by Proposition 2.3 it is *SO*-continuous on $M(\Gamma)$. Thus (A') implies (I').

That (I') implies (K') is clear since $M_d(\Gamma) \subset M(\Gamma)$.

Let (L') be satisfied. Extend the *WO*-continuous linear functional

$$\lambda \rightarrow \int_\Gamma f d\lambda \quad \text{on} \quad M_d(\Gamma)$$

to a *WO*-continuous linear functional on all of $M(\Gamma)$. By Proposition 2.3, this linear functional will be *w*-continuous and hence has the form

$$\lambda \rightarrow \int_\Gamma g d\lambda, \quad \text{for some} \quad g \in M(G)^\wedge.$$

By considering λ a unit point measure, it follows that $f = g$. Hence (A') is satisfied. ■

2.5. *Remark.* Theorems 2.2 and 2.4 hold with $L^1(\Gamma)$ replacing $M(\Gamma)$.

Chapter III. Characterization of Sidon sets

Let G be compact and $E \subset \Gamma$ (discrete) be such that to every bounded (and continuous) function ϕ on E there corresponds a measure $\mu \in M(G)$ such that $\mu^\wedge = \phi$ on E . Then E is called a *Sidon set* (9), p. 121):

3.1. PROPOSITION. Let $E \subset \Gamma$, Γ discrete. Then E is a Sidon set if and only if $M(G)^{\wedge}|E = C^B(E)$.

Proof. If E is a Sidon set, then $M(G)^{\wedge}|E = C^B(E)$ and hence $M(G)^{\wedge-}|E = C^B(E)$.

Suppose $M(G)^{\wedge-}|E = C^B(E)$. To show that E is a Sidon set. If E were not a Sidon set, then there would exist $\phi \in C^B(E)$, $\phi(y) = \pm 1$, such that for $\mu \in M(G)$, $\|\mu^{\wedge} - \phi\|_{\infty, E} \geq 1$ ((9), p. 123). Hence $\phi \in C^B(E)$ but $\phi \notin M(G)^{\wedge-}|E$, and therefore E is a Sidon set. ■

3.2. THEOREM. Let Γ be discrete. The following are equivalent for $E \subset \Gamma$:

(S₁) E is a Sidon set.

(S₂) For all $f \in C^B(E)$, if $\{\lambda_n\} \subset M(E)$, $\|\lambda_n^{\wedge}\|_{\infty} \leq 1$, and $\lambda_n^{\wedge}(x) \xrightarrow{n} 0$, for all $x \in G$, then

$$\int_E f d\lambda_n \xrightarrow{n} 0.$$

(S₃) For all $f \in C^B(E)$, if $\{\lambda_n\} \subset M(E)$, $\|\lambda_n\| \leq 1$, and $\lambda_n^{\wedge}(x) \xrightarrow{n} 0$, for all $x \in G$, then

$$\int_E f d\lambda_n \xrightarrow{n} 0.$$

Proof. Let E be a Sidon set and $f \in C^B(E)$. There is $\mu \in M(G)$ such that $\mu^{\wedge}|E = f$. Hence

$$\int_E f d\lambda = \int_E \mu^{\wedge} d\lambda = \int_{\Gamma} \mu^{\wedge} d\lambda = \int_G \lambda^{\wedge} d\mu \quad \text{for } \lambda \in M(E).$$

Let $\{\lambda_n\} \subset M(E)$ be such that $\|\lambda_n^{\wedge}\|_{\infty} \leq 1$ and $\lambda_n^{\wedge}(x) \xrightarrow{n} 0$ for all $x \in G$. Then the Lebesgue dominated convergence theorem implies that

$$\int_E f d\lambda_n = \int_G \lambda_n^{\wedge} d\mu \xrightarrow{n} 0.$$

Clearly, (S₂) implies (S₃).

Let (S₃) be satisfied. If E is not a Sidon set, we may find $\lambda_2 \in M(E)$, $\|\lambda_2\| = 1$ and $\|\lambda_2^{\wedge}\|_{\infty} \leq \frac{1}{2}$ ((9), p. 121). We may assume that $F_2 = \text{support of } \lambda_2$ is finite. Since F_2 is finite and E is not a Sidon set, then $E \setminus F_2$ is not a Sidon set. Similarly there exists $\lambda_3 \in M(E \setminus F_2)$, $\|\lambda_3\| = 1$, $F_3 = \text{supp } \lambda_3$ finite, and $\|\lambda_3^{\wedge}\| \leq \frac{1}{3}$. Likewise, there exists

$$\lambda_{n+1} \in M(E \setminus F_2 \cup F_3 \cup \dots \cup F_n).$$

$$\|\lambda_{n+1}\| = 1, \quad F_{n+1} = \text{supp } \lambda_{n+1} \text{ finite} \quad \text{and} \quad \|\lambda_{n+1}^{\wedge}\|_{\infty} \leq \frac{1}{n+1}.$$

Let $f \in C^B(E)$, $\|f\|_{\infty, E} \leq 1$ be defined such that

$$\int_{F_n} f|F_n d\lambda_n = 1.$$

Now $\{\lambda_n\} \in M(E)$, $\|\lambda_n\| = 1$, $\lambda_n^{\wedge}(x) \xrightarrow{n} 0$ for all $x \in G$, but

$$\int_E f d\lambda_n = 1.$$

This is contradiction and so E must be a Sidon set. ■

3.3. PROPOSITION. Let Γ be discrete. The following are equivalent for $E \subset \Gamma$.

(S₁) E is a Sidon set.

(S₄) For all $f \in C^B(E)$, $\lambda \rightarrow \int_E f d\lambda$ is SO-continuous on $M(E)$.

Proof. Let E be a Sidon set. Since G is compact, the SO -topology is equivalent to the sup-norm topology on G . Let $\lambda_n \in M(E)$ be such that $\|\lambda_n\|_\infty \xrightarrow{n} 0$ and $f \in C^B(E)$. There exists $\mu \in M(G)$ such that $\mu^\wedge|E = f$. Hence

$$\left| \int_E f d\lambda_n \right| = \left| \int_G \mu^\wedge d\lambda_n \right| = \left| \int_G \lambda_n^\wedge d\mu \right| \leq \|\lambda_n^\wedge\|_\infty \|\mu\| \xrightarrow{n} 0.$$

Hence $\lambda \rightarrow \int_E f d\lambda$ is SO -continuous on $M(E)$.

Let (S_4) be satisfied. If E is not a Sidon set, then there exists $\{\lambda_n\} \subset M(E)$ such that $\|\lambda_n\| = 1$ and $\|\lambda_n^\wedge\|_\infty \leq (1/n)$. We may assume that $\text{supp } \lambda_n$ is finite. It follows as in the proof of Proposition 3.2 that $\lambda \rightarrow \int_E f d\lambda$ is not SO -continuous. This contradicts (S_4) . Hence E is a Sidon set. \blacksquare

Let Z denote the group of integers and let $P \subset Z$ contain arbitrarily long arithmetic progressions. P is repeating in the sense that there is an infinite set $S \subset Z^+$ such that for each $s \in S$ there is a $p \in P$ and $q \in Z$ such that $q \geq 1$ and

$$\{p+q, p+2q, \dots, p+sq\} = P_s \subset P.$$

We call such a repeating set an R -set.

It is known that a Helson set (i.e. a compact Sidon set) P on the line R cannot contain arbitrarily long arithmetic progressions ((9), p. 117). The following proposition is the corresponding result for Sidon sets.

3.4. PROPOSITION. *Let $P \subset Z$ be an R -set. Then P is not a Sidon set.*

Proof. Let P be the given R -set. Let $S \subset Z^+$ be the associated infinite set such that for each $s \in S$ there is a $p \in P$ and $q \in Z$ such that $q \geq 1$ and

$$P_s = \{p+q, p+2q, \dots, p+sq\} \subset P.$$

We may assume that for $s_1, s_2 \in S$, $P_{s_1} \cap P_{s_2} = \emptyset$. For $s \in S$, let

$$\lambda_s = \frac{1}{s} \sum_{n=1}^s e^{-in \log n} \delta_{p+nq},$$

where δ_{p+nq} is the unit point measure at $p+nq$. Then $\|\lambda_s\| = 1$ and there exists $M > 0$ such that

$$\begin{aligned} \|\lambda_s^\wedge\|_\infty &= \sup \left\{ \left| \frac{1}{s} \sum_{n=1}^s e^{-in \log n} e^{-i(p+nq)\theta} \right| : 0 \leq \theta \leq 2\pi \right\} \\ &= \sup \left\{ \left| \frac{1}{s} \sum_{n=1}^s e^{in \log n} e^{inq\theta} \right| : 0 \leq \theta \leq 2\pi \right\} \leq \frac{1}{s} \cdot M \cdot s^{\frac{1}{2}} = \frac{M}{s^{\frac{1}{2}}} \xrightarrow{s} 0 \end{aligned}$$

((11), vol. I, p. 199). Let $f \in C(P)$ be such that $f(p+nq) = e^{in \log n}$.

That P is not a Sidon set is now immediate from Theorem 3.2. \blacksquare

3.5. Remark. Theorem 3.2 could be derived by pairing $M(G)^\wedge|E$ and $M(E)$ as in Chapter I.

3.6. COROLLARY. *A Sidon set does not contain arbitrarily long arithmetic progressions.*

Proof. This is a rewording of Proposition 3.4. \blacksquare

3.7. Remark. Corollary 3.6 is a result due to Rudin ((10), p. 216).

REFERENCES

- (1) BOURBAKI, N. *Espaces vectoriels topologiques*, Ch. III-V (Hermann, Paris, France, 1964).
- (2) BUCK, R. C. Bounded continuous functions on a locally compact space. *Michigan Math. J.* 5 (1958), 95-104.
- (3) DAY, M. *Normed linear spaces* (second printed corrected). (Academic Press Inc., New York, N.Y., 1962).
- (4) DUNFORD, N. and SCHWARTZ, J. *Linear operators*, Part I. (Interscience Publishers, New York, N.Y., 1958).
- (5) EDWARDS, R. E. Uniform approximation on non-compact spaces. *Trans. Amer. Math. Soc.* 122 (1966), 249-276.
- (6) HEWITT, E. A survey of abstract harmonic analysis. *Abstract harmonic analysis, surveys in applied mathematics. IV*, pp. 105-168 (J. Wiley and Sons, Inc., 1958).
- (7) KÖTHER, G. *Topologische lineare raume I*. (Springer-Verlag, Berlin, 1960).
- (8) RICKART, C. *General theory of Banach algebras* (D. Van Nostrand Company, Inc., Princeton, 1960).
- (9) RUDIN, W. *Fourier analysis on groups* (Interscience Publishers, New York, N.Y., 1962).
- (10) RUDIN, W. Trigonometric series with gaps. *J. Math. Mech.* 9 (1960), 203-228.
- (11) ZYGMUND, A. *Trigonometric series* (Cambridge University Press, New York, N.Y., 1959).