# $SU(n)/SO(n)$ : Landweber's manifolds

Recently, Peter Landweber wrote to ask some questions about the manifolds $SU(n)/SO(n)$. These are quite interesting manifolds and I thought I would tell you a bit about them. At the same time, I will include $U(n)/O(n)$

To begin, one has fibrations

$$SU(n)/SO(n) \xrightarrow{i} BSO(n) \xrightarrow{\otimes \mathbb{C}} BSU(n)$$

$$U(n)/O(n) \xrightarrow{i} BO(n) \xrightarrow{\otimes \mathbb{C}} BU(n)$$

This gives $n$-plane bundles $E_n$ over $U(n)/O(n)$ and $SU(n)/SO(n)$ with $E_n \otimes \mathbb{C}$ being a trivial complex bundle. These are the universal (oriented) $n$-plane bundles with trivial complexification.

If you consider the space of real $n$-dimensional subspaces $V \subset \mathbb{C}^n$ with the property that $iV = V^\perp$ is the orthogonal complement of $V$ one has what is called the <u>Lagrangian Grassmannian</u> : $\Lambda_n = U(n)/O(n)$ and $E_n$ consists of the pairs — a subspace and a vector in that subspace. $S\Lambda_n = SU(n)/SO(n)$ is called the <u>special Lagrangian Grassmannian</u>.

If you consider mod 2 cohomology for the fibrations one has

$$H^*(BSO(n)) = \mathbb{Z}_2[w_2, w_3, \ldots, w_n] \longleftarrow H^*(BSU(n)) = \mathbb{Z}_2[c_2, c_3, \ldots, c_n]$$

$$H^*(BO(n)) = \mathbb{Z}_2[w_1, w_2, \ldots, w_n] \longleftarrow H^*(BU(n)) = \mathbb{Z}_2[c_1, c_2, \ldots, c_n]$$

and one has $(\otimes \mathbb{C})^*(c_i) = w_i^2 \quad (c(E \otimes \mathbb{C}) = w(E \otimes \mathbb{C}) = w(E \oplus E) = w(E)^2)$

from which one has

$$H^*(U(n)/O(n)) = \frac{\mathbb{Z}_2[w_1, \ldots, w_n]}{(w_1^2, \ldots, w_n^2)} \quad \text{and} \quad H^*(SU(n)/SO(n)) = \frac{\mathbb{Z}_2[w_2, \ldots, w_n]}{(w_2^2, \ldots, w_n^2)}.$$

given as the exterior algebras on the Stiefel–Whitney classes of the bundles $E_n$.

One has $\dim U(n)/O(n) = 1 + 2 + \cdots + n = \frac{n(n+1)}{2} = \binom{n+1}{2}$ and $\dim SU(n)/SO(n) = 2 + \cdots + n = \binom{n+1}{2} - 1$.